STRONGLY MINIMAL STEINER SYSTEMS III: PATH GRAPHS AND SPARSE CONFIGURATIONS

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Abstract. We introduce a uniform method of proof for the following results. There are, for *each* of the following conditions, $2^{\aleph_0}$ families of elementarily equivalent Steiner sytems, satisfying: i) (extending [CGGW10]) each Steiner triple system is ∞-sparse and has a uniform but not perfect path graph; ii) (extending [CW12] each Steiner k-system (for $k = p^n$) has a uniform path graph (infinite cycles only) iii) extending [Fuj06a], each is anti-Pasch (anti-mitre); iv) has an explicit quasi-group structure, v) every model is 2-transitive. Each family has $\aleph_0$ countable models and one model of each uncountable cardinal.

In this paper we expound some applications of the Hrushovski construction of strongly minimal sets to the combinatorics of infinite Steiner systems. We reformulate (Section 2) the notion of sparse configurations [CGGW10, Fuj06a] in terms of the δ-function fundamental to the Hrushovski construction and give uniform accounts of the existence of anti-Pasch and anti-mitre Steiner triple systems (STS). While the examples of strongly minimal pure Steiner systems (M, R) admit no definable 'truly binary' operation with infinite domain [BV21], we construct (Section 3) strongly minimal quasigroups which induce q-Steiner systems (line length q) for q a prime power. We extend (Section 4) the notion of (a, b)-cycle graph $G_M(a, b)$ of an infinite STS [CW12] to *path-graphs* of q-Steiner systems induced by quasigroups. Rather than *ad hoc* examples, we provide a method to construct first order theories and thus infinite families of countable models exhibiting various combinatorial properties. In particular, the countable models of these theories are arranged in a *tower*, a countable increasing sequence $\langle M_i : i < \omega\rangle$. The structure of $G_{M_0}(a, b)$ depends heavily on whether $\mathrm{acl}_{\mathrm{M}_0}(\emptyset) = \emptyset$. In various cases $G_{M_0}(a, b)$ may have only finite cycles, only infinite cycles or a mixture. In Section 5, we construct 2-transitive models, which so have uniform path graphs. For this we must alter different sets of the parameters for a Hrushovski discussion that we describe in Notation 0.4. We construct in several ways theories of Steiner systems where every model is 2-transitive.

A first order theory T is strongly minimal if every definable subset of every model of T is finite or co-finite. Three prototypical examples are the theories of: the integers with successor, rational addition, and the complex field. Each model of T determines a combinatorial geometry (matroid) given by algebraic closure. Zilber conjectured these examples were canonical; each such geometry was discrete, vector space like, or field like. Hrushovski refuted this conjecture by an intricate extension of Fraïssé's construction of countable homogeneous universal models. Building on

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[BP20] we vary his construction to obtain the Steiner systems with the properties noted.

A *linear space* is collection of points and lines such that two points determine a line, a minimal condition to call a structure a geometry. A linear space is a Steiner k-system if every line (block) has cardinality k. We showed in Section 2 of [BP20] that linear spaces can be naturally formulated in a one-sorted logic with single ternary 'collinearity' predicate and proved:

Fact 0.1 ([BP20])**.** *For each k, with $3 \leqslant k < \omega$, there are $2^{\aleph_0}$ strongly minimal theories T_μ (depending*[1] *on an integer valued function μ) of infinite linear spaces in the one-sorted vocabulary τ whose models are Steiner k-systems.*

These theories are model complete and satisfy the usual properties of counterexamples to Zilber's trichotomy conjecture. Their acl*-geometries are non-trivial, not locally modular, and the theory cannot interpret a group.*

Finite Steiner systems were first defined in the 1840's and developed by such mathematicians as Steiner, Bose, Skolem, and Bruck. Much of the history of Steiner systems interacts with the general study of non-associative algebraic systems such as quasigroups. A quasigroup is a structure with a single binary operation whose multiplication table is a Latin Square (each row or column is a permutation of the universe) [Ste56]. This is the third paper in a series developing the properties of strongly minimal Steiner systems. Existence is shown in [BP20]. Drawing on universal algebra and combinatorics, we [Bal21] found that the restriction to prime power cardinality of the universe that is essential to existence in the finite for the existence of quasigroups coordinatizing q-Steiner systems is replaced by prime power block length for strongly minimal Steiner systems.

Theorem 0.2. *(Theorem 3.6) For each q and each of the T_μ in Theorem 0.1 with line length $k = q = p^n$ and certain varieties of quasigroups V, there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that interprets a strongly minimal q-Steiner system.*

The following theorem from [BV21] suggests a finer classification of flat strongly minimal sets.

Theorem 0.3. *(*[BV21, Theorem 0.2]*) Let T_μ be among the family of strongly minimal (Steiner systems) theories in* [Hru93] *(*[BP20]*). If μ is triplable*[2] *T_μ does not admit a 'non-trivial' definable binary function and so does not interpret a quasigroup. Even without the triplable hypothesis, there is no definable commutative binary function.*

These counterexamples with Hrushovski's 'flat geometries' [BP20, Definition 6.2] have generally been regarded as an undifferentiated class of exotic structures. However, there are both Steiner systems and quasigroups are among them [BP20, Bal21] and [BV21] shows that the structural distinctions arise by varying the function μ. In fact, the family of 'Hrushovski constructions of strongly minimal sets' depend on five parameters.

Notation 0.4. A *Hrushovski sm-class* is determined by a quintuple $(\sigma, \boldsymbol{L}^*, \boldsymbol{L}_0, \epsilon, \mathbf{U})$. $\boldsymbol{L}^*$ is a collection of finite structures in a vocabulary σ, not necessarily closed under

[1] The theory of course depends on the line length k; but it is coded by μ so we suppress the k.

[2] $\mu(A/B) \geqslant 3$ if $\delta(B) \geqslant 2$

substructure[3]. ϵ is a function from a specified collection of finite σ-structures to natural numbers satisfying the conditions imposed on δ in Definition 1.2.3. $\boldsymbol{L}_0$ is a subset of $\boldsymbol{L}^*$ defined using ϵ. From such an ϵ, one defines notions of $\leqslant$, primitive and good pair. Hrushovski gave one technical condition on the function μ counting the number of realizations of a good pair that ensured the theory is strongly minimal rather than ω-stable of rank ω. We consider the class $\mathbf{U}$ as the collection of functions μ satisfying a specific condition provides way to index a rich group of distinct constructions. As explained in Definition 1.1.3, from $\boldsymbol{L}_0, \epsilon$ and $\mu \in \mathbf{U}$, one defines $\boldsymbol{L}_\mu$ and $\hat{\boldsymbol{L}}_\mu$. (For any collection $\boldsymbol{L}$ of finite structures, we write $\hat{\boldsymbol{L}}$ for the collection of direct limits of structures in $\boldsymbol{L}$.) Thus one obtains a strongly minimal theory T_μ and a generic structure $\mathcal{G}_\mu$.

We rely heavily on [BP20, BV21, Bal21], sketching basic arguments and emphasizing how the parameters of the last paragraph are changed to get the specific result.

For convenience, one usually specifies in $\boldsymbol{L}^*$ that the relations are symmetric; but to reach important cases such as quasigroups and Steiner systems one adds the relevant axioms to this starting point. And $\boldsymbol{L}^*$ is made $\forall\exists$ axiomatizable to create quasigroups. Working in linear spaces with a 'geometric' ϵ in [Pao20] is vital to obtain Steiner systems. In this paper, to obtain Steiner systems which are (e.g. anti-Pasch, ∞-sparse, 2-transitive) we both vary the class U of admissible μ-functions and change the way that the class of finite structures $\boldsymbol{L}_0$ is determined by the relevant δ playing the role of ϵ.

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1. Background

We show how modifications of the most basic Hrushovski construction provides examples in Steiner systems. [BP20, 2.1, 2.2] summarises the role of strongly minimal sets in model theory and the bi-interpretability of a one-sorted (used here) and two-sorted approach to Steiner system. This paper elaborates the method for uses in combinatorics. [Bal] provides a somewhat outdated survey of vastly wider study of modifications of the construction to study e.g. fusions, 'bad' fields, Spencer-Shelah random graphs and higher levels of stability classification. Section 1.1 outlines the general setting emphasizing the parameters that can be varied to get specific behaviors. Remark 1.1.9 reminds us of the original context; Section 1.2 lays out the notation for studying linear spaces.

1.1. **The Hrushovski framework**

The basic ideas of the Hrushovski construction are i) to modify the Fraïssé construction by replacing substructure by a notion of strong substructure, defined using a predimension δ (Definition 1.2.3) so that independence with respect to the dimension induced by δ is a combinatorial geometry[4] and ii) to employ an *algebrizing function* μ to bound the number 0-primitive extensions of each finite structure so that closure in this geometry is algebraic closure.

[3] They were in [Hru93] but not here.

[4] The requirement that the range of this function is well-ordered is essential to get the exchange property in the geometry; using rational or real coefficients yields a stable theory and the dependence relation of forking [BS96].

A Steiner (t, k, v)-system is a pair (P, B) such that $|P| = v$, B is a collection of k element subsets of P and every t element subset of P is contained in exactly one block. Since we are primarily interested in infinite structures, we omit the v unless it is crucial and so, by Steiner k-system I mean Steiner $(2, k)$ system of arbitrary cardinality. A *groupoid* (also called a *magma*) is a structure $(A, *)$ with one binary function $*$.

Unfortunately, while the extensive literature on Hrushovski constructions contains the same fundamental notions related in a fairly standard way, the notation is not standard. So we quickly list our terminology.

We give an abstract formulation of the construction of generic model due to [KL92]. This provides a common framework for the Fraïssé and Hrushovski constructions which does *not* require the class $\boldsymbol{L}$ to be closed under substructure and is essential in Section 3. For the general discussion in this section we work in a finite relational vocabulary σ.

