The Rocky Romance of Model Theory and Set Theory

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Eilat meeeting in memory of Mati Rubin

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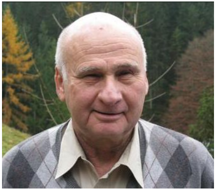
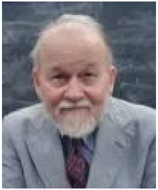
Goal: Maddy 

In Second Philosophy Maddy writes,

> *The Second Philosopher sees fit to adjudicate the methodological questions of mathematics – what makes for a good definition, an acceptable axiom, a dependable proof technique?– by assessing the effectiveness of the method at issue as means towards the goal of the particular stretch of mathematics involved.*

We discuss the choice of definitions of model theoretic concepts that reduce the set theoretic overhead:

Entanglement


Such authors as Kennedy, Magidor, Parsons, and Väänänen have spoken of the entanglement of logic and set theory.

It depends on the logic

There is a deep entanglement between (first-order) model theory and cardinality.

There is No such entanglement between (first-order) model theory and cardinal arithmetic.
At least for stable theories; more entanglement in neo-stability theory.

There is however such an entanglement between infinitary model theory and cardinal arithmetic and therefore with extensions of ZFC.

Equality as Congruence

Any text in logic posits that:
Equality '=' is an equivalence relation:

Further it satisfies the axioms schemes which define what universal algebraists call a congruence.

The indiscernibility of identicals

For any x and y, if x is identical to y, then x and y have all the same first order properties.
For any formula ϕ: $\forall \mathbf{x} \forall \mathbf{y}[\mathbf{x} = \mathbf{y} \rightarrow (\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y}))]$

Equality as Identity

The original ‘sin’

The inductive definition of truth in a structure demands that the equality symbol be interpreted as identity:

$$M \models a = b \text{ iff } a^M = b^M$$

The entanglement of model theory with cardinality is now ordained! This is easy to see for finite cardinalities.

$$\phi_n : (\exists x_1 \ldots x_n) \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge (\forall y) \bigvee_{1 \leq i \leq n} y = x_i$$

is true exactly for structures of cardinality n.

Entanglement with infinite Cardinality

Three examples of the entanglement of first order logic with cardinality.

1. Downward Löwenheim Skolem –not so much
2. Upward Löwenheim Skolem
   Yes! Look at the proof.
3. Only finite structures are categorical.

Shelah: Set theory and first order model theory

Shelah  Jensen 

During the 1960s, two cardinal theorems were popular among model theorists. . . . Later the subject becomes less popular; Jensen complained when I start to deal with gap n 2-cardinal theorems, they were the epitome of model theory and as I finished, it stopped to be of interest to model theorists.

Two Questions

I. Why in 1970 did there seem to be strong links of even first order model theory with cardinal arithmetic and axiomatic set theory?

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I. Why in 1970 did there seem to be strong links of even first order model theory with cardinal arithmetic and axiomatic set theory?

II. Why by the mid-70's had those apparent links evaporated for first order logic?

I. Is there model theory without axiomatic set theory?

The role of saturation

Definition

The model M is saturated if for every set $A \subset M$, every $p \in S(A)$ is realized in M.

Facts/ Proof Scheme

1. In general the continuum hypothesis is needed to construct saturated models.
2. (Keisler, GCH) Elementarily equivalent models have isomorphic elementary extension (ultrapowers).
3. (Ax-Kochen-Ershov) Original solution of the Artin conjecture used CH to get a saturated model and absoluteness of the algebraic hypothesis and conclusions to get proof in ZFC.

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

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Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$. And we write $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ if *every theory* that admits (κ, λ) also admits (κ', λ').

Set Theory Intrudes Morley 

Theorem: Vaught

$(\beth_\omega(\lambda), \lambda) \to (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Theorem: Morley's Method

Suppose the predicate is defined not by a single formula but by a type:
$(\beth_{\omega_1}(\lambda), \lambda) \to (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Both of these results need replacement; the second depends of iterative use of Erdös-Rado to obtain countable sets of indiscernibles.

