# Determined Theories and Limit Laws

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**Abstract**

We provide a new model theoretic technique for proving 0-1 and convergence laws. As an application, we obtain a new (slightly less computational) proof of convergence laws due to Spencer and Thoma for the probability functions: $p_n^l = \frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$.

## 1 Introduction

Consider the class $K_n$ of all graphs on the set of vertices $\{1, 2, ...n\}$. One can define a probability space on $K_n$ by assigning each graph in $K_n$ equal probability, to form a uniform probability space. Fagin [5] proved that the probability of any property expressible in first order logic holding of a graph in $K_n$ converges to 0 or 1 as $n$ goes to infinity. We say that first order logic has a 0-1 law for the uniform probability space on finite graphs.

A probability space can be formed from the class $K_n$ by assigning a probability $p_n$ to the existence of an edge between any two vertices. Given a sentence $\phi$, let $p_n(\phi)$ be the probability that $\phi$ is satisfied by a graph in $K_n$. In the case studied by Fagin, the uniform probability is induced by an edge probability of 1/2. A family of edge measures $p_n$, $\{p_n\}$, on graphs of size $n$ obeys a 0-1 law (for first order logic) if for each first order sentence $\phi$, $\lim_{n\to\infty} p_n(\phi)$ is either 0 or 1. More generally the family $p_n$ has a convergence law if each such limit converges.

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**Definition 1.1** Given a family of edge measures $p_n$, the *almost sure theory* $Th$ is the set of sentences $\phi$ such that $\lim_{n\to\infty} p_n(\phi) = 1$. We refer to the models of $Th$ as the limit models.

Notice that the almost sure theory is complete if and only if first order logic has a 0-1 law for $\{p_n\}$. We are concerned with the various methods used to prove 0-1 laws. Let $L$ be the language of symmetric irreflexive graphs. One method is to prove the 0-1 law by induction on the complexity of $L$-formulas. Another method is to show that the almost sure theory $Th$ is complete. In the case of uniform probability, this can be accomplished by proving that $Th$ is $\aleph_0$-categorical. More generally one can show that all countable models of $Th$ are elementarily equivalent. This can be done by the use of quantifier elimination or Ehrenfeucht-games.

Shelah and Spencer [6] proved the 0-1 law for first order logic with edge probability $p_n = n^{-\alpha}$ for irrational $\alpha$, $0 < \alpha < 1$. Baldwin and Shelah [4] provided an alternative proof, without using Ehrehfreuht-games or quantifier elimination arguments to show completeness. Baldwin [3] abstracted this argument into the definition of a *determined* theory. In this paper, we generalize this method to deal with convergence. We use it to give a new proof of convergence for the edge probability $p_n^l$ studied by Thoma and Spencer in [7]:

$$p_n^l = \frac{ln(n)}{n} + \frac{l \cdot ln(ln(n))}{n} + \frac{c}{n}$$

where $l$ is an arbitrary fixed nonnegative integer, and $c$ is a positive constant.

Our interest in these edge measures arose from the fact that the limit models induced from the family $\{p_n^l\}$ are similar to the models of the first order theories considered for purely model theoretic reasons in [1], [2]. In particular, these limit models are rather simple from a model theoretic standpoint. They decompose into components which are 'almost' trees; the completions of the almost sure theory can be seen to be $\omega$-stable. In this range of probabilities, the parameter $l$ determines the possibility of the limit model admitting an '$r$-isolated point', a vertex of degree $r$. There is none if $r < l$ and infinitely many if $r > l$, but for $r = l$, the number of vertices of degree $r$ is not determined. In essence, fixing this number determines a completion of the almost sure theory. Each completion is finitely axiomatizable over $Th$, so the probability of each completion can be computed, which in turn, allows one to compute the probability for each sentence in $L$.

Spencer and Thoma proved:

**Theorem 1.2** *1. A graph $G$ satisfies the almost sure theory $Th_l$ for $p_n^l$ iff:*

*(a) For all finite $A$ in $G$, the number of vertices in $A$ is strictly less than the number of edges.*

*(b) For all $t \geq 1$ and $m \geq 3$, $t$ copies of an $m$-cycle can be embedded in $G$.*

(c) *For all integers $r$, $s$, $t \geq 0$, $s \geq r$, there does not exist a pair of vertices $x$, $y \in G$ such that $x$ has degree $r$ and $y$ has degree $s$ and the distance from $x$ to $y$ is equal to $t$.*

(d) *For all integers $r$, $t \geq 0$ and $m \geq 3$ there does not exist a vertex $x \in G$ of degree $r$ that is of distance $t$ from an $m$-cycle.*

(e) *For all integers $r$, $0 \leq r < l$, there does not exist a vertex of degree $r \in G$.*

(f) *For all integers $t \geq 1$ and $r > l$, there exist $t$ vertices of degree $r \in G$.*

2. *Moreover, for any integer $s$, if the sentence $\sigma_s^l$ asserts "there exist precisely $s$ vertices of degree $l$", $\lim_{n\to\infty} p_n^l(\sigma_s^l)$ exists.*

3. *For each $l$ and $s$, $Th_l \cup \sigma_s^l$ is complete.*

From this analysis of graphs they established a convergence law for edge probability $p_n^l$.