Notation 1.1.1. *(1) For any class $\boldsymbol{L}$ of finite structures, $\hat{\boldsymbol{L}}$ denotes the collection of structures of arbitrary cardinality that are direct limits*[5] *of models in $\boldsymbol{L}$.*

(2) Let σ be a finite relational vocabulary. A class $(\boldsymbol{L}_0, \leqslant)$ of finite structures, with a transitive relation $\leqslant$ on $\boldsymbol{L}_0 \times \boldsymbol{L}_0$ is called smooth *if $B \leqslant C$ implies $B \subseteq C$ and for all $B \in \boldsymbol{L}$ there is a collection $p^B(\mathbf{x})$ of universal formulas with $|\mathbf{x}| = |B|$ and for any $C \in \boldsymbol{L}_0$ with $B \subseteq C$,*

$$B \leqslant C \leftrightarrow C \models \phi(\mathbf{b})$$

for every $\phi \in P^B$ and $\mathbf{b}$ enumerates B.

We write B is strongly embedded in C if an isomorphic image B' of C satisfies $B' \leqslant C$.

(3) A structure A is a $(\boldsymbol{L}_0, \leqslant)$-union if $A = \bigcup_{n<\omega} C_n$ where each $C_n \in \boldsymbol{L}_0$ and $C_n \leqslant C_{n+1}$ for all $n < \omega$. If A is a $(\boldsymbol{L}_0, \leqslant)$-union, $B \subseteq A$, $B \in \boldsymbol{L}_0$, we say $B \leqslant A$ if $B \leqslant C_n$ for all sufficiently large n.

*(4) A structure A is an $(\boldsymbol{L}_0, \leqslant)$-*generic *if A is a $(\boldsymbol{L}_0, \leqslant)$-union and for any $B \leqslant C$ each in $\boldsymbol{L}_0$ and $B \leqslant A$ there is a $\leqslant$-embedding of C into A.*

$\leqslant$ is read 'strongly embedded'. The crucial fact is:

Fact 1.1.2 ([KL92])**.** *If $(\boldsymbol{L}_0, \leqslant)$ is a smooth class of finite structure with only countably many elements that satisfies $\leqslant$-amalgamation and $\leqslant$-joint embedding there is a unique countable generic $\mathcal{G}_{\boldsymbol{L}_0}$ for $(\boldsymbol{L}_0, \leqslant)$*

Axiom 1.1.3. *Let δ be a map from a collection finite σ-structures into N. Let $\boldsymbol{L}^*$ be a collection of such structures closed under isomorphism. We write $A \leqslant B$ if for every C with $A \subseteq C \subseteq B$, $\delta(C/A) \geqslant 0$. We require that $\boldsymbol{L}^*, \hat{\boldsymbol{L}}^*, \delta$ satisfy the following requirements. First, $\mathbf{L_0}$ is the collection of finite B such that:*

(1) $\delta(\emptyset) = 0$

(2) If $B \in \boldsymbol{L}^*$ and $A \subseteq B$ then $\delta(A) \geqslant 0$.

(3) If A, B, and C are disjoint then $\delta(C/A) \geqslant \delta(C/AB)$.

(4) $(\boldsymbol{L}^*, \delta)$ admits canonical amalgamations in the following sense.

[5] If $\boldsymbol{L}$ is closed under substructure so is $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{L}}$ is axiomatized by a universal sentence in $L_{\omega_1,\omega}$ ($L_{\omega,\omega}$ if the vocabulary is relational.).

Definition 1.1.4. Canonical Amalgamation *For any class $(\boldsymbol{L}_0, \epsilon)$, if $A \cap B = C$, $C \leqslant A$ and $A, B, C \in \boldsymbol{L}^*$, G is a* free (or canonical) amalgamation*, $G = B \oplus_C A$ if $G \in \boldsymbol{L}^*$, $\epsilon(A/BC) = \epsilon(A/C)$ and $\epsilon(B/AC) = \epsilon(B/C)$ Moreover, $\epsilon(A \oplus_C B) = \epsilon(A) + \epsilon(B) - \epsilon(C)$ and any D with $C \subseteq D \subseteq A \oplus_C B$ is also free. Thus, $B \leqslant G$.*

Disjoint union is the canonical amalgamation for the basic Hrushovski construction and Definition 1.2.4 gives the appropriate notion satisfying Axiom 1.1.3.5 for linear spaces. Axiom 1.1.3.2 can be rephrased as: $B \subseteq C$ and $A \cap C = \emptyset$ implies $\epsilon(A/B) \geqslant \epsilon(A/C)$; so we can make the following definition.

Definition 1.1.5. *Extend ϵ to $d : \hat{\boldsymbol{L}}^* \times \boldsymbol{L}^* \to N$ by for each $N \in \hat{\boldsymbol{L}}^*$, $d(N, A) = \inf\{\epsilon(B) : A \subseteq B \subseteq_\omega N\}$, $d_N(A/B) = d_M(A \cup B) - d_M(B)$. We usually write $d(N, A)$ as $d_N(A)$ and omit the subscript N when clear.*

What Hrushovski called *self-sufficient* closure is in the background.

Definition 1.1.6. *(1) For $N \in \hat{\boldsymbol{L}}^*$ and $A \in \boldsymbol{L}^*$, we say $A \subseteq N$ is strong in N and write $A \leq N$ if $d(N/A) \geqslant 0$.*
(2) For any $A \subseteq B \in \boldsymbol{L}^*$, the intrinisic (self-sufficient) closure of A, denoted $\mathrm{icl_B(A)}$ *is the smallest superset of A that is strong in B.*

Note that in the current situation icl(B) is finite if B is. The following definition describes the pairs $B \subseteq C$ such that eventually tp(C/B) will be an algebraic set (realized only finitely often).

Definition 1.1.7. *Let $A, B \in \boldsymbol{L}^*$ with $A \cap B = \emptyset$ and $A \neq \emptyset$.*

(1) B is a primitive extension *of A if $A \leqslant B$ and there is no $A \subsetneq B_0 \subsetneq B$ such that $A \leqslant B_0 \leqslant B$.*
  B is a k-primitive extension *if, in addition, $\epsilon(B/A) = k$.*
  We stress that in this definition, while B may be empty, A cannot be.
(2) We say that the 0-primitive pair A/B is good *if there is no $B' \subsetneq B$ such that (A/B') is 0-primitive. (This notion was originally called a minimal simply algebraic or m.s.a. extension.)*
(3) If A is 0-primitive over B and $B' \subseteq B$ is such that we have that A/B' is good, then we say that B' is a base *for A (or sometimes for AB).*
(4) If the pair A/B is good, then we also write (B, A) is a good pair*.*

Definition 1.1.8. *(1) Let $\mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{L}^*$ a number $\mu(\beta) = \mu(B, C) \geqslant \epsilon(B)$.*
(2) For any good pair (B, C) with $B \subseteq M$ and $M \in \hat{\boldsymbol{L}}^*$, $\chi_M(B, C)$ denotes the number of disjoint copies of C over B in M. A priori, $\chi_M(B, C)$ may be 0.
(3) Let $\boldsymbol{L}_\mu$ be the class of structures M in $\boldsymbol{L}^*$ such that if (B, C) is a good pair $\chi_M(B, C) \leqslant \mu((B, C))$.

Up to this point, we have denoted the rank function by ϵ to indicate it is being treated entirely axiomatically. *We switch to* δ to emphasize that (Hrushovski's definition or Paolini's) may be used but trust to context for the reader to know which.

Remark 1.1.9 (The basic Hrushovski construction)**.** *In the original context* [Hru93]*, σ consists a single ternary relation R and $\delta(A) = |A| - r(A)$ where $r(A)$ is the number of triples $\boldsymbol{a}$ from A satisfying $R(\boldsymbol{a})$. $\boldsymbol{L}^*$ is all finite σ-structures and $\boldsymbol{L}_0$ is those $A \in \boldsymbol{L}^*$ with $\emptyset \leqslant A$ and $\mathcal{U}$ is as in Definition 1.1.8 with the relevant choice of δ.*

1.2. **Linear Spaces**

In this section we outline the adaptation of Remark 1.1.9 that generates most of the examples in this paper. For the remainder of the paper we will deal at various times with two vocabularies τ, with a single ternary relation symbol, R, and τ' with a second ternary relation H, which will be the graph of a binary function $*$

Definition 1.2.1. *A τ-structure (M, R) is*

(1) a 3-hypergraph *if R holds only of distinct triples and in any order.*
(2) a linear space *if it is a 3-hypergraph in which two points determine a unique line. That is, each pair of distinct points in contained in unique maximal R-clique (line). That is, all triples from the line satisfy R.*
*(3) A linear space is a k-*Steiner system *if all lines have the same length k.*

Thus our finite structures will in general be partial k-Steiner systems (lines may not have full length) for some k. We use the words 'block' and 'line' interchangeably and often fail to distinguish when the line has full length. When this is important, we may write clique to denote a subset of a line, i.e., a maximal clique.

Definition 1.2.2. *(1) For $\ell \subseteq A$, we denote the cardinality of a clique ℓ by $|\ell|$, and, for $B \subseteq A$, we denote by $|\ell|_B$ the cardinality of $\ell \cap B$.*
(2) We say that a non-trivial line ℓ contained in A is based *in $B \subseteq A$ if $|\ell \cap B| \geqslant 2$, in this case we write $\ell \in L(B)$.*
(3) The nullity of a line *ℓ contained in a structure $A \in \mathbf{K}^*$ is:*

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

Now we define our geometrically based pre-dimension function [Pao20].