In the other direction, the notion of indiscernibles is imported into Set Theory by Silver to define $O^{\#}$.

Set Theory Becomes Central

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Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

The links dissolve

Why did it stop? Lachlan  Bays 

Revised Theorem: solved in ZFC

Suppose

1. [Shelah, Lachlan $\approx$ 1972] T is stable
2. or [Bays 1998] T is *o*-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if T admits (κ, λ) then T also admits (κ', λ').

Ask the right question

$P(\kappa, \lambda, T)$ means, 'there is a (κ, λ)-model of T.'

Reversing the question

set theorist:
For which cardinals does $P(\kappa, \lambda, T)$ hold for all theories ?

model theorist:
For which theories does $P(\kappa, \lambda, T)$ hold for all cardinals ?

Really, Why did it stop?

Definition

[The Stability Hierarchy:] Fix a countable complete first order theory T.

1. T is stable in χ if $A \subset M \models T$ and $|A| = \chi$ then $|S(A)| = |A|$.
2. T is
    1. ω-stable[a] if T is stable in all χ;
    2. superstable if T is stable in all $\chi \geq 2^{\aleph_0}$;
       That is, for every A with $A \subset M \models T$, and $|A| \geq 2^{\aleph_0}$, $|S(A)| = |A|$
    3. stable if T is stable in all χ with $\chi^{\aleph_0} = \chi$;
    4. unstable if none of the above happen.

[a]This 'definition' hides a deep theorem of Morley that T is ω-stable if and only if it stable in every infinite cardinal.

The autonomy of first order model theory

Thesis 1: Formalize specific areas

Thesis 1: Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

Examples of Thesis 1

1. The Ax-Kochen-Ershov proof of the Artin conjecture proceeds by identifying complete theories of Henselian valued fields.
2. Differentially closed fields
   - i. differential nullstellensatz
   - ii. existence of prime models
   - iii. prime models need not be minimal
   - iv. Classically (Painlev'e) interesting solutions of second order ode are strongly minimal sets
   - v. comparison of model theoretic ranks with Kolchin rank.

Thesis 2: Compare Formalizations

Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Stability is Syntactic

Definition

T is stable if no formula has the order property in any model of T.

ϕ is unstable in T just if for every n the sentence
$\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_j) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_j)$ is in T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

More precisely

While the stability spectrum function is another function about cardinality,
The notions defining the hierarchy are all absolute.

1. ω-stability (Morley rank defined: Π^1_1)
2. superstability (D-rank defined: Π^1_1)
3. stability (no formula has the order property: arithmetic)

So what? Sacks 

Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

Stability Hierarchy

Theorem

[Stability spectrum theorem] Every complete first order theory falls into one of the 4 classes just defined.
If T is stable then it has a saturated model in exactly the cardinals in which is stable.

Neo-stability and o-minimality

Neo-stability and o-minimality extend the range of applicability to included valued fields, combinatorics, learning theory, etc. etc.

The success of the hierarchy

A crucial consequence of stability is the ability to define family of dimensions and classify structures.

The stability classification of T gives detailed information about the fine structure of definable sets in each model of T.
This information is encoded by stability ranks that are in many cases (e.g. algebraic geometry) the same as those arising in other content areas.

A sophisticated theory for studying the interactions of these various dimensions has had applications in many fields.

Mathematically relevant areas of mathematics can be axiomatized by complete first order theories of various stability classes.

Model theory entangles with Algebra and Geometry

Theorem (Hrushovski 1989) Let T be a stable theory. Let $\tilde{p} \not\perp \tilde{q}$ be stationary, regular types and let n be maximal such that $\tilde{p}^n \perp^a \tilde{q}^\omega$. Then there exist p almost bidominant to $\tilde{p}$ and q dominated by $\tilde{q}$ such that:

- $n = 1$ q is the generic type of a type definable group that has the regular action on the realizations for p.
- $n = 2$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as an affine line.
- $n = 3$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as a projective line.
- $n \geq 4$ is impossible.