**Theorem 1.3** *Let $lim_{n\to\infty} p_n^l(\sigma_s^l) = q_s^l$. For any $L$-sentence $\theta$, there exists a finite set $I$ of nonnegative integers such that $lim_{n\to\infty} p_n^l(\theta) = \Sigma_{i\in I} q_i^l$ or $lim_{n\to\infty} p_n^l(\theta) = 1 - \Sigma_{i\in I} q_i^l$.*

This paper is a step in isolating the 'model theoretic' from the 'probabilistic' components of proofs of limit laws on finite models. In Section 2 we give a general definition of an indexed closure operator and a determined theory. Relying on the probability arguments of Spencer and Thoma for parts 1 and 2 of Theorem 1.2, we give in Section 3 a different model theoretic proof of part 3 of Theorem 1.2 and of Theorem 1.3. In particular, the fact (and computation showing it) that $\Sigma_{i<\omega} q_i^l = 1$ which is a part of the Spencer-Thoma argument is avoided here.

We write $Mod(T)$ for the class of models of a theory $T$. The collection of finite subsets of a set $X$ is denoted by $S_\omega(X)$. For any model $M$, and for any $\overline{a} \in M^r$, $\theta(M, \overline{a})$ denotes the set of solutions of $\theta(x, \overline{a})$ in $M$. We denote the length of a tuple $\overline{a}$ by $lg(\overline{a})$.

## 2 Indexed closure and determined theories

The key to the method of determined theories is a way of breaking the algebraic closure of a finite set into a (possibly infinite) sequence of finite sets by using an indexed closure operator. We give here a general notion of such a closure operator and use it to provide a method to prove not only 0-1 but also convergence laws. The closure operator of Definition 2.2 which is used in Section 3 to prove Theorem 3.15 and Theorem 3.17 is a special case.

If cl is a function from $\omega \times S_\omega(M) \to S_\omega(M)$, we write $\mathrm{cl}_M^i(\overline{a})$ for $cl(i, \overline{a})$.

**Definition 2.1** An *indexed closure operator* cl for a theory $T$ is a function, which for each $M \models T$, maps $\omega \times S_\omega(M) \to S_\omega(M)$ and has the following properties.

1. For any model $M \in Mod(T)$ and $\overline{a} \subseteq \bar{b} \in M$, for $j < i < \omega$, $\mathrm{cl}^j_M(\overline{a}) \subseteq \mathrm{cl}^i_M(\overline{a})$ and $\mathrm{cl}^i_M(\overline{a}) \subseteq \mathrm{cl}^i_M(\bar{b})$.

2. For any $M, N \in Mod(T)$, if for some $s$, $\mathrm{cl}^s_M(\emptyset) \simeq \mathrm{cl}^s_N(\emptyset)$ then for all $0 \le i \le s$, $\mathrm{cl}^i_N(\emptyset) \simeq \mathrm{cl}^i_M(\emptyset)$.

We extend the notation by writing $\mathrm{cl}^\omega_M(\overline{a})$ for $\cup_{i<\omega}\mathrm{cl}^i_M(\overline{a})$. In the example considered in this paper, for all $M \in Mod(T)$, there exists a $k < \omega$, such that for all $s > k$, $\mathrm{cl}^k_M(\overline{a}) = \mathrm{cl}^s_M(\overline{a})$ so $\mathrm{cl}^\omega_M(\overline{a}) = \mathrm{cl}^k_M(\overline{a}) = \mathrm{acl}_M(\overline{a})$, the algebraic closure of $\overline{a}$ in $M$. Following is the indexed closure operator we will use in this paper. While it provides a natural way to 'layer' the algebraic closure, there is an unfortunate lack of monotonicity, described in Example 3.5, which requires us to treat closure over the empty set with special care.

**Definition 2.2** For $k \le \omega$, let the $k$-closure of $\overline{a}$ in $M$, $cl^k_M(\overline{a})$, be the set of solutions in $M$ of all the formulas $\theta(x, \overline{a})$ where the quantifier rank of $\theta$ and $|\theta(M, \overline{a})|$ are each less than $k$.

The following fact, shown by straightforward calculation [4], is fundamental for the kind of argument used here.