Definition 1.2.3. *We define the appropriate $\boldsymbol{K}^*$ and $\boldsymbol{K}_0$.*

(1) Every $(A, R) \in \boldsymbol{K}^*$ is a finite linear spaces.
(2) For $(A, R) \in \boldsymbol{K}^*$ let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

(3) Moreover $(A, R) \in \boldsymbol{K}_0$ if for any $A' \subseteq A, \delta(A') \geqslant 0\}$.
(4) $(\boldsymbol{K}_0, \delta)$ satisfies the conditions on ϵ given in Section 1.1

The explicit definition of the free amalgamation *in this context* is:

Definition 1.2.4. [BP20, Lemma 3.14] *Let $A \cap B = C$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_C B$ as follows:*

(1) the domain of D is $A \cup B$;
(2) a pair of points $a \in A - C$ and $b \in B - C$ are on a non-trivial line ℓ' in D if and only if there is line ℓ based in C such that $a \in \ell$ (in A) and $b \in \ell$ (in B). Thus $\ell' = \ell$ (in D).

We single out a type of good pair that provides the line-length invariant for the Steiner systems.

Notation 1.2.5 (Line length)**.** *We write $\boldsymbol{\alpha}$ for the isomorphism type of the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$. Note (Lemma 5.18 of* [BP20]*) lines in models of T_μ have length k if and only if $\mu(\boldsymbol{\alpha}) = k - 2$.*

If one restricts the counting functions to $\mathcal{U}$ (Definition 1.1.8), Steiner triple systems are excluded. Since they are a key topic, the $\mathcal{U}$ is slightly altered from Definition 1.1.8 to admit them.

Definition 1.2.6 ($\mathcal{U}^{\mathrm{ls}}$)**.** *Let $\mathcal{U}^{\mathrm{ls}}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$ a number $\mu(\beta) = \mu(B, C) \geqslant \delta(B)$.*

(i) a number $\mu(\beta) = \mu(B, C) \geqslant \delta(B)$, if $|C - B| \geqslant 2$;
(ii) a number $\mu(\beta) \geqslant 1$, if $\beta = \boldsymbol{\alpha}$.

We will omit the superscript ls, when it is clear we working with linear spaces.

2. Omitting configurations in Steiner triple systems

There is a long history of studying finite Steiner triple systems that omit specific configurations, e.g. Pasch. We derive such results for infinite Steiner triple systems by variants on our general construction. We begin by examining the connection between the Pasch configuration [Fuj06b] and the group configuration from model theory. The notion of an ∞-sparse system uniformizes these anti-x constructions [CGGW10]. We first find specific amalgamation constructions that give the strongly minimal Steiner systems omitting certain configurations by varying the class U of acceptable bounds on algebraicity. We next obtain ∞-sparseness, by enforcing the uniformity with δ, and finally with more drastic restrictions on the class $\boldsymbol{K}_0$.

2.1. Anti-Pasch and Anti Mitre

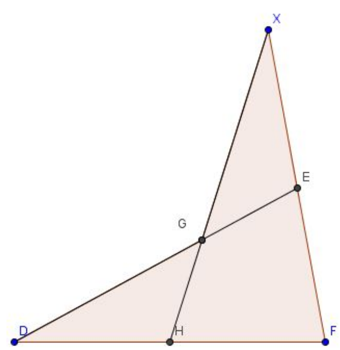

Figure 1. Pasch configuration

Diagram 2.1 is known in the study of Steiner triple systems as the *Pasch configuration*. This same diagram, interpreting the lines as representing algebraic closure, is known to model theorists as the *group configuration*: here the acl-dimension of the set of 6 points is 3; any triple of non-collinear points are independent; each point has acl-dimension 1, and each line has acl-dimension 2. Hrushovski's proof, described for the Steiner system case in [BP20, Corollary 6.3], that no T_μ interprets an infinite group originated the model theoretic argument that the group configuration in the algebraic closure geometry implies the existence of a definable infinite group. We give a more direct argument for:

Fact 2.1.1. *The strongly minimal quasigroups whose existence is proven in Section 3 have no infinite definable associative subquasigroup.*

Proof. Let G be a definable infinite subquasigroup of $\mathcal{G}_\mu$ with associative multiplication that is generated by three algebraically independent elements, say D, G, H as in Figure 2.1. Now $DG = E$ so, by associativity, $F(DG) = FE = X$. Similarly, $H = FD$ implies $(FD)G = HG = X$ so the lines HG and FE intersect in X. In any T_μ the algebraic closure dimension of a closed subset A is $d(A) = \delta(A)$. So if A is the six points of the configuration we should have $\delta(A) \geqslant d(A) = 3$. But the actual calculation[6] gives $\delta(A) = 2$. So the Pasch configuration is omitted. $\blacksquare_{2.1.1}$

In particular, a strongly minimal quasigroup constructed in this way can never be a group. Nevertheless, in general there will be many realizations of a Pasch configuration P in a strongly minimal Steiner triple system, since $\delta(P) = 2 \geqslant 0$. Indeed any pair of points extends to a Pasch configuration in the generic model. Fact 2.1.1 shows in general it cannot extend to an infinite subquasigroup. But we prove more; by specializing the construction, we can guarantee there is no occurrence of the Pasch-configuration. That is, the quasigroup is anti-Paschian.

We extend the considerable investigation (e.g. [Fuj06a]) of finite anti-Pasch quasigroups by providing infinite examples. We need the following notion.

Definition 2.1.2 (R-closure)**.** *Let (M, R) be a τ-structure. We define the R-closure, $\mathrm{cl_R(X)}$, for $X \subset M$. Define inductively $X = X_0$ and for each n, $c \in X_{n+1}$ if $a, b \in X_n$ and $R(a, b, c)$. Now $\mathrm{cl_R(X) = X_N}$, where N (possibly ω) is where the inductively defined sequence X_n terminates. A set X is R independent if no element is in the R-closure of the others.*

Lemma 2.1.3. *The subclass of $\boldsymbol{K}_0^P$ of those finite structures with 3-element lines that omit the Pasch configuration satisfies amalgamation.*

Proof. We can reformulate the problem by setting $\boldsymbol{\rho}$ as the isomorphism type of the good pair (A/B) in Figure 2.1, taking $\{F, H\}$ as the base B and $A = \{X, D, E, G\}$ as a good extension. We use the standard $\boldsymbol{K}_0$ for linear space. But we modify a $\mu \in \mathcal{U}$ by setting $\mu(A/B) = 0$. We must show $\boldsymbol{K}_\mu$ has amalgamation.

Fixing notation as in the proof of amalgamation in [BP20, 5.11], consider structures with (E/D) a good pair, $D \subseteq F$ and all in $\boldsymbol{K}_0$; we want to amalgamate F and E over D. Note that every non-trivial line that intersects $E - D$ is contained in E and has two elements in $E - D$. This holds, as if the line intersects $F - D$ then it has 4 points by Definition 1.2.4. But, if it intersects D in 2 points E is not primitive over D. Thus F is R-closed in G. The key property of the Pasch configuration is that each point not in the base is on a 3-element line that intersects the base. This implies that if there is an embedding of the Pasch configuration P in G, the image of the base EH is contained in D. (Otherwise there would be a line from $F - D$ to $E - D$.) But since the Pasch configuration is R-generated by the base along with any other point, we have $A \subseteq F$ if $A \cap F \neq \emptyset$ and $A \subseteq E$ if not. Either violates the hypothesis that F and E omit the Pasch configuration. $\blacksquare$

Applying Lemma 2.1.3 any model of T_μ satisfies:

Corollary 2.1.4. *There is a strongly minimal anti-Pasch Steiner triple system.*

[6]Hrushovski isolates geoemtries supporting this calculation as 'flat'.

Similar arguments construct anti-mitre and anti-mia configurations. The mitre configuration is shown in Figure 2.1. Letting abc be the bottom line, $c'b'a'$ the middle, and x the vertex, the diagram represents the left self-distributive law:

$$x(ab) = (xa)(xb).$$

Namely the self distributive law implies naming a' as xc and c' as xa the lines ac', bb', ca' intersect at x. This $(5, 7)$-configuration [Fuj06b] is called a *mitre*[7]; The only other $(5, 7)$-configuration, (*mia*), is obtained by adding a point between the two points on the base of the Pasch configuration and creating a new line. By constructing ∞-sparse configurations below we simultaneously omit the Pasch, mitre, and mia configurations.

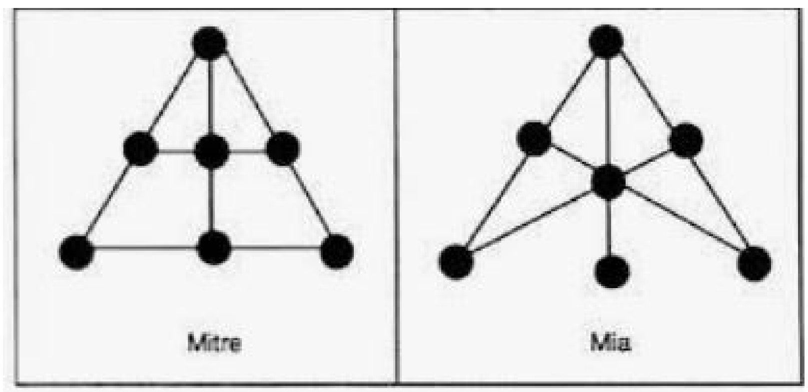

FIGURE 2. Mitre and mia configurations

Lemma 2.1.5. *The subclass $\boldsymbol{K}_0^M$ of $\boldsymbol{K}_0$ consisting of those finite structures with* 3*-element lines that omit the mitre (or those omitting the mia) configuration satisfies amalgamation.*

Proof. We use the first paragraph of the proof of Lemma 2.1.3. The argument there that the base is contained in D, here yields only that two points of the base are in $E - D$. Say $a \in F$ and $b, c \in E - D$. We violate F closure unless the point $F - E$ is c'. Now if the pivot x is in $E - D$, we violate the R-closure of F. But if $x \in F$ and a' or b' is in F, $a'xa$ or $b'xb$ violates that E is primitive over D. While if either is in $E - D$, $a'b'c'$ violates R-closure of F. The proof of the mia case offers nothing new. ∎

Thus we construct structures which have no instances of associativity or self-distributivity anywhere and every left multiplication by an element not on a line fails to preserve lines.