The Entanglement with group and field theory: Importance

The hypotheses are purely model theoretic.

There is no assumption that a group or ring is even interpretable in the theory.

The conclusion gives precise kinds of group and field actions that are *definable* in the given structures.

There are important consequences in model theory, diophantine geometry, differential fields, ...

Model theory and mathematics: Hrushovski

Hrushovski ICM talk 1998

Instead of defining the abstract context for the [stability] theory, I will present a number of its results in a number of special and hopefully more familiar, guises: compact complex manifolds, ordinary differential equations, difference equations, highly homogeneous finite structures. Each of these has features of its own and the transcription of results is not routine; they are nonetheless readily recognizable as instances of a single theory.

Thesis 3: the centrality of vocabulary choice and logic

Thesis 3 The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical development of first order logic have been more important in other areas of modern mathematics than such developments for other logics.

Bourbaki

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We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Entanglement of (Infinitary) Model Theory and Axiomatic Set Theory

Beyond stable first order theories

Some Examples

Vasey eliminates the use of the replacement axiom for 'forking = dividing' in simple theories.

Shelah Taming the $\leq_{\text{univ}}$ ordering in simple theories.

Malliaris-Shelah Keisler order

1. lowness is a dividing line among simple theories;
2. SOP_2 implies maximality;
3. Set theoretic by-product: $\mathfrak{p}$ and $\mathfrak{t}$ are equal.

The role of infinitary logic

In 1970, model theory and axiomatic set theory seemed intrinsically linked. Shelah wrote

> *". . . in 69 Morley and Keisler told me that model theory of first order logic is essentially done and the future is the development of model theory of infinitary logics (particularly fragments of $L_{\omega_1,\omega}$). By the eighties it was clearly not the case and attention was withdrawn from infinitary logic (and generalized quantifiers, etc.) back to first order logic."*

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3. Entanglement:

Entanglement of model theory and Infinitary Logic

‘Algebraic’ versus ‘Combinatorial properties’

Examples

Interpreting a field in a plane is certainly algebraic.
It is controversial whether interpreting a ternary ring is ‘only’ combinatorial.

Few notions of ‘algebraic interest’ have been axiomatized in infinitary logic.
Perhaps the relation between locally free and free.

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Certainly an involvement between ‘algebraic’ properties and consistency results in set theory indicates involvement.

Amalgamation Spectrum

Theorem

[BKL] For every $r \geq 1$, the class $\mathbf{At^r}$ satisfies:

1. there is a model of size $\aleph_r$, but no larger models;
2. every model of size $\aleph_r$ is maximal, and so 2-amalgamation is trivially true in $\aleph_r$;
3. disjoint 2-amalgamation holds up to $\aleph_{r-2}$;
4. 2-ap fails in $\aleph_{r-1}$.

More technically, amalgamation for elementary submodels in $\hat{\boldsymbol{K}}^r$ also fails in $\aleph_{r-1}$.

The Amalgamation spectrum

The *finite amalgamation spectrum* of a complete sentence ϕ is the set X_ϕ of $n < \omega$ and $\mathrm{mod}(\phi)$ satisfies amalgamation in $\aleph_n$.

Many examples: X_ϕ is $\emptyset$ or ω.

This is the first example of a complete sentence an aec where the spectra was not: all, none, or just $\{0\}$.

Question

Can the amalgamation spectrum of a complete sentence of $L_{\omega_1,\omega}$ have a proper alternation?

Amalgamation:upper bound on Hanf number

Theorem (B-Boney)

Let κ be strongly compact and $\boldsymbol{K}$ be an AEC with Löwenheim-Skolem number less than κ.