**Lemma 2.3** *For any first order $T$ in a finite relational language, there exists a function $f$ of $|A|, m, n$, such that for any $M \models Th_l$ and any embedding of $A$ into $M$, $cl^m_M(\mathrm{cl}^n_M(A)) \subseteq cl^{f(|A|,m,n)}_M(A)$.*

**Definition 2.4**

1. We write $\simeq$ for isomorphism. We say $\mathrm{cl}^k_M(\overline{a}) \simeq^s \mathrm{cl}^k_{M'}(\overline{a}')$ if $\mathrm{cl}^k_M(\overline{a}) \simeq \mathrm{cl}^k_{M'}(\overline{a}')$ by an isomorphism taking $\overline{a}$ to $\overline{a}'$ and $\mathrm{cl}^k_M(\emptyset) \simeq \mathrm{cl}^k_{M'}(\emptyset)$

2. For an integer $k$ and a theory $T$, a formula $\theta(\overline{x})$ is *determined* by its $k$-closure in $T$ if for any $M, M' \models T$ and for any $\overline{a} \in M^r$ and $\overline{a}' \in M'^r$, if $\mathrm{cl}^k_M(\overline{a}) \simeq^s \mathrm{cl}^k_{M'}(\overline{a}')$, then $M \models \theta(\overline{a})$ if and only if $M' \models \theta(\overline{a}')$.

3. The theory $T$ is *determined* if for any formula $\theta(\overline{x})$, there is an integer $k_\theta$ such that $\theta(\overline{x})$ is determined by its $k_\theta$-closure in $T$.

# 3 An Application

In this section we consider the almost sure theory studied by Spencer and Thoma in [7], whose axioms are the theory $Th_l$ of the introduction. Let $Th_{l,s}$ denote the theory which consists of $Th_l$ plus the axiom "there exists $s$ isolated vertices of degree $l$" and $Th_l^{>0}$ denote the theory which consists of $Th_l$ plus the axiom "there exists an isolated vertex of degree $l$".

**Notation 3.1** A tree in which every vertex has infinite degree is denoted by $T$ and called a *complete tree.* A *hairy cycle* is a cycle with a complete tree attached to every vertex of the cycle. Let $H_n$ denote a hairy cycle whose cycle is of size $n$. An isolated component, denoted $I_n$ is a tree which contains one point with degree $n$ and all others have infinite degree.

As was pointed out in [7] it is easy to check that each model of $Th_l$ is a direct sum of the following components:

1. For every integer $i$ greater than one, infinitely many components each containing one cycle of size $i$ and every vertex has infinite degree.
2. For every $r > l$, infinitely many components which do not contain a cycle and every vertex has infinite degree except one, which has degree $r$.
3. Any (possibly finite) number of components which do not contain a cycle and every vertex has infinite degree.
4. For some $s > 0$, $s$ copies of components which do not contain a cycle and every vertex has infinite degree except one vertex which has degree $l$.

More formally:

**Lemma 3.2** *Let $M$ be a countable model of $Th_l$, then there exists an $s$, $0 \leq s < \omega$, and $j$, $0 \leq j \leq \omega$, such that $M$ has the following form:*

$$\Sigma_{1<i<\omega} H_i^{(\omega)} \oplus \Sigma_{l<i<\omega} I_i^{(\omega)} \oplus T^{(j)} \oplus I_l^{(s)}$$

For $0 \leq s \leq \omega$, we denote the model with the form above and $s$ copies of $I_l$ by $M_s$.

**Remark 3.3** From Lemma 3.2 we observe the following facts about algebraic closure. For any model $M$ of $Th_{l,s}$ with $s < \omega$:

1. $\mathrm{acl}_M(\emptyset)$ consists of the isolated vertices of degree $l$, and their neighbors.
2. $\mathrm{acl}_M(a) = \mathrm{acl}_M(\emptyset)$ unless $a$ is on component which contains a cycle or an isolated point.
   (a) In the first case $\mathrm{acl}_M(a)$ is the union of $\mathrm{acl}_M(\emptyset)$ with all points on the path from $a$ to the cycle.
   (b) In the second case $\mathrm{acl}_M(a)$ is the union of $\mathrm{acl}_M(\emptyset)$ with all points on the path from $a$ to the isolated point.
3. $\mathrm{acl}_M(a, b) = \mathrm{acl}_M(a) \cup \mathrm{acl}_M(b)$ unless $a$ and $b$ are on the same component; in that case it also includes all points on the path from $a$ to $b$.

4. For any set $A$, $\mathrm{acl}_M(A) = \bigcup_{a,b\in A} \mathrm{acl}_M(a,b)$.

**Definition 3.4** Let $G^l_n$, for $n > 0$, be the direct sum of $n$ components, each consisting of one vertex with $l$ neighbors. To ease notation, define $G^l_\omega$ to be equal to the empty set.

