Encouraging cooperative solution of mathematics problems

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September 29, 1995

Abstract

To help students adjust to group problem solving, questions can be posed with different students receiving different 'clues'. Several such problems are presented here and their use in workshops is described.

Keywords: cooperative learning, many-piece problems

Beginning with Treisman's development of the Professional Development Program (PDP) at the University of California, Berkeley [5], a number of universities across the country have begun programs that aim to improve performance in elementary mathematics courses by encouraging collaborative/cooperative student effort on challenging problems. In Treisman's study, students involved in these workshops, African-American students especially, performed significantly better in the freshman calculus course than similar students without the workshops. These results have been replicated at the University of Illinois at Chicago during the last two years. Such results are expected in the light of research on cooperative learning such as that summarized by [4]. Students who work together to clarify questions, discuss and select problem-solving strategies, and resolve controversies usually demonstrate greater gains in concept development and problem-solving abilities than similar students who work alone. Cooperative learning has been associated with an increase in problem-solving skills of college students [2].

The University of Illinois at Chicago has about 20,000 studemts. About 10% of the students taking Precalculus and Calculus are enrolled in auxiliary PDP workshops. The courses meet for three lectures and two recitation sections per week. The workshops meet for two further two hour sessions. About 10% of the class registration is African American and 15% Hispanic. These groups comprise about 1/2 the students in the PDP workshops. Each workshop is led by a TA and an undergraduate assistant. The presence of two facilitators in the room makes a valuable change in the center of gravity of the classroom. These leaders meet for several periods of training at the beginning of the year and occasionally while classes continue. All training is conducted in the style that we intend for a PDP class. Problems (both mathematical and pedagogical) are discussed in small groups and reported to the meeting. The coordinator plays a further important role in the program by helping students with the bureaucracy, counseling, as well as general management.

A distinguishing feature of PDP programs is the use of additional class time to present students with conceptual problems that are often too hard for a single student to master. In developing such a program at a commuter campus certain additional obstacles arise. It is important to try to develop class cohesion and a willingness to cooperate as early in the semester as possible. Commuting students rarely see each other outside of class. Thus the formation of a group of students centered on mathematical interests is even more important than at a residential institution. Moreover, we want the students to work cooperatively on the solution of the problems from the beginning. Ice-breaking exercises, such as requiring each student to introduce another to the class after a brief interview, help to instill a sense of membership in the group. Such a session is described under Example 5 below.

The level of interaction we seek is not the same as individually solving the problem and then comparing answers or just doing separate parts of a problem which can be combined to answer a larger question. (These are also valid techniques in a course of this sort, but they do not require the intense communication between students that we are trying to generate.)

The many-piece problem (adapted from the EQUALS project [3]) is one way to require this kind of interaction. Each member of a group (usually 4 students) is presented with a clue to the problem. Each student may read the clue aloud, but is not allowed to show the clue to someone else. The group must work from the information provided orally to solve the problem. Groups of 5 or 6 can also by accommodated by extra clues. These may be redundant; if only four students are in the group, the extra clues are not used.

Originally these usually were logical puzzles, in which students might be trying to determine, among five people, which was the doctor, lawyer, mechanic, secretary, and physicist. In keeping with the equity goals of EQUALS, the doctor, physicist and mechanic often turned out to be females. A recent publication [3] includes many-piece problems on a variety of subject matters and levels.

Here is an icebreaking problem that doesn't come in many pieces. Each pair of people at your table should shake hands. How many hand shakes were there? What if each pair in the room shook hands?

Students who are not used to working together can be encouraged to do so through logic puzzles or recreational topics. However, we present here a number of examples to show that the technique can be used to study material in the normal curriculum. We have discovered that the information in more complex problems can often be divided up to meet our major requirement: Each student in the group should have information needed to solve the problem. Thus, the participation of every student is required for a correct group solution.

1 Some Examples of Many-piece Problems

1.1 Example 1. Here is problem that might appear in an Intermediate Algbra class.

The Simpsons are converting a section of a warehouse into an apartment. They are decorating an open 20' x 30' area that will serve as both living room and dining room. They have bought a new 9' x 12' carpet for the living room area. Mrs. Simpson's mother gave them an antique braided circular rug, 11 feet in diameter, that they plan to use in the dining area. The Simpsons will paint only the floor area not covered by the carpets. How much floor area needs to be painted?

Converted to many-piece format, in this case with some editing, we have a problem for a group of four:

The instructor gives these Guidelines: 'Here is a problem for your group. Each one of you will have a piece of paper with part of the problem. You may share the information by telling or reading aloud. You may not show your piece to anyone else. When the group agrees on the solution, and everyone in the group can explain it, raise your hands.'

A. The Simpsons are converting a section of a warehouse into an apartment. Mrs. Simpson's mother gave her an antique braided circular rug, 11 feet in diameter, that she plans to use in the dining area.