[7]In the diagram, x is the top point. Label the middle line a, b, c and the bottom line c', b', a'. Diagram taken from [CFMP17].

2.2. **Sparse Configurations in 3-Steiner systems**

In, for example [CGGW10, page 116], an $(n, n+2)$ configuration in a Steiner triple system (STS) is a substructure (A, R) of $n+2$ points with n lines. That is, $\delta(A) = 2$. They say a system is ∞-*sparse* if there are no $(n, n+2)$ configurations with $n \geqslant 4$. We reformulate 'sparse' in terms of δ.

Definition 2.2.1. *A Steiner triple system (M, R) is ∞-sparse if there is no $A \subseteq M$ with $|A| \geqslant 6$ and $\delta(A) = 2$.*

Note that the Pasch, mitre, and mia configurations are all forbidden in an ∞-sparse STS. [CGGW10] construct by a four page inductive construction of finite approximations, $2^{\aleph_0}$ non-isomorphic countable ∞-sparse systems. we modify the construction in [BP20] by restricting $\boldsymbol{K}_0$ to get ∞-sparse STS of every infinite cardinality.

Definition 2.2.2. *Let $\boldsymbol{K}_0^{sp}$ be the subclass of $\boldsymbol{K}^*$ (linear spaces) such that for every $B \subseteq A$:*

$$(\#)\ |B| > 1 \ \rightarrow\ \delta(B) > 1\ \&\ |B| > 3 \rightarrow \delta(B) > 2.$$

and, as in [BP20] *take* $\mathbf{U}$ *as* $\mathcal{U}^{sp}$*, those* $\mu \in \mathcal{U}$ *which can be achieved in* $\boldsymbol{K}_0^{sp}$*.*

Condition # implies there are no 4 element lines in a member of $\boldsymbol{K}_0^{sp}$ so $\mu(\boldsymbol{\alpha}) = 1$ and the generic model will be a Steiner triple system.

Theorem 2.2.3. *The system $(\boldsymbol{K}_0^{sp}, \leqslant)$ has $\leqslant$-amalgamation. And so for any $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu^{sp}$ has $\leqslant$-amalgamation.*

Proof. Let $C \leqslant_{sp} A$ and $C \leqslant_{sp} B$. Linear space amalgamation (Definition 1.2.4) cannot introduce any relation between $A - C$ and $B - C$, as this would produce a 4-element line. But then it is clear that # is preserved in the amalgam. We use the first clause of # to avoid B with $\delta(B) = 1$. Now the proof from [BP20] applies to give amalgamation for $\boldsymbol{K}_\mu$ if $\mu \in \mathcal{U}^{sp}$. $\blacksquare_{2.2.3}$

Theorem 2.2.4. *There are continuum many μ such that*

(1) T_μ is strongly minimal (so $\aleph_1$-categorical);
(2) Every model of T_μ is an ∞-sparse Steiner triple system;
(3) T_μ has countably many countable models.

Proof. As in [BP20], for any μ satisfying Definition 2.2.2, the associated T_μ is a Steiner triple system. But by omitting A with $\delta(A) = 2$ and $|A| \geqslant 6$, the structure is ∞-sparse. $\blacksquare_{2.2.4}$

Lemma 4.4 shows that the associated (a, b)-graphs are not perfect.

3. Constructing strongly minimal quasigroups

While Fact 0.1 shows there are strongly minimal k-Steiner systems for every k, [GW75, Bal21] imply that there can be quasigroups only when k is a prime power. Our strongly minimal k-Steiner systems (M, R) can admit a definable binary function [BV21] only under very strong additional hypotheses (Theorem 0.3). Nevertheless, there are strongly minimal quasigroups which induce k-Steiner systems when k is a prime power. For this result we need the generality of Fact 1.1.2, as we will axiomatize $\boldsymbol{L}^*$ ($\boldsymbol{K}^*$ here) with $\forall\exists$-sentences. We sketch a different proof than that detailed in [Bal21] of the existence of strongly minimal quasigroups.

The coordinatizing result rests primarily on work of [GW75, Ste56, Ś61] and others who achieved a 'coordinatization' of such Steiner systems by quasigroups. The contribution here is that although, for $k > 3$, this coordinatization is not a bi-interpretation, the Steiner system never interprets a quasigroup, we can in fact demand for $k = q = p^n$ the existence of a Steiner k-system that is interpreted in a strongly minimal quasigroup. The key to this is the relationship of so-called $(2, k)$ varieties [Pad72, GW75] to a two-transitive finite structure and thus eventually to the reconstruction of a finite field. Following [GW80] we call the quasigroups which arise when k is a prime power q, *block algebras.*

A variety is a collection of algebras (structures in a vocabulary with only function/constant symbols and no relation symbols) that is defined by a family of equations. The essential characteristic of the equational theories below is that each defining equation involves only two variables. In particular, none of the varieties are associative.

Definition 3.1. [Smi07] *A quasigroup* $(Q, *)$ *is a groupoid*[8] $(A, *)$ *such that for* $a, b \in Q$*, there exist unique elements* $x, y \in Q$ *such that both*

$$ax = b, ya = b.$$

The general notion is a universal Horn class, not a variety. But an (r, k) variety of groupoids is a quasigroup [Qua92].

Definition 3.2. [Pad72] *The variety* V *is an* (r, k) *variety if every* r*-generated subalgebra of any* $A \in V$ *is isomorphic to the free* V*-algebra on* r *elements and has cardinality* k

Fact 3.3. [GW75] *Given a (near)-field*[9] $(F, +, \cdot, -, 0, 1)$ *of cardinality* q *and a primitive element* $a \in F$*, define a multiplication* $*$ *on* F *by* $x * y = y + (x - y)a$*. An algebra* $(A, *)$ *satisfying the 2-variable identities of* $(F, *)$ *is in a* $(2, q)$*-variety of* block algebras *over* $(F, *)$*.*

This is one of 5 equivalent characterizations of such a variety in [Pad72]. Obviously, the collection of r-generated subalgebras $A \in V$ form an Steiner (r, k)-system; we need a third: the automorphism group of any r generated algebra is strictly (i.e. sharply) r-transitive.

Fix two vocabularies $\tau = \{R\}$ and τ' with two ternary relations symbols R, H. For each $(2, q)$-variety V quasigroups, we construct a strongly minimal theory of quasigroups (in V) that induce q-Steiner systems. We use H as the graph of the quasigroup operation in V, $*$, to make our amalgamation class contain only finite structures (as in [BC19]). We define the base class of finite structures as follows.

Definition 3.4. *[*$\boldsymbol{K}^*$*] Fix a prime power* q *and a* $(2, q)$*-variety* V *of quasigroups (e.g. a block algebra from Fact 3.3). Let* F_2 *denote the free algebra in* V *on* 2 *generators. Let* $\boldsymbol{K}^* = \boldsymbol{K}^*_V$ *be the collection of finite* (H, R)*-structures* A *such that*

(1) (A, R) *is a linear space;*

(2) $(\forall a_1, a_2, a_3)(\exists b_1, \ldots b_{q-3})[R(a_1, a_2, a_3) \to \bigwedge_{i \leqslant q-3} R(a_1, a_2, b_i)$*;*

[8] In the background literature on quasigroups, a *groupoid* is simply a set with a binary operation. So, I use this notation although it is no longer common.

[9] A near-field is an algebraic structure satisfying the axioms for a division ring, except that it has only one of the two distributive laws.

(3)
$$(\forall a_1, a_2, a_3, b_1, \dots b_{q-2})[(R(a_1, a_2, a_3) \wedge \bigwedge_{i \leqslant q-3} R(a_1, a_2, b_i)) \to \bigvee_{i,j \leqslant q_2} b_i = b_j].$$

(4) *If $A' \restriction R$ is a maximal clique (line) with respect to R (necessarily $|A'| = q$), $A' \restriction H$ is the graph of the free algebra $F_2 \in V$.*

Note that the definition of linear space implies that any triple satisfying R in $A' \in \boldsymbol{K}_{\mu,V}^{\tau'}$ extends to a line in A' of exactly length q. Since V is axiomatized by 2-variable equations, if $A' \in \boldsymbol{K}_{\mu,V}^{\tau'}$, $A' \restriction H$ is the graph of an algebra in V.

Definition 3.5. (1) *For a τ-structure (A, R) $\delta_\tau(A)$ is defined as for linear spaces in Definition 1.2.3. Now for each $A' \in \boldsymbol{K}^* = \boldsymbol{K}_V^*$, let $A = A' \restriction R$ and $\delta_{\tau'}(A') = \delta_\tau(A)$ and induce $\leqslant'$ from δ'.*

(2) $\boldsymbol{K}_{0,V}^{\tau'} = \{A' \in \boldsymbol{K}_{\mu,V}^{\tau'} : \delta_{\tau'}(A') \geqslant 0\}$

(3) *A μ' mapping $\boldsymbol{K}_{0,V}^{\tau'}$ into Z is in $\mathcal{U}_{\tau'}$ if it satisfies $\mu(A/B) \geqslant \delta_{\tau'}(B)$¿ Let $D \in (\boldsymbol{K}_{\mu,V}^{\tau'}, \leqslant')$ if and only if $\chi_D(A/B) \leqslant \mu(A/B)$.*

Note that B/A is a good pair, just when B'/A' is a good pair. Since both the restriction $\delta(A) \geqslant 0$ and the bound imposed by μ are universally axiomatized it is easy to check that $(\boldsymbol{K}_{\mu,V}^{\tau'}, \leqslant')$ is smooth. However it is AE-axiomatized because of clause 3.4.2. Thus the main difficulty in proving the following is establishing amalgamation.