It is easy to see that for any $M \models Th^{>0}_l$, there exists an $s$, $0 < s \leq \omega$ such that $\mathrm{cl}^\omega_M(\emptyset) \simeq G^l_s \simeq \mathrm{cl}^t_M(\emptyset)$ for $t \geq s$. The following property of $k$-closure in models of $Th_l$ is crucial. Let $M$ be a model of $Th_l$, and let $n = |acl_M(\emptyset)|$; if $k < n$, $\mathrm{cl}^k_M(\emptyset) \simeq \emptyset$. So, if $M_0$ models $Th_{l,0}$, $\mathrm{cl}^k_M(\emptyset) \simeq \emptyset \simeq \mathrm{cl}^k_{M_0}(\emptyset)$. In particular, for any $k < \omega$ we have $\mathrm{cl}^k_{M_\omega}(\emptyset) \simeq \emptyset \simeq \mathrm{cl}^k_{M_0}(\emptyset)$ and $M_\omega \not\equiv M_0$. Thus the theory $Th_l$ is not determined. However, we will show that $Th_{l,0}$ and $Th^{>0}_l$ are each determined with respect to our notion of closure.

The following example shows why we had to treat the closure of the empty set in a special way in Definition 2.4. It could easily be modified to show the theories in question were not determined if we omitted this special case.

**Example 3.5** *Consider the models $M_1$ and $M_2$ with the notation set after Lemma 3.2. Let $a$ and $a'$ be neighbors of neighbors of isolated vertices in $M_1$ and $M_2$ respectively. Then the $l+2$ closure of $a$ and the $l+2$ closure of $a'$ are isomorphic. Both consist of the neighbors of the isolated point near $a$, ($a'$) respectively. But the $l+2$ closure of the empty set is empty in $M_2$ and contains the neighbors of the isolated point in $M_1$.*

**Notation 3.6** We adopt the following notation. For any $a$ in $M$, let $C_M(\overline{a})$ be the union of the components in $M$ which intersect $\overline{a}$. Denote the number of free variables plus the quantifier rank of a formula $\theta$ by $qr^*(\theta)$.

**Definition 3.7** If $\theta(\overline{y})$ is quantifier free, $k_\theta = (l+1)\cdot \mathrm{lg}(\overline{y})$. If $\theta(\overline{y})$ is the formula $(\exists x)\phi(x,\overline{y})$ let $k_\theta$ be the least integer $k > \max(3k_\phi, (l+1)qr^*(\phi))$ and such that for any element $b$ in $\mathrm{cl}^{k_\phi}(\overline{a})$, $\mathrm{cl}^{k_\phi}(b,\overline{a}) \subseteq \mathrm{cl}^{k_\theta}(\overline{a})$.

*Our main induction concerns a formula $\phi(x,\overline{y})$; we denote $\exists x\phi(x,\overline{y})$ by $\theta(\overline{y})$.*

The condition $k_\theta > (l+1)qr^*(\phi)$ guarantees that if $cl^{k_\theta}_M(\emptyset) \simeq cl^{k_\theta}_{M'}(\emptyset)$ and they each contain an isolated point, then they contain the same number of isolated points.

The main result is to show in Theorem 3.9 that the theories $Th^{>0}_l$ and $Th_{l,0}$ are determined. This argument is simply a different way to organize the back-and-forth argument showing each $Th_{l,s}$ is complete. We require one technical definition.

**Definition 3.8** *For any $M$ and $\overline{a}, b \in M$, let $d(b, \mathrm{cl}^{k_\phi}_M(\overline{a}))$ be the shortest distance from $b$ to $\mathrm{cl}^k_M(\overline{a})$. For any $M$ and $\overline{a} \in M$, let*

$$D^\phi_{M,\overline{a}} = \max\{d(b, \mathrm{cl}^{k_\theta}_M(\overline{a})) : b \in \phi(M,\overline{a})\}$$

.

**Theorem 3.9** *Suppose both $M$ and $M'$ are models of $Th_l^{>0}$ or both are models of $Th_{l,0}$. For any $\theta(\overline{x})$ in $L$ (with arity $r$), and any $\overline{a}$, $\overline{a}$ in $M^r$ and $M'^r$ respectively, there exists $k_\theta$ such that if $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s cl_{M'}^{k_\theta}(\overline{a}')$ then $M \models \theta(\overline{a})$ if and only if $M' \models \theta(\overline{a}')$.*

Proof: The lemma follows by induction on the complexity of formulas. Let $\theta(\overline{y}) = \exists x\phi(x;\overline{y})$. Choose $k_\theta$ as in Definition 3.7. Let $\overline{a}$, $\overline{a}'$ be in $M$ and $M'$ respectively. We need to show, if $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s cl_{M'}^{k_\theta}(\overline{a}')$ and $\theta(\overline{a})$ holds in $M$, then we can choose $b$, satisfying $\phi(x,\overline{a})$, such that there exists $b'$ for which $\mathrm{cl}_M^{k_\phi}(\overline{a},b) \simeq^s cl_{M'}^{k_\phi}(\overline{a}',b')$ (equivalently, since $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s cl_{M'}^{k_\theta}(\overline{a}')$, $\mathrm{cl}_M^{k_\phi}(\overline{a},b) \simeq cl_{M'}^{k_\phi}(\overline{a}',b')$) whence by induction $M \models \phi(b,\overline{a})$ if and only if $M' \models \phi(b',\overline{a}')$.