B. The Simpsons have bought a new 9' x 12' carpet for the living room area. How much floor area needs to be painted?

C. The Simpsons will paint only the floor area not covered by the carpets. Make a scale drawing of a possible layout on graph paper.

D. They are decorating an open 20' x 30' area that will serve as both living room and dining room. Graph paper might be helpful in solving this problem.

After the group has agreed that they have solved the problem, it is important that the instructor checks to see that it is an agreement by consensus, rather than a majority solution. The instructor may then choose a group member to explain the solution to him within the small group. The student should eventually be comfortable enough, with the help of group members, to explain the group's solution to the entire class.

1.2 Example 2. A Syllogism

This is freely adapted from one of numerous syllogisms of 2 through perhaps 12 lines by Lewis Carroll. It could be used in many ways. The many clever examples by Carroll are to be found in his 'Symbolic Logic' [1].

Each student (of 4 in this particular group) is given one of the following phrases. He/she should read it aloud but not show it to the others. Together, all the students at the table try to come up with the simple conclusion which is implied when all statements are taken together.

> No students under 18 are admitted to this school as boarders.
>
> All the industrious students have red hair.
>
> None of the day-students get A's.
>
> None but those under 18 are idle.

A more current version might be:

> No students not serious about basketball are admitted to this school on scholarship.

All the industrious students have purple socks.

None of the paying students ever shoot the final basket.

None but those not serious about basketball are idle.

A five line syllogism from the same source might be:

Promise-breakers are untrustworthy.

Wine-drinkers are very communicative.

A man who keeps his promises is honest.

No teetotalers are pawnbrokers.

One can always trust a very communicative person.

The complicated language (e.g. double negatives) force the students to discuss the meanings of text and their problems of analysis in a nonthreatening situation. (No one can understand the first one without effort.) Both important mathematical distinctions ($A \to B$ versus $B \to A$) and the value of considering the problem setter's situation (Apparently no one in England drinks only whiskey.) will come up in the discussion. Posing the syllogisms without stating the conclusion allows for a number of correct intermediate solutions. The groups might be asked to exchange their solutions to illustrate this. Still another extension (especially in high school) would be to have the students write their own syllogisms on the same pattern but with more contemporary situations.

This same structure can be used for problems that both address the pedagodical goals outlined above and move the students into the heart of the course. Examples 3 and 4 were designed for a Precalculus course and Example 4 for a Calculus course.

1.3 Example 3. The Sears Tower

Here are the instructions given the teaching assistant.

This is a set of problems designed to encourage cooperation while working on a nontrivial problem. There are 4 sets of 6 clues. Put the students in groups of 4, 5, or 6 and give each member of each group one of the clues. Ideally the groups have 5 members each and clue 6 (which is identified to the T.A. as irrelevant by an asterisk) is omitted. For a group of 4, put the fifth clue (the question) in the middle of the table. The four groups each get slightly different information. One gets the height of the Sears tower in feet without the antennas; a second gets the height with the antennas; these two need to compare to get the height of the antennas (in feet). The other pair will do the same things but the answer is in meters. (The height of the tower without antennas is correct; but the distances on the ground are only approximate and the height of the antennas was sheer guesswork.) At the end have the metric and English units groups compare their answers. Note that translation between the two systems is most easy by the approximation that a kilometer is 5/8 of a mile. (Sometimes .6 is used.)

Here are the clues for the first group. As indicated in the instructions for the teaching assistant, the other sets of clues are slight variants (e.g. meters instead of feet).

A1 The Sears Tower is located 4 blocks East of Halsted Street on Jackson.

A2 The Corner of Halsted and Polk (Chicago Circle Center) is three blocks south of Jackson.

A3 There are eight Chicago city blocks to a mile. Each mile contains 5280 feet.

A4 A straight line from the top of the Sears tower (not including the television antennas) to the corner of Halsted and Polk makes an angle of 23.77 degrees with the ground.

A5 How many feet high is the Sears tower (without antennas)? How tall is each antenna? (You will need help from another group to answer the last part.)

A6 * It is $\sqrt{2}$ miles from Chicago Circle Center to Comiskey Park.

1.4 Example 4. The following problems do not require as much interaction at first. But the second stage is the most important. The students all have graphing calculators.

Six students will work on each problem, one on part a), one on part b) etc. If there are less than six students in your group, each do one part then split up the remaining parts.

Problem 1) a) Calculate $\sin 20°$, calculate $\sin {}^2 20°$, calculate $\sin 40°$, calculate $\sin^2 20°$.

Problem 1) b) Calculate $\cos 20°$, calculate $\cos^2 20°$, calculate $\cos 40°$, calculate $\cos^2 20°$.