In [Bal21], we give a different construction which involves a μ which counts good pairs in τ and μ' which counts good pairs in τ'. We write μ' here to emphasize that μ' counts good pairs of τ' structures and for compatibility with the earlier notation.

Theorem 3.6. *For each $q = p^n$ and each $\mu \in \mathcal{U}$ and each $(2, q)$ variety of quasi-groups V there is a strongly minimal theory of quasigroups $T_{\mu,V}$ that interprets a strongly minimal q-Steiner system.*

Proof. We now show the amalgamation for $(\boldsymbol{K}_{\mu,V}^{\tau'}, \leqslant')$, as in Lemma 5.11 and Lemma 5.15 of [BP20]. Consider a triple D, E, F in $\boldsymbol{K}_{\mu,V}^{\tau'}$ as in Lemma 2.1.3. That is, $D \subseteq F$ and E is 0-primitive over D'. Since E is primitive over D, although there may be a line contained in the disjoint amalgam G with two points in each of D and $F - D$, each line that contains 2 points in $E - D$ can contain at most one from D. If a line contains three points from D, since D satisfies Definition 3.4.2 it is contained in D. Thus, there is no issue with defining the relation H on the disjoint amalgamation. If μ' requires some identification for some (B, C), just as in [BP20], it is because the (relational) τ'-structure BC is DE and (Note the 'further' in [BP20, Lemma 5.10].) there is a copy of C over B in F.

The blocks of the Steiner system are the 2-generated $*$-subalgebras. Now the strong minimality of the generic follows exactly as in Lemmas 5.21 and 5.23 of [BP20] and we have proved Theorem 3.6. ∎

4. Strongly minimal block algebras, towers, and path graphs

The notion of an *(a,b)-cycle graph* is widely studied for finite Steiner *triple systems.* [CW12, CGGW10] consider the notion for infinite Steiner *triple* systems and prove the existence of infinite perfect and uniform Steiner triple systems.

We generalize this notion to consider infinite q-Steiner systems that are induced from strongly minimal $(2, q)$*-quasigroups* with q a prime power [BV21, Bal21]. They could be given either as quasigroups (only symbol *) or built from $(2, q)$-varieties in $\tau' = \langle *, R\rangle$ as in Section 3. In either case *we write* $*$ *for the quasigroup multiplication operation in the infinite models,* H *for the graph of* $*$*, and* $R(a, b, c)$ *for collinearity,* each of a, b, c is in the q-element quasigroup generated by the other two, which is a definable relation in $(Q, *)$. We use the $*$ operation to inductively construct a path on points.

QUESTION 4.0.1. *We use our special situation heavily below. What can be said about the path graphs of arbitrary q-Steiner systems induced by quasigroups?*

4.1. **Path Graphs**

Finite Steiner systems Q are often studied via the cycle graph over various ab; the pairs (c, d) from $Q - \overline{ab}$ are colored red or blue depending on whether a or b lies on the line $\overline{cd}$. Then a path is generated by choosing a point d off $\overline{ab}$ and starting with $\overline{ad}$ and inductively choosing the line of a different color through the third point on the current line. We extend this idea to q-Steiner systems. It is immediate that 3-paths do not intersect; so for strongly minimal 3-Steiner systems the definitions reduce to those in [CW12]. However, such disjointness is no longer immediate when $q > 3$ leading to the more complicated description of paths in Definitions 4.1.3 and 4.1.4. In order to carry out the analysis[10], we exclude from the graph, not just $\overline{ab}$ but the larger finite set $\mathrm{icl(a,b)}$, the smallest subset containing a, b that is strong in M.

Definition 4.1.1. *Consider a Steiner system* $(M, *, R)$ *determined by a q-block algebra* $(M, *)$ *(Definition 3.2). For any* $a, b \in M$*, we will write* $G_M(a, b)$ *for the graph determined by the pair* $a, b \in M$*.*

1. *The domain of* $G_M(a, b)$ *is* $M - \mathrm{icl(ab)}$*.*
2. *For* $x, y \notin \mathrm{acl(a,b)}$*, there is an edge colored a (resp., b) joining x to y if and only if* $R(a, x, y)$ *(resp.,* $R(b, x, y)$*).*

Remark 4.1.2. *There is an edge coloured*[11] *a (resp., b) joining x to y if and only if* $a * x = y$ *(resp.,* $b * x = y$*).*

We have partitioned the lines (R-cliques) that intersect $\{a, b\}$ into a and b lines. Two lines with distinct colors can intersect in at most one point.

We introduce certain *paths* and then in Section 4.3 *fans* in the graph that under appropriate hypotheses cover (most of) the domain of the graph.

Definition 4.1.3. *Let* $M \models T_{\mu',V}$ *with* $\mu \in \mathcal{U}^{\mathrm{ls}}$ *(Definition 1.2.6). Consider a q-block algebra* $(M, *)$ *with associated path graph* $G_M(a, b)$*.*

1. *For any* a, b*, we write* $\overline{ab}$ *to denote the line of length q generated by* $\{a, b\}$*.*
2. *For* $d_0 \notin \overline{ab}$ *we define a sequence, denoted* $P_{ab\mathbf{d}}$ *generated by* $d_1 \in M - \mathrm{acl(ab)}$ *over* $\{a, b\}$ *as follows.*

   The path $P_{ab\mathbf{d}}$ *is the sequence* $\mathbf{d} = d_1, \ldots d_m$ *such that* $a * d_{2i+1} = d_{2i+2}$ *and* $b * d_{2i+2} = d_{2i+3}$ *for* $0 \leqslant i \leqslant m$*.*

---

[10] This guarantees that the generator d_0 satisfies $d(d_0/\mathrm{icl(ab)}) \geqslant 0$.

[11] Note that if $q = 3$, this is the same as collinearity and we return to the framework of [CW12].

(3) *The* envelope, $P^e_{ab\mathbf{d}}$*, of the* path, $P_{ab\mathbf{d}}$*, with* $\mathbf{d} = d_1 \dots d_m$*,is the union of the lines*[12] $\overline{d_i, d_{i+1}}$ *for* $1 \leqslant i < m$*. Note that if* i *is odd (even),* a *(*b*) is on* $\overline{d_i, d_{i+1}}$*.*

Note that for e is on an a-edge, $a * e$ is on the same line (and similarly for b). Thus, the lines of the Steiner system are cliques of the path graph. But, if e with $e \neq a$ and $e \notin \mathrm{acl(ab)}$ is on an a-edge multiplying e by b begins the generation of a distinct path, $P_{ba\mathbf{e}}$ in the graph. We will show such a path is either an infinite chain or 'cycles' by generating a 0-primitive extension of ab.

Definition 4.1.4. *(1) There are two possibilities when this process is iterated forward* m *times.*

(a) An (a, b)-chain *of length* m *is a path* $P_{ab\mathbf{d}}$ *with* $\mathbf{d} = d_0 d_1, \dots d_m$ *such that* $a * d_{2i+1} = d_{2i+2}$ *and* $b * d_{2i+2} = d_{2i+3}$ *for* $0 \leqslant i \leqslant m$
and: for $j > i + 1$ *the lines* $\overline{d_i d_{i+1}}$*,* $\overline{d_j d_{j+1}}$ *do not intersect. Thus* $\delta(P_{ab\mathbf{d}}) = \delta(P^e_{ab\mathbf{d}})$*. Note that* m *counts the number of lines in the path. We write* $\boldsymbol{\sigma_m}$ *for the isomorphism type of an* m*-chain. Note that, as in the 3-Steiner system case, the length of an* m*-chain must be divisible by* 4*.*

(b) At some stage the new line generated by a, d_{2i+1} *or* b, d_{2i+2} *intersects one of the earlier lines in the envelope of the path. In this case, we stop the construction with the new line. The result is an* m-pseudo-cycle*, an envelope* $P_{ab\mathbf{d}}$*, such that for exactly one pair* (i, j) *with* $0 \leqslant i \leqslant m$ *and* $j > i + 1$ *the lines* $\overline{d_i d_{i+1}}$*,* $\overline{d_j d_{j+1}}$ *intersect.*
We write $\boldsymbol{\gamma_s}$ *for an isomorphism type of an* s*-pseudo-cycle* $P_{ab\mathbf{d}}$ *and* $P^e_{ab\mathbf{d}}$ *for the isomorphism type of its*

(c) If the process continues infinitely we call the result an infinite chain.

(2) Note that the construction of path through d_0 *could equally well begin with the first line a* b*-line. In this case, we introduce a finicky notation. The* P_{bad_0} *path through* d_0 *starts with a* b*-line.*

Recall the construction stops as soon as there is a loop but may be infinite. In the pseudo-cycle case $P^e_{ab\mathbf{d}}$ contains a minimal pseudo-cycle, which is 0-primitive over ab. Thus, each triple a, b and $d \notin \mathrm{icl(ab)}$, determine a unique maximal path P_{abd} beginning with an a-edge; it may be a pseudocycle (perhaps starting with a different d') or an infinite chain. *While formally we have defined* pseudo-cycles *to emphasize the return need be back to the initial point, we will often write* cycle *for short.*

Within the algebraic closure of ab analysis by the graph structure is more complicated. As, since any two points determine a line implies there are $c \in \mathrm{acl(ab)}$ such that $d(c/ab)$ remains 0 even when c is an intersection point of many lines. Thus, in Section 4.2 we study inside acl(ab) the graph over icl(ab) and in Section 4.3 work over acl(ab).

4.2. **Inside** acl(ab)**: Many Finite paths**

This subsection analyzes the structure of $G_M(a, b)$ when M is a prime model that is algebraic over the empty set and for arbitrary M the stucture of $\mathrm{acl_M(ab)} - \mathrm{icl_M(ab)}$. Section 4.3 describes the properties of (a, b)-path graph off acl(ab).