Now, all possible cases are handled by the next 4 lemmas. The major division depends on whether $D_{M,\overline{a}}^{\phi} > k_\phi$. Within each side of this dichotomy, there are several cases depending on the disjoint cases: $\mathrm{cl}^{k_\phi}(b)$ is $\{b\}$, or contains a cycle, or contains an isolated point.

First we consider the case where $D_{M,\overline{a}}^{\phi}$ is large and $\mathrm{cl}^{k_\phi}(b)$ is either $\{b\}$ or contains a cycle.

**Lemma 3.10** *Let $M, M' \models Th_l$ and $\overline{a} \in M^r$. Suppose $D_{M,\overline{a}}^{\phi} > k_\phi$. Fix $b \in M$ for which $d(b, \mathrm{cl}_M^{k_\theta}(\overline{a})) = D_{M,\overline{a}}^{\phi}$. Suppose $\mathrm{cl}_M^{k_\phi}(b) = \{b\}$ or $\mathrm{cl}_M^{k_\phi}(b)$ contains a cycle. If $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}_M^{k_\phi}(\overline{a},b) \simeq^s \mathrm{cl}_{M'}^{k_\phi}(\overline{a}',b')$.*

Proof: By assumption, $\mathrm{cl}_M^{k_\phi}(b)$ is either just $b$ or the vertex $b$ and a cycle of cardinality less than $k_\phi$ and the vertices on a path, of length less than $k_\phi$, from $b$ to this cycle. In the first case, choose $b'$ to be a vertex of infinite degree, on a component which does not intersect $\overline{a}'$. In the second case, choose one of the infinitely many components in $M'$ which contains an n-cycle and does not intersect $\overline{a}'$, and choose $b'$ on this component with the same distance from the n-cycle as $b$ is from the $n$-cycle on the component where $b$ resides. In both cases, the result follows since $\mathrm{cl}^{k_\phi}(\overline{a},b) = \mathrm{cl}^{k_\phi}(\overline{a}) \cup \mathrm{cl}^{k_\phi}(b)$ and similarly for $\overline{a}', b'$. $\square$

Now we consider the case where $D_{M,\overline{a}}^{\phi}$ is large and $\mathrm{cl}^{k_\phi}(b)$ contains an isolated point. There are two traps which must be avoided in the following proof: $M = M_\omega$ and $M' = M_0$, $M = M_i$ and $M' = M_j$ where $j$ is much greater than $k_\theta$ is much greater than $i$. We avoid the first by restricting to $Th_l^{>0}$; this is permissible since there are no isolated points in models of $Th_{l,0}$ and so the case can occur only for $Th_l^{>0}$. The second is dealt with by using $\simeq^s$.

**Lemma 3.11** *Let $M, M' \models Th_l^{>0}$ and $\overline{a} \in M^r$. Suppose $D_{M,\overline{a}}^{\phi} > k_\phi$. Fix $b \in M$ for which $d(b, \mathrm{cl}_M^{k_\theta}(\overline{a})) = D_{M,\overline{a}}^{\phi}$. Suppose there is an isolated vertex* *contained in* $\mathrm{cl}_M^{k_\phi}(b)$. *If* $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, *there exists a* $b' \in M'$ *such that* $\mathrm{cl}_M^{k_\phi}(\overline{a}, b) \simeq^s \mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b')$.

Proof. Since $D^\phi_{M,\overline{a}} > k_\phi$, $\mathrm{cl}_M^{k_\phi}(\overline{a}, b) = \mathrm{cl}_M^{k_\phi}(\overline{a}, b) \cup c_M^{k_\phi}(b)$. If there is an isolated point $c'$ in $M' - \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, this is easy, we can map $b$ to a point near $c'$. Specifically, since $k_\theta > l+1$, none of the neighbors of $c'$ can be in $\mathrm{cl}_{M'}^{\theta}(\overline{a}')$ either. Therefore, there exists a $b'$ such that $\mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b') = \mathrm{cl}_M^{k_\phi}(\overline{a}', b') \cup \mathrm{cl}_M^{k_\phi}(b')$ and $\mathrm{cl}_{M'}^{k_\phi}(b') \simeq \mathrm{cl}_M^{k_\phi}(b)$. So $\mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b') \simeq \mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b')$.