Problem 1) c) Calculate $\tan 20°$, calculate $\tan^2 20°$, calculate $\tan 40°$, calculate $\tan^2 20°$.

Problem 1) d) Calculate $\sec 20^\circ$, calculate $\sec^2 20^\circ$, calculate $\sec 40^\circ$, calculate $\sec^2 20^\circ$.

Problem 1) e) Calculate $\cot 20^\circ$, calculate $\cot^2 20^\circ$, calculate $\cot 40^\circ$, calculate $\cot^2 20^\circ$.

Problem 1) f) Calculate $\csc 20^\circ$, calculate $\csc^2 20^\circ$, calculate $\csc 40^\circ$, calculate $\csc^2 20^\circ$.

Compare your answers. What relationships can you find between them? (There are several important ones)

Problem 2) a) Calculate $\sin 2$ (radians), calculate $\sin^2$ 2, calculate sin 4, calculate $\sin^2$ 4.

Problem 2) b) Calculate cos 2 (radians), calculate $\cos^2$ 2, calculate cos 4, calculate $\cos^2$ 4.

Problem 2) c) Calculate tan 2 (radians), calculate $\tan^2$ 2, calculate tan 4, calculate $\tan^2$ 4.

Problem 2) d) Calculate sec 2 (radians), calculate $\sec^2$ 2, calculate sec 4, calculate $\sec^2$ 4.

Problem 2) e) Calculate cot 2 (radians), calculate $\cot^2$ 2, calculate cot 4, calculate $\cot^2$ 4.

Problem 2) f) Calculate csc 2 (radians), calculate csc 4, calculate $\csc^2$ 2, calculate $\csc^2$ 4.

Compare your answers. What relationships can you find between them? (There are several important ones)

Problem 3) What is the difference between problems 1) and 2)? Do the relationships you are discussing depend on whether the angle is measured in radians or degrees?

1.5 Example 5. Finally, here is a problem given to calculus students on the first day of class.

Clue 1. $x < 0$.

Clue 2. $|x - 1| \leq 11/2$.

Clue 3. $|x + 1| > 2$.

Clue 4. $y = mx + b$ has slope -3 and x-intercept $2/3$.

Clue 5. Find the maximum and minimum value of y. Sketch a figure illustrating this problem.

The initial workshop section in calculus began by assigning students randomly to groups of four or five. Pairs were then assigned to interview each other asking about name, major, home etc. Then the members of the group exchanged this information and finally one person reported for the group to the room at large. Example 5 was then given to the groups (one clue per person) with no further instructions. Some groups had to be nudged towards the notion of minimizing y on the domain determined by the restraints on x, but this sort of hint wasn't given until the students had had five or ten minutes to work out the meaning of the problems. In one class, after forty-five minutes, three of the groups had solved the problem - complete to a common insistence that the minimum of the given decreasing function on a open interval existed but was 'a little bit bigger than' the value at the end point. The fourth group had solved the system of inequalities but was not really sure how to interpret the minimization problem.

1.6 Conclusion. The combination of icebreaking exercises, the clear specification that cooperation was demanded, and the use of many-piece puzzles enabled the students to form their groups more quickly and produced more fruitful work on difficult problems. At the end of the semester, even a teaching assistant somewhat skeptical of 'social engineering' agreed that this strategy had paid off.

References

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John Baldwin received his Ph.D. degree in Mathematical Logic from Simon Fraser University in 1971. He joined the University of Illinois at Chicago in 1973. Eight students have completed their Ph.D.'s under his direction. He has published over 60 papers and one book in model theory. Since 1990, he has been Director of the Professional Development Program at UIC.

After teaching at high schools and junior colleges Roberta Dees earned her Ph.D. in Mathematics Education from the University of Florida in 1980. She taught at Purdue University-Calumet and at Indiana Unversity's campus in Malaysia before joining the staff of the University of Illinois at Chicago in 1990 as an Associate Professor. Her research focuses on Cooperative Learning. With Baldwin she directs the College Preparatory Mathematics Program. This NSF-funded program features the cooperation of the University of Illinois at Chicago and seven high schools (soon to add two more universities and another 10 high schools). It provides training in methods of cooperative learning for high school teachers and summer school and academic year classes for high school students taught by high school teachers using these methods.

Professor David Foulser wrote his Ph.D. thesis on projective planes at the University of Michigan in 1963. He has been at UIC since 1965 and has been active in teacher education through the 80's. He has been primarily responsible for the Calculus segment of PDP.

Professor David Tartakoff obtained his Ph.D. in partial differential equations in 1969 from the University of California at Berkeley. He has been at UIC since 1978 and is an active researcher in the fields of partial differential equations and several complex variables. During the last two years he and Baldwin introduced the use of graphing calculators in the precalculus course with Tartakoff writing most of the problems for the associated PDP sections.