[12] We may sometimes write $P_{ab\mathbf{d}}$ when $P_{ab\mathbf{d}} - \{a, b\}$ is more precise; this is the usual ambiguity in describing good pairs C/B; technically B and C are disjoint.

We now use the model theoretic machinery about strongly minimal sets.

Remark 4.2.1 (Towers)**.** Two prototypical properties of a strongly minimal theory T are: a) the existence of a unique *generic type* over the model whose restriction to any set has infinitely many solutions and, as a result, if T has at least two non-isomorphic countable models, b) the arrangement of the countable models into a tower. Let $\langle M_j : 0 \leqslant j < \omega + 1\rangle$ be the tower (elementary chain) of countable models of $T_{\mu',V}$, with M_0 the prime model[13]; then M_ω is isomorphic to the generic structure $\mathcal{G}_{\mu,V}$ [BP20, Lemma 5.29]. One might think each M_n is prime with an acl-basis of cardinality n. We show in Lemma 4.2.3) for any strongly minimal Steiner system $M \models T_{\mu',V}$ with $\mu \in \mathcal{U}$, $\mathrm{acl}(\emptyset) \neq \emptyset$ if and only $\mathrm{acl}(\emptyset)$ is infinite. However, in Section 5 we provide choices of $T_{\mu',V}$ where M_0 has dimension 2 and so M_n has dimension $n + 2$.

The cycles (using only partial lines of length three) played an important role in [BP20]. We constructed the $2^{\aleph_0}$ distinct theories T_μ in [BP20, Lemma 4.11], by showing (in the vocabulary $\tau = \{R\}$) there were a countable family of $4n$-cycles (actually back to the same element) that are mutually non-embeddible and 0-primitive over 2-element sets. Varying the argument slightly shows as s increases the $\boldsymbol{\gamma}_s$ (Definition 4.1.3.(1b)) induce infinitely many mutually non-imbeddible primitives in $\boldsymbol{K}_\mu$ over a two element set that is strongly embedded. We also noted in [BP20, Lemma 4.11] that there are infinitely many mutually non-embeddible primitives in $\boldsymbol{K}_\mu$ over the empty set and similarly over a 1-element set.

$\mathcal{U}^{\mathrm{ls}}$ allows μ that forbid the realization of specific good pairs $B/\emptyset$. In [BP20], we showed the algebraic closure of the empty set was infinite if the generic contained a copy of the Fano plane, – the unique 7-element projective plane, F. So setting $\mathcal{F}$ as the collection of $\mu \in \mathcal{U}$ with $\mu(F/\emptyset) > 0$ guarantees $\mathrm{acl}_{\mathrm{M}}(\emptyset)$ is infinite for any $M \models T_\mu$. We retain the name $\mathcal{F}$ but make it a much larger subset of $\mathcal{U}^{\mathrm{ls}}$.

Notation 4.2.2. *Let $\mathcal{F}$ be the set of $\mu' \in \mathcal{U}^{\mathrm{ls}}$ such that $\mu(C/\emptyset) > 0$ for some good pair $C/\emptyset$.*

Lemma 4.2.3. *If $\mu' \in \mathcal{F}$ and $M \models T_{\mu',V}$ then $\mathrm{acl}_{\mathrm{M}}(\emptyset)$ is infinite.*

Proof. It is easy to see that any C that is 0-primitive over $\emptyset$ must contain two intersecting lines so three non-collinear points exist. Noting that the only use in Lemma 5.27 of [BP20] of the assumption that the Fano plane is imbedded in M is to guarantee that there are three non-collinear point in a subset of M that is 0-primitive over $\emptyset$, we get an infinite algebraic closure here. The construction of an infinite tower of 0-primitive extensions uses only that $\mu \in \mathcal{U}^{\mathrm{ls}}$. ∎

In Section 5.1, we give several examples of strongly minimal quasigroups where the dimension of the prime model is 2.

Lemma 4.2.4. *If $\mu \in \mathcal{U}$, for any $a, b \in M \models T_\mu$ with $\mathrm{acl}_{\mathrm{M}_0} \neq (\emptyset)$ there are infinitely many disjoint (over the finite $\mathrm{icl}_{\mathrm{N}}(\mathrm{ab})$) finite cycles in $G_N(a, b)$, where N is a copy of the prime model of T_μ with $N \supseteq \{a, b\}$.*

Proof. Such an N exists by Lemma 4.2.3. Fix $\mathcal{D}$ as $\mathrm{icl}_{\mathrm{M}}(\mathrm{a}, \mathrm{b})$. For each i there is a pseudocycle C_i that is a primitive extension over icl(ab) based on ab with length $4i$. The structure with domain $\mathcal{D} \cup C_i$ is denoted $\mathcal{A}_i$. Since $\mu(C_i/ab) \geqslant \delta(ab) = 2$,

[13] The prime model of T is the unique model that can be elementarily embedded in each model.

there is an embedding of $\mathcal{A}_i$ into the saturated (also generic) model M_ω. But $N \prec M_\omega$ and is algebraically closed so the image of C_i is in N. Now, since the C_i are 0-primitive over $\mathrm{icl_N(ab)}$, the $\mathcal{A}_i$ are disjoint over $\mathcal{D}$.

∎

QUESTION 4.2.5. *Can the prime model contain an infinite chain? Is there any decomposition by chains of the prime model? Compare these questions with the alternative decomposition of the prime model by taking the union of tree decomposition by normal subsets in* [BV21].

4.3. **Over** $\mathrm{acl(ab)}$ **all paths are infinite**

We study those paths in $G_M(a,b)$ that are generated by $d_0 \notin \mathrm{acl_M(a,b)}$. We justify in Lemma 4.3.2 the following notation:

Notation 4.3.1. *For $d_0 \notin \mathrm{acl_M(a,b)}$, P_{abd_0} ($P^e_{abd_0}$) denotes the (envelope of) the longest path generated by beginning with ad_0. This path may be infinite.*

Lemma 4.3.2. *Suppose $d_0 \notin \mathrm{acl(a,b)}$.*

(1) $d(d_0/ab) = 1$; the path generated by d_0 is infinite.
(2) Distinct a-edges in the path $P_{ab\mathbf{d}}$ cannot intersect; but each a-edge intersects $q-1$ b-edges.
(3) If $P_{ab\mathbf{d}}$ is an infinite path then for every $X \subseteq P^e_{ab\mathbf{d}}$, $d(X/ab) = 1$.
(4) If $P_{ab\mathbf{d}}$ an infinite path there is exactly one e on $P^e_{ab\mathbf{d}}$ that is on an a-line and P_{bae} is an infinite path (Recall Definition 4.1.4.2).

Proof. 1) If $d_0 \in M - \mathrm{acl(a,b)}$, $d(d_0/ab) = 1$; otherwise $d_0 \in \mathrm{acl(ab)}$. If $P_{ab\mathbf{d}}$ is finite, it is because some $C \subseteq P_{ab\mathbf{d}}$ is a pseudocycle. But the $\delta(C/ab) = 0$ and $d_0 \in \mathrm{acl(ab)}$.

2) If $\overline{ad_{2i}}$ is a line in $P_{ab\mathbf{d}}$ then for any element $x \in \overline{ad_{2i}}$, $a * x \in \overline{ad_{2i}}$. But for each of the $q-1$ non-trivial star terms, $t(x,y)$, $b * (t(a,d_0)$ generates a new line.

3) Suppose (without loss) that $d_0 \subseteq X \subseteq P^e_{ab\mathbf{d}}$ and $d(X/\mathrm{acl(ab)}) = 0$. Then $d_0 \in \mathrm{acl(ab)}$. Two paths generated by distinct $d_i \notin \mathrm{acl(ab)}$ can intersect in one point; $d(P_{abd_0} \cup P_{abd'_0}) = 1$. But if there are two points of intersection $d_0 \in \mathrm{acl(ab)}$.

4) For any such e there is a line determined by $b, b*e$. But this line generates an infinite path only if $d(e/ab) = 1$. Now apply 2). ∎

With these results in hand we see that actually a, b, d_0 generate a fan of lines.

Definition 4.3.3. *The* fan *generated by abd_0 is defined by induction.*

(1) $F^0_{abd_0}$ consists of all points on envelopes of paths generated by a line ae where e is on a b edge of P_{abd_0} or by a line be with e on an a-edge of $P^e_{bad_0}$;
(2) $F^{n+1}_{abd_0}$ consists of all points on envelopes of paths generated by lines ae where e is on a b edge of $P^n_{abd_0}$ or by a line be with e on an a-edge of ${P^e}^n_{bad_0}$;
(3) The fan $F_{abd_0} = \bigcup_{n<\omega} F^n_{abd_0}$.

Note that $F_{abe} = F_{baf}$ if e and f are both on the same line in $M - \mathrm{acl(a,b)}$ through a (or through b).