We are left with the case that all isolated points of $M'$ are in $\mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$. But then all isolated points of $M'$ are in $\mathrm{cl}_{M'}^{k_\theta}(\emptyset)$. Since $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, $\mathrm{cl}_M^{k_\theta}(\emptyset) \simeq \mathrm{cl}_{M'}^{k_\theta}(\emptyset)$: this is the essential use of $\simeq^s$ instead of $\simeq$. Since $\mathrm{cl}_{M'}^{k_\theta}(\emptyset)$ contains all the $t > 0$ isolated points in $M'$, $\mathrm{cl}_M^{k_\theta}(\emptyset)$ and therefore $\mathrm{cl}_M^{k_\theta}(\overline{a})$ contains all the $t > 0$ isolated points in $M$. So $b \in C_M(\overline{a})$. Let $\overline{a}_0 = C_M(b) \cap \overline{a}$. Since $d(b, \mathrm{cl}_M^{k_\theta}(\overline{a})) = D^\phi_{M,\overline{a}} > k_\phi$, the isolated point $c$ of $C_M(b)$ is not in $\mathrm{cl}_M^{k_\theta}(\overline{a}_0)$. So we can map $b$ to a $b'$ on the component of $\overline{a}'_0$ so that $\mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b') \simeq \mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b')$.

Now we turn to the cases where $D^\phi_{M,\overline{a}}$ is small; first, suppose $\mathrm{cl}^{k_\phi}(b) = \{b\}$.

**Lemma 3.12** *Let* $M, M' \models Th_l$ *and* $\overline{a} \in M^r$. *Suppose* $D^\phi_{M,\overline{a}} \leq k_\phi$. *Fix* $b \in M$ *for which* $d(b, \mathrm{cl}_M^{k_\theta}(\overline{a})) = D^\phi_{M,\overline{a}}$. *For any* $\overline{a}' \in M'^r$, $b \in M$ *if* $\mathrm{cl}_M^{k_\phi}(b) = \{b\}$ *and* $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, *there exists a* $b' \in M'$ *such that* $\mathrm{cl}_M^{k_\phi}(\overline{a}, b) \simeq^s \mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b')$.

Proof: We may assume that $b \notin \mathrm{cl}_M^{k_\theta}(\overline{a})$, otherwise the result follows immediately from the second requirement in defining $k_\theta$. First we claim there is at most one path whose vertices are in $\mathrm{cl}_M^{k_\phi}(b, \overline{a}) - \mathrm{cl}_M^{k_\theta}(\overline{a})$ from $b$ to the $k_\theta$-closure of $\overline{a}$ in $M$. Suppose not, then we claim that any vertex $c$ which lies on a fork of the path from $b$ to the $\mathrm{cl}_M^{k_\theta}(\overline{a})$ is in fact in $\mathrm{cl}_M^{k_\theta}(\overline{a})$, which is a contradiction.

So assume there are two paths in $\mathrm{cl}_M^{k_\phi}(b, \overline{a})$ from $c$ to $\mathrm{cl}_M^{k_\theta}(\overline{a})$, one going to say, $a_1$ in $\overline{a}$, and the other going to say, $a_2$ in $\overline{a}$, ($a_1$ could be equal to $a_2$) with lengths $k_1$ and $k_2$ respectively. Note, since both paths are in $\mathrm{cl}^{k_\phi}(b, \overline{a})$, then we may assume both $k_1$ and $k_2$ are less than $k_\phi$ which is less than $k_\theta$. Thus there is at most one other vertex with distance $k_1$ to $a_1$ and distance $k_2$ to $a_2$ (if $a_1 = a_2$, $c$ is the only vertex, since a component of a model of $Th_l$ can have at most one cycle). Thus $c$ satisfies a formula with only two solutions (or one solution if $a_1 = a_2$) and quantifier rank less than $k_\phi$. So $c \in \mathrm{cl}_M^{k_\theta}(\overline{a})$. This proves the claim.

Assume the shortest path from $b$ to $\mathrm{cl}_M^{k_\phi}(\overline{a})$ is of length $k_0$ and the nearest vertex in $\mathrm{cl}_M^{k_\phi}(\overline{a})$ to $b$ is $a_0$. Let $a'_0$ be in $\mathrm{cl}_{M'}^{k_\phi}(\overline{a}')$ such that $a'_0$ corresponds to $a_0$ in the isomorphism from $\mathrm{cl}_M^{k_\theta}(\overline{a})$ to $\mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$.

We need to choose a vertex $b' \in M'$ a vertex and a path in $M'$ of size $k_0$ none of whose vertices are in $\mathrm{cl}_{M'}^{k_\phi}(\overline{a}')$ except for $a'_0$. This is immediate if $a'_0$ has infinite degree; $a_0$ and $a'_0$ have the same degree. If $a_0$ has finite degree then all the neighbors of $a_0$ are in $\mathrm{cl}_M^{k_\phi}(\overline{a})$ since $k_\phi > l$. But then $k_0$ was not chosen minimal. So $a_0$ and thus $a'_0$ has infinite degree and we can choose $b'$.

