MOTIVATION

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I learned the following example from Magidor. There is a sentence ϕ of $L_{\omega_1,\omega}$ that for a cofinal set of λ below the first measurable, ϕ has no maximal model in λ.

The point of these papers is to find a ϕ which is complete.

{incompex}

Example 0.1. Consider a class $\boldsymbol{K}$ of 3-sorted structures where: P_0 is a set, P_1 is a boolean algebra of subsets of P_0 (given by an extensional binary E) and P_2 is just a set; $\{F_n : n < \omega\}$ is a family of unary functions which assigns to each $c \in P_2$, a sequence $F_n(c) \in P_1$. Demand: $\bigwedge_n F_n(c) = F_n(d)$ implies $d = c$. Let $\psi \in L_{\omega_1,\omega}$ axiomatize $\boldsymbol{K}$. We claim M is a maximal model of $\mod(\psi)$ with cardinality λ if $\lambda <$ first measurable, $|P_0^M| = \lambda$, $P_1^M = \mathcal{P}(P_0^M)$, and P_2^M codes each sequence in ${}^{\omega}(P_1^M)$ via the F_n.

Suppose for contradiction that N with $M \precneqq_{\omega_1,\omega} N$ witnesses non-maximality, then the choice of M and the demand imply that there must be an element $a^* \in P_0^N - P_0^M$. Then $D = \{b \in P_1^M : E(a^*, b)\}$ is a non-principal ultrafilter on $\lambda = P_0^M$. To see that D is non-principal, note that if some $b' \in P_1^M$ generated D, then $b' \lneqq a^*$, contrary (by elementary extension) to their both being atoms.

Since D is $\aleph_1$-incomplete (as λ is not measurable) there exists a sequence $\langle b_n : n < \omega\rangle$ of elements of P_1^M with empty intersection. Since each countable sequence of subsets of P_0^M is coded as $\langle F_n^M(c) : n < \omega\rangle$ for some $c \in P_2^M$, there is a $d \in P_2^M$ with $F_n(d) = b_n$ for each n. Thus, $M \models \neg(\exists x) \bigwedge E(x, F_n(d))$, while $N \models \bigwedge E(a^*, F_n(d))$. This contradicts $M \prec_{\omega_1,\omega} N$.

There are $2^{\aleph_0}$ 2-types over the empty set, given, for each $X \subset \omega$, via (c, d) realizes p_X iff $X = \{n : F_n(c) \cap F_n(d) \neq \emptyset\}$. This implies no sentence satisfied by M can be complete, since a minor variant of Scott's characterization of countable models shows that a sentence ψ is complete if and only if only countably many $L_{\omega_1,\omega}$-types over $\emptyset$ are realized in models of ψ. In Section **??** we modify this example to obtain a complete sentence.

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What are the difficulties making ϕ complete (and preserving maximality)?