As in Lemma 4.3.2, we see immediately that if two fans intersect in a single point their union is a larger (not definable) subset of rank 1:

Lemma 4.3.4. *Two fans can intersect in at most one point.*

Theorem 4.3.5. *If M is countable and* $\dim(M/N) = 1$*, then for any $a, b \in M$, M is a union of fans over N. Inductively, the conclusion applies to any $M' \succ M$.*

Proof. Let $\langle e_i : i < \omega \rangle$ enumerate $N-M$. Fix any $a, b \in N$, choose $e_0 = d_0 \in M-N$ and let F_0 be the fan F_{abd_0}. Now for each n, let d_{n+1} be e_j for the least j such that $e_j \notin N \cup F_n$. Clearly $\bigcup_{n<\omega} F_n \cup N = M$. Since the dimension N/M is 1, there will be algebraic relations among the fans. However, any two can intersect in at most one point and by construction there graph edges (a or b lines) that are not in one of the listed fans. However, many instance of R are not in the graph. ∎

4.4. **No Perfect Path graphs**

Cameron and Webb [CW12] extend to infinite structures the notion of a perfect Steiner triple system as one in which each cycle graph $G(a, b)$ is a single cycle. They find $2^{\aleph_0}$ countable such Steiner triple systems. In line with Definition 4.1.1, we can extend this definition to any q-block algebra. However, we show none of the q-Steiner systems constructed here are perfect. Clearly there can be no uncountable perfect Steiner k system in any reasonable sense since whatever replaces cycle will be countable. We will take the weakest plausible notion, which includes a single path or a fan; we show no such complex covers $M - \mathrm{acl}(\mathrm{ab})$, when $M \models T_{\mu'',V}$. In Theorem 4.3.5, we covered $M - N$ by at most $|M - N|$ fans, but not finitely many.

Definition 4.4.1 (Perfect)**.** *If $(M, *, R) \models T_{\mu',V}$ for some $\mu \in \mathcal{U}$ and $(2, q)$-variety V, we say $(M, *, R)$ is a* perfect q-Steiner system *if for some finitely generated R-closed set (Definition 2.1.2) $X = M - \mathrm{acl}(\mathrm{ab})$.*

Since every line in a Steiner system associated with a q-Steiner system is two-generated as a quasigroup, we can think of R-closure as finding the generated sub-quasigroup. Omer Mermelstein suggested the key idea for the proof for the following result. We write T_μ for the theory here since this argument works for any of the variants.

Lemma 4.4.2. *If M is a model of T_μ with $\mu \in \mathcal{U}$, $A \leqslant M$, and $|M - A|$ is infinite, then M has infinite R-dimension.*

Proof. We first show that if C is 0-primitive over A and $a \notin C \cup A$, A^*, the R-closure of Aa, does not intersect C. Note by induction that every finite $E \subseteq (A^* - A)$ satisfies $\delta(E/A) = 0$. Now, fix an enumeration A^* such that $e_j \in \mathrm{cl_R}(\{\mathrm{e_i : i < j}\} = \mathrm{E_j}$. Suppose for contradiction $A^* \cap C \neq \emptyset$ and choose the least k with $e_k \in C \cap A^*$. But then e_k witnesses an edge between C and E_{j+1}; this implies $\delta((E_j \cup C)/A) < 0$, contrary to hypothesis.

There are infinitely many incomparable 0-primitives C_j over A ([BP20, Lemma 4.11]; choose successively, a seed a_j in C_j. Applying the first paragraph, we see the $\mathrm{cl_R(Aa_j)}$ are mutually disjoint. By constructing $\langle A_j, A_j^* \rangle$ by the procedure of the last paragraph, we witness infinite R-dimension. ∎

Since a perfect Steiner system is the R-closure of finitely many elements, we have immediately from Lemma 4.4.2:

Corollary 4.4.3. *If $(M, *, R) \models T_{\mu',V}$ for some $\mu \in \mathcal{U}$ and $(2, q)$-variety V, $(M, *, R)$ is not a Steiner perfect system.*

QUESTION 4.4.4. *In* [BV21]*, we show the definable closure of a strongly minimal system (M, R) is essentially unary if T_μ is triplable (Footnote 2). Models of $T_{\mu',V}$*

have $*$ as a non-trivial binary function. But, assuming μ' is triplable, are there any binary functions that are not polynomials in $*$?

5. Varying the construction for combinatorial motives

In Section 4, we studied theories $T_{\mu',V}$ of quasigroups built by a Hrushovski style construction as in Section 3 where $\mu' \in \mathcal{U}$ and for any $M \models T_{\mu',V}$, $\mathrm{acl_M}(\emptyset) \neq \emptyset$. In this section, we make major modifications to the construction to consider subsets where algebraic closure has few pseudo-cycles and to find 2-transitive structures.

In Section 4.3 we found examples where all cycles were infinite when we took the domain of the path graph as $M - \mathrm{acl(ab)}$. But in Section 4.2 with domain $M_0 - \mathrm{icl(ab)}$ we always had finite cycles and the existence of infinite cycles in the prime model is an open problem. In this section we restrict to the domain, $M - \mathrm{iclab}$. We first (Section 5.1) modify the construction to be able to specify which, if any, finite cycles occur. In Section 5.2 we introduce the notion of a uniform (The isomorphism type of $G_M(a,b)$ does not depend on the choice of a,b.) q-Steiner system (generalizing [CW12, CGGW10]). Then by different methods in Sections 5.3 and 5.4 construct families of 2-transitive and hence uniform q-Steiner systems.

We use two model theoretic methods to solve some problems suggested from the study of cycle graphs in [CW12]. These methods modify the theory $T_{\mu',V}$ either by changing μ or, more drastically, restricting the class $\boldsymbol{K}_0$ of finite structures. And then we combine the two in Section 5.4.

5.1. **All paths are infinite**

In this section, we find $T_{\mu'',V}$ whose models have no finite cycles. It is then easy to allow certain specified lengths. The key point here is to vary the class $\mathcal{U}^{\mathrm{ls}}$ from Definition 1.2.6 maintaining the amalgamation so the resulting generic model is strongly minimal but preventing finite cycles. As before we work in a vocabulary $\{H,R\}$, where R is collinearity in a linear space and in the generic model H is the graph of a quasigroup operation $*$. We introduce a set $\mathcal{B}$ of μ'' obtained by modifying $\mu' \in \mathcal{U}'$ to μ'' by changing the value *only on* the isomorphism types good pairs $C/\{a,b\}$ which are pseudo-cycles. As $\mathcal{B} \cap \mathcal{U}^{\mathrm{ls}} = \emptyset$, apparent contradictions between here and Section 4 are resolved.

Definition 5.1.1. *Recall from Definition 3.4 that $\boldsymbol{\gamma_n}$ denotes an isomorphism type of a pseudo-cycle over a two element set. Let $\mathcal{B}$ denote the set of μ'' obtained by for every n, redefining each $\mu' \in \mathcal{U}_\tau$ to μ'' by setting $\mu''(\boldsymbol{\gamma_n}) = 0$ for each n.*

We define a class $\boldsymbol{K}'_{\mu'',V}$ whose generic has only infinite cycles. Thus there are no finite cycles in any model of $T_{\mu'',V}$.

Lemma 5.1.2. *If $\mu'' \in \mathcal{B}$, the class of $\tau' = \{*,R\}$-structures $\boldsymbol{K}^{\tau'}_{\mu'',V}$ from Definition 3.4 has the $\leqslant$-amalgamation property. If $\mu'' \in \mathcal{B}$, every model of $T_{\mu'',V}$ has only infinite cycles.*

Proof. We must check that we can complete the amalgamation while insisting that for each n, $\boldsymbol{\gamma_n}$ is omitted. For this we must slightly vary the proof of Lemma 5.10 in [BP20], whose notation we follow. Let $F,E \in \boldsymbol{K}_{\mathcal{B},\mu}$. Now, let $G = E \oplus_D F$, where (D,E) is a good pair (with $|E-D| > 1$) and $((a,b),C_k)$ is a good pair witnessing $\boldsymbol{\gamma}_k$ (So C_k is a pseudo-cycle.). The difficulty is that the good pair (C_k/B) does not satisfy the requirement $\mu(C_k/B) \geqslant \delta_{T'}(B)$. We gave a separate argument to show no $\boldsymbol{\gamma}_k$ blocks amalgamation; the result then follows without change. There are no realizations of the good pair $\boldsymbol{\gamma}$ in any of D, E, F; we must show it is not realized in G. The crux is that, by definition of $((a,b), C_k)$, for any k, i, each $c_i \in C_k$ is on a separate triple in R with each of a and b. Now if $(a,b) \subseteq F$ (compare Case B.1 of [BP20]), each C_i must be contained in F or else there is a clique (ac_ic_{i+1}), modulo renaming, with two elements in F and one in $E-F$ contradicting the primitivity of E over D. If one of a, b, say a is in $E-F$ then for each i, $C_i \subseteq E$ or the line between a and c_i is based in D (Definition 3.11 of [BP20]) and that is clearly impossible, since it contradicts that E is primitive over D; so each $C_i \subseteq E$. But now, since E doesn't realize γ_n, b must be in $F-D$ and $C_i \cap (E-D) \neq \emptyset$; we get the same contradiction. So $C_i \subseteq D$. But now $a \in E-D$ is on a line based on $C_i \subseteq D$, contradicting the primitivity of E over D. Thus for any $M \models T_{\mu'',V}$, $a, b \in M$ and $d_0 \notin \mathrm{icl(ab)}$, P_{abd_0} is infinite. So we finish. ■

A simple variant on the argument for Corollary 5.3 of [BP20] (Replace 'for every n' in Definition 5.1.1 by 'for $n \in X^c$'.) shows we can omit arbitrary sets of $\boldsymbol{\gamma_n}$:

Theorem 5.1.3. *For any $X \subseteq \omega$ of numbers divisible by 4 and $\mu \in \mathcal{U}$, we can construct still another variant μ^X of μ such that models of $T_{\mu^X,V}$ realize an n-pseudo-cycle if and only if $n \in X$.*

One cannot simply modify $\mathcal{U}$ to say all points have trivial algebraic closure and carry out the amalgamation argument. Omer Mermelstein provided the following counterexample, showing some restriction, such as to the γ_n, is necessary for Lemma 5.1.2. We provide an amalgamation diagram where the good pair C/B does not appear in any of the components but is in the amalgam. Nevertheless, we give several examples in later sections where $\mathrm{acl}_{\mathrm{M}_0}(\emptyset) = \emptyset$.