Finally by the first claim, $\mathrm{cl}_M^{k_\phi}(b, \overline{a})$ is contained in $\mathrm{cl}_M^{k_\phi}(\overline{a})$ along with $b$ and a path of length at most $k$ from $b$ to $\mathrm{cl}_M^{k_\theta}(\overline{a})$ and $\mathrm{cl}_{M'}^{k_\phi}(b', \overline{a}')$is contained in $\mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$ along with $b'$ and a path from $b'$ to $\mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$ of the same length. Therefore we have $\mathrm{cl}_M^{k_\phi}(b, \overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\phi}(b', \overline{a}')$ $\square$

Finally, we consider the case where $D_{M,\overline{a}}^{\phi}$ is small and $\mathrm{cl}^{k_\phi}(b) = \{b\}$ contains a cycle or an isolated point.

**Lemma 3.13** *Let $M, M' \models Th_l$. Fix $\overline{a} \in M^r$, $\overline{a}' \in M'^r$, $b \in M$ with $D_{M,\overline{a}}^{\phi} \leq k_\phi$. If there is an isolated vertex or a cycle contained in $\mathrm{cl}_M^{k_\phi}(b)$ then the isolated vertex or the cycle is contained in $\mathrm{cl}_M^{k_\theta}(\overline{a})$. In particular, if $\mathrm{cl}_M^{k_\theta}(\overline{a}) \simeq^s \mathrm{cl}_{M'}^{k_\theta}(\overline{a}')$, there exists a $b' \in M'$ such that $\mathrm{cl}_M^{k_\phi}(\overline{a}, b) \simeq^s \mathrm{cl}_{M'}^{k_\phi}(\overline{a}', b')$.*

Proof: Note that $\overline{a}$ satisfies the formula which asserts: there is a path of length at most $k_\phi$ to a point $x$ and there is a path of length at most $k_\phi$ from $x$ to a cycle of length at most $k_\phi$ (or to an $l$-isolated point). Since $k_\theta > 3k_\phi$ and $l < k_\phi$ the result follows.

This completes the proof of Theorem 3.9. We now apply this result to computing the probabilities of sentences with respect to $p_n^l$.

**Definition 3.14** *Let $\sigma_s^l$ be the sentence: "there exists exactly $s$ vertices of degree $l$". Define $q_s^l$ to be the limit probability of $\sigma_s^l$ (the existence of the limit is shown in [7].) Note this limit also depends on the constant $c$ in the definition of $p_n^l$.*

**Theorem 3.15** *For every non-negative integer $s$ and for $s = \omega$, $Th_{l,s}$ is a complete theory. Furthermore these are all possible completions of the almost sure theory $Th_l$.*

Proof: First we show $\{Th_{l,s} : 0 < s \leq \omega\}$, is the set of all possible completions of the theory $Th_l^{>0}$. Fix an integer $s$ or let $s = \omega$. It is clear that if $M$ and $M'$ model $Th_{l,s}$, then $\mathrm{cl}_M^{\omega}(\emptyset) \simeq \mathrm{cl}_{M'}^{\omega}(\emptyset) \simeq G_s^l$. (Remember, by convention, $G_\omega^l = \emptyset$.) Furthermore, for all $t \geq s$, $\mathrm{cl}_M^t(\emptyset) \simeq \mathrm{cl}_{M'}^t(\emptyset) \simeq G_s^l$, since $t$ is large enough to capture the algebraic closure of $M$ and $M'$. Finally, for all $q < s$ $\mathrm{cl}_M^q(\emptyset) \simeq \mathrm{cl}_{M'}^q(\emptyset) \simeq \emptyset$. Thus, since for all $t$, $0 < t \leq \omega$, $\mathrm{cl}_M^t(\emptyset) \simeq \mathrm{cl}_{M'}^t(\emptyset)$, Theorem 3.9 implies $M \equiv M'$. Therefore $Th_{l,s}$ is complete. Since $Th_l^{>0}$ is determined, $\{Th_{l,s} : 0 < s \leq \omega\}$ is the set of all completions of $Th_l^{>0}$.

We note now that $Th_{l,0}$ is the theory $Th_l$ plus the negation of the axiom "there exists an isolated vertex"(recall, this axiom plus $Th_l$ is $Th_l^{>0}$). Since $Th_{l,0}$ is determined, and the algebraic closure of the empty set of any model of

$Th_{l,0}$ is empty, $Th_{l,0}$ is complete. Thus we have now all possible completions of $Th_l$. □

The existence of $k^*$ below follows from our characterization of the closure of the empty set in models of $Th_l$.

**Definition 3.16** For any $k$, let $k^*$ be the least integer $s$ greater than or equal to $k$ such that for any $t \geq s$, $\mathrm{cl}^k_{M_t}(\emptyset) = \emptyset$.