Example 5.1.4. Let B consist of five points $a, b_1, \ldots b_4$ and C consist of four points $c_1, \ldots c_4$, where $R(c_i, b_i, c_{i+1})$ for $i = 1, \ldots, 3$, $R(a, b_2, b_3)$, and $R(c_4, c_1, b_4)$. Then C is 0-primitive over B. But now if we let $D_0 = \{a, c_2, c_4\}$, $D_1 = \{b_1, c_1, b_4\}$ and $D_2 = \{b_2, c_3, b_3\}$ we have $D_0 \leqslant D_1$ and $D_0 \leqslant D_2$, but BC appears in the amalgam.

5.2. **Uniform** $G(a,b)$

[CW12] call a Steiner system uniform if all the cycle graphs $G_M(a,b)$ are isomorphic. [CGGW10] construct $2^{\aleph_0}$ countable uniform sparse infinite Steiner triple systems. We obtain $2^{\aleph_0}$ families of countable uniform infinite Steiner systems for each prime power q; we have not considered the extension of 'sparse' to q-Steiner systems.

We adapt the notions of uniform [CW12] to accommodate q-Steiner systems. Recall (Definition 4.1.1) that the domain of $G_M(a,b)$ is $M - \mathrm{icl(a,b)}$. We will consider cases where $\mathrm{acl(a,b)}$ is both finite and infinite.

Definition 5.2.1 (Uniform)**.** *We say a model $(M, *, R)$ of $T_{\mu',V}$ is* uniform*, if for any (a,b), (a',b'), $G_M(a,b) \simeq G_M(a',b')$.*

Here is a sufficient condition for uniformity.

Lemma 5.2.2. *(1) If $(M, *, R)$ is a model of a theory T generated by a Hrushovski class (Definition 0.4) of linear spaces such that every two element set A satisfies $A \leqslant M$, the automorphism group of $(M, *, R)$ acts 2-transitively on (M, R).*

(2) *Clearly, if the automorphism group of* $(M, *, R)$ *acts* 2*-transitively on* $(M, *, R)$*,* $(M, *, R)$ *is uniform.*

Proof. Since all pairs (a, b) are isomorphic and each is embedded strongly in the generic $\mathcal{G}$, the result is immediate for $\mathcal{G}$. But this transitivity extends to all models since if one model of a complete theory has a single 2-type, all models do. And, each model of a strongly minimal theory is finitely homogeneous (e.g. [BL71, Theorem 5]). ∎

5.3. 2-transitive M, 3-Steiner systems, Changing $\boldsymbol{K}$

In Section 5.1 we showed that, by modifying the set of possible μ, we could ensure that there were no finite pseudo-cycles. The Steiner system in Section 5.1 was far from uniform as there were many 2-types, e.g. pairs with non-isomorphic algebraic closures. (We only restricted those primitive extensions that were pseudo-cycles.)

We have dealt with two variants of the Hrushovski construction. We constructed generics in both τ and τ', with the same basic construction. But in the more general context of Definition 0.4 we can restrict $\boldsymbol{K}_0$ before beginning the construction and realize the hypothesis of the general statement of Lemma 5.2.2.1.

In Section 5.2 of [Hru93], Hrushovski proves there are $2^{\aleph_0}$ strongly minimal τ-structures with pairwise non-isomorphic associated combinatorial geometries. He achieves this by ensuring that algebraic dependence of a triple a, b, c is equivalent to $R(a, b, c)$. Mermelstein pointed out to me that these structures are in fact Steiner triple systems. We will see that they are 2-transitive and every cycle is infinite. Example 5.3.1 is considerably more restrictive than the linear space examples; it not only forces that two points determine a line but also that every full line has 3 points. In Theorem 5.4.2 we show less drastic surgery on the [BP20] construction still allows us to find uniform $G(A, B)$-graphs when $q > 3$.

Example 5.3.1. [Hru93, Example 5.2] We denote the theories described in this example by $T_{H,\mu}$. The dimension function δ_H is the usual: $\delta_H(A) = |A| - |R|$, where $|R|$ is the number of 3-element subsets of A satisfying R and strong submodel is defined in usual way. The novelty is in use of the δ-condition to define $\boldsymbol{K}_0^H$. Namely, the collection of finite structures C such that every subset B of $C \in \boldsymbol{K}_0^H$ with power at most 3 is strong in C:

$$(*)\ \boldsymbol{K}_0^H = \{A \colon B \subseteq A \wedge |B| \leqslant 3 \rightarrow B \leqslant A\}.$$

Since the amalgamation of Hrushovski's basic example added no edges, this subclass also has amalgamation by the same amalgam. For each μ, $\boldsymbol{K}_{H,\mu}$ is to $\boldsymbol{K}_0^H$ as $\boldsymbol{K}_\mu$ is to $\boldsymbol{K}_0$ (Definition 1.2.3).

Now to define a linear space say that a line is a triple satisfying R. Two points determine a line as $R(a, b, c) \wedge R(a, b, c) \wedge \neg R(a, b, d)$ makes $\delta(\{a, b, c, d\}) = 2 < \delta(\{b, c, d\})$. Since any non-trivial 0-primitive over a two element set contains 3 non-collinear points, (*) implies the algebraic closure of two points is the third point on the line they determine. Thus there are two quantifier-free configuration of three points: dependent, independent. Since, by $(*)$, both configurations are strong in the generic, they determine as in Lemma 5.2.2 the two possible 3-types. Similarly property $(*)$ of this Hrushovski example makes it a Steiner triple system[14].

[14] This example will not permit lines with longer length by modifying μ. As, there can be no 4-clique, ℓ, since with the Hrushovki definition $\delta(\ell) = 0$ while δ of two points is 2.

Here we write cycle since we are dealing with a Steiner-triple cycle and no path can be a proper pseudo-cycle as opposed to a cycle.

Fact 5.3.2. *For any μ and any $(M,R) \models T_{H,\mu}$, (M,R) is a strongly minimal uniform Steiner triple system. In fact, the algebraic closure of any pair is the third point on the line through a,b and so each cycle is infinite.*

Proof. As noted in the description of Example 5.3.1, in (M,R) the algebraic closure of a pair is the line through them. Since there are only two 3-types of tuples extending (a,b), any two d_i that are not on the line ab are isomorphic over a,b and thus the cycles they generate are isomorphic. The last claim is immediate since all points not on the line are automorphic over ab. Since any potential finite pseudo-cycle over a,b is in $\mathrm{acl(ab)} = \{\mathrm{a,b,c}\}$, where $R(a,b,c)$, there are no finite pseudo-cycles. ∎

5.4. 2-transitive q-Steiner systems; Changing $\boldsymbol{K}_0$ and $\boldsymbol{U}$

We turn to a different method[15] obtain uniformity results for Steiner q-systems for any prime power $q \geqslant 3$ and to restrict the number of finite cycles. We combine a variant of the Hrushovki's Example 5.3.1 with varying μ to control a second fundamental invariant: number of cycles.

Definition 5.4.1. *We write $\boldsymbol{K}_0^2$ for the class of linear spaces such that*

$$(**)\ |B| \leqslant 2 \textit{ implies } B \leqslant A$$

for every finite linear space $A \in \boldsymbol{K}_0^2$ containing B. We write $\boldsymbol{K}_{\mu''}^2$ for the class determined by $**$ and $\mathcal{B}$ and for any variety of quasigroups T with strongly minimal theory $T^2_{\mu'',V}$ associated by Theorem 3.6.

As in Example 5.3.1, $(**)$ and Lemma 5.2.2 imply every two element set is strong, so each model is 2-transitive. There are two differences from Example 5.3.1: i) the strong substructure notion is with respect to the δ in [BP20] and so we can vary the line length; ii) we don't kill the entire (non-trivial) algebraic closure of each 2-element set but explicitly forbid only the finite cycles. We note below that we can allow finitely many cycles over each pair (a,b).

Theorem 5.4.2. *If $\mu \in \mathcal{B}$ (Definition 5.1.1), $\boldsymbol{K}_\mu^2$ has amalgamation, the generic (and hence every countable model) is uniform and has no finite paths.*

Proof. The amalgamation follows *mutatis mutandis* from Lemma 5.1.2. Note that $(**)$ implies every two element set is strong, so each model is 2-transitive. This holds in every model by Lemma 5.2.2; hence $G_M(a,b)$ is uniform. Finite paths are blocked, since $\mu \in \mathcal{B}$. ∎

As we modified Lemma 5.1.3, we modify the proof of Theorem 5.4.2 to get:

Theorem 5.4.3. *If $\mu'' \in \mathcal{B}$ then for any variety V and for any model $(M,*,R)$ of $T^2_{\mu'',V}$ and any (a,b), both $\mathrm{acl_M}(\emptyset) = \emptyset$ and $(M,*,R)$ is uniform.*

Further, for any finite set X of pairs, (n_i,m_i) with n_i divisible by 4, we can construct a theory T^2_X such that if $(M,*,R) \models T_X$ and $(a,b) \in M$, $G_m(a,b)$ has m_i cycles of length n_i.

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[15] This approach of restricting primitives over very small sets to establish various amounts of transitivity of the non-Desguaresian plane appears in [Hru93, Bal95].

QUESTION 5.4.4. *Can we have all cycles in the prime model finite by insisting exactly one isomorphism type of a pseudocycle is consistent, say, a 4-pseudocycle?*

QUESTION 5.4.5. In [Bal94] (using the methods of this section) a rank 2 $\aleph_1$-categorical non-desarguesian projective planes is coordinatized by a ternary ring that is not linear. The non-linearity means that while the quasi-groups for both addition and multiplication are definable, they cannot be composed to give the ternary $t(x, y, z) = xy + z$ that arises in a division ring. That is, the plane is at the lowest level in the Lenz-Barlotti hierarchy. *Could similar but less radical surgery yield $\aleph_1$-categorical non-desarguesian projective planes that are higher in that hierarchy?*

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