We write $p^l(\theta)$ for $lim_{n\to\infty} p^l_n(\theta)$ if it exists. We use the standard notation $p^l(\theta|\sigma^l_i)$ for $\frac{p^l(\theta\wedge\sigma^l_i)}{p^l(\sigma^l_i)}$. In probabilistic terms, $p^l(\theta|\sigma^l_i)$ is the probability of $\theta$ conditioned by $\sigma^l_i$. Stated informally in terms of this particular application, $p^l(\theta|\sigma^l_i)$ is the probability of $\theta$ holding in a model of the theory $Th_{l,i}$. In the proof of the following theorem, we emphasize the special role that $\sigma^l_0$ plays.

**Theorem 3.17** *For any $L$-sentence $\theta$, there exists a finite set $I$ of positive integers such that $p^l(\theta)$ is $\Sigma_{i\in I} q^l_i$ or $1 - \Sigma_{i\in I} q^l_i$.*

Proof: Since for all $i \neq j$ and for all $n$, $p^l_n(\sigma^l_i \wedge \sigma^l_j) = 0$ and since Theorem 3.15 states that every completion of $Th_l$ is of the form $Th_{i,l}$, we can write $p^l_n(\theta) = \Sigma_{i\in\omega}[p^l_n(\theta|\sigma^l_i) \times p^l_n(\sigma^l_i)]$ or $p^l_n(\theta) = 1 - \Sigma_{i\in\omega}[p^l_n(\neg\theta|\sigma^l_i) \times p^l_n(\sigma^l_i)]$. Furthermore, since by Theorem 3.15, $Th_{l,i}$ is complete, the terms $p^l(\theta|\sigma^l_i)$ and $p^l(\neg\theta|\sigma^l_i)$ are either 1 or 0. Hence the limit of any addend in the above sums is either $q^l_i$, $1-q^l_i$ or 0. To ensure that the limit probability $p^l(\theta)$ exists, we need to show that in one of the two sums all but a finite number of addends can be ignored. That is, we will show that there exists a finite set $I$ such that for all $i \in I$, $Th_{l,i} \models \theta$ or there exists a finite set $I$ such that for all $i \in I$, $Th_{l,i} \models \neg\theta$.

We now consider only the limits of each of the above addends. We need to treat the term the case of $\sigma^(\sigma^l_0)$ separately from the $\sigma^(\sigma^l_i)$ for $i > 0$. By Theorem 3.9, there exists a finite $k_\theta$ such that the $k_\theta$-closure of the empty set determines $\theta$ in $Th^{>0}_l$. Without loss of generality, assume that $M_{k^*_\theta} \models \theta$ (an analogous argument works if $M_{k^*_\theta} \models \neg\theta$). We observe that for all $j \geq k^*_\theta$, $M_j \models \theta$. Consider the set $I'$ (possibly empty) of positive integers bounded by $k^*_\theta$ such that for all $i \in I'$, $Th_{l,i} \models \neg\theta$. If this set is empty we conclude by Lemma 2.2 and Theorem 3.15, that $Th^{>0}_l$ proves $\theta$. In this case, it is clear that $lim_{n\to\infty} p^l(\theta) = 1 - q^l_0$ or 1, depending on whether $Th_{l,0} \models \theta$. If, however, the set $I'$ is not empty $Th^{>0}_l$ proves $\vee_{i\in I'} \sigma^l_s \leftrightarrow \neg\theta$. Thus $p^l(\neg\theta)$ is $\Sigma_{i\in I'} q^l_i$, plus the contribution from $Th_{l,0}$, which is $q^l_0$ if $\sigma^l_0 \to \neg\theta$ and 0 if $\sigma^l_0 \to \theta$. So, if we let $I = I' \cup \{0\}$, $p^l(\theta)$ is either $1 - \Sigma_{i\in I'} q^l_i$ or $1 - \Sigma_{i\in I} q^l_i$ and both are in the correct form. Finally we note that this limit exists since the limit of every addend exists and $I$ is finite. □

## 4 Conclusion and Questions

We have provided another proof of the convergence law for the edge probability $p_n^l$ considered in [7]. Our analysis allows for one less probability calculation. But the argument depends very heavily on Lemma 3.9 which seems to be an unusual and overly strong condition. In particular, it implies that in every model the algebraic closure of the empty set is finite. This seems to be a necessary condition for this type of argument to work. Basically, the key is to be able to compute the probability of assertions, '$\mathrm{acl}_M(\emptyset)$ has form $X$'. Can a general method of showing convergence be developed by adding this hypothesis?

A natural way to continue these investigations would be to see whether this method extends to show the more general result proved in [7]. Namely to extend to the probability:

$$p_n^{l,k} = \frac{ln(n) + l \cdot ln(ln(n)) + c}{kn}.$$

In particular, can convergence be proved using exactly the indexed closure operator of this paper?

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