- If $\boldsymbol{K}$ satisfies $AP(< \kappa)$ then $\boldsymbol{K}$ satisfies AP.
- If $\boldsymbol{K}$ satisfies $JEP(< \kappa)$ then $\boldsymbol{K}$ satisfies JEP.
- If $\boldsymbol{K}$ satisfies $DAP(< \kappa)$ then $\boldsymbol{K}$ satisfies DAP.

Amalgamation: lower bound

The best lower bound for the disjoint amalgamation property is $\beth_{\omega_1}$.

1. Incomplete Sentences
   1. (B-Kolesnikov-Shelah) disjoint embedding up to $\aleph_\alpha$ for every countable α but did not have arbitrarily large models.
   2. (Kolesnikov & Lambie-Hansen) disjoint embedding up to $\aleph_\alpha$ for every countable α and arbitrarily large models.
2. (Complete Sentences) Baldwin-Koerwein-Laskowski) At least trivially the amalgamation spectrum does not have to be an interval.
   Disjoint amalgamation and even amalgamation fail in $\aleph_{r-1}$ but holds (trivially) in $\aleph_r$; there is no model in $\aleph_{r+1}$.

GALOIS TYPES: Algebraic Form

Suppose $\boldsymbol{K}$ has the amalgamation property.

Definition

Let $M \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- locality Any increasing chain of types has at most one upper bound;
- tameness two distinct types differ on a finite set;
- compactness an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\boldsymbol{K}$ is (χ, μ)-*tame* if for any $N \in \boldsymbol{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Hanf number for locality

Definition

- κ is δ-measurable if there is a uniform, δ-complete ultrafilter on κ.
- κ is almost measurable if it is δ-measurable for all $\delta < \kappa$.

Theorem (Shelah)

If every AEC with Löwenheim-Skolem number less than κ is κ-local, then κ is almost measurable.

Hanf numbers of tameness

Boney 

Definition

- κ is (δ, λ)-strongly compact for $\delta \leq \kappa \leq \lambda$ if there is a δ-complete, fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.
- κ is (δ, ∞)-strongly compact if it is (δ, λ)-strongly compact for all δ with $\delta < \kappa$.
- κ is almost strongly compact if it is (δ, ∞)-strongly compact for all $\delta < \kappa$.

Theorem (Boney-Unger)

Let κ be uncountable such that $\mu^\omega < \kappa$ for every $\mu < \kappa$. If every AEC with Löwenheim-Skolem number less than κ is κ-tame, then κ is almost strongly compact.

Maximal models

Theorem: B-Souldatos

There are complete sentences of $L_{\omega_1,\omega}$, with

1. maximal models in κ and κ^+.
2. Assume for simplicity that $2^{\aleph_0} > \aleph_\omega$. For each $n \in \omega$, there is a complete $L_{\omega_1,\omega}$-sentence ϕ'_n with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\aleph_n}$.
3. Assume κ is a homogeneously characterizable cardinal and for simplicity let $2^{\aleph_0} \geq \kappa$. Then there is a complete $L_{\omega_1,\omega}$-sentence ϕ_κ with maximal models in cardinalities 2^λ, for all $\lambda \leq \kappa$.

Everything below $\beth_{\omega_1}$.

Maximal Models and Measurable cardinals

Clearly no sentence of $L_{\omega_1,\omega}$ has a maximal above a measurable cardinal.

Theorem: B-Shelah – in progress

There is a complete sentence ϕ of $L_{\omega_1,\omega}$ such that for cofinally many λ below the first measurable
there is a maximal model M of ϕ with cardinality λ.

One Completely General Result

Theorem: ($2^\lambda < 2^{\lambda^+}$) (Shelah)

Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. For any Abstract Elementary class, if amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality λ^+.

Is $2^\lambda < 2^{\lambda^+}$ needed?

Is $2^{\lambda} < 2^{\lambda^+}$ needed?

Let $\lambda = \aleph_0$:

a. Definitely not provable in ZFC: There are $L(Q)$-axiomatizable examples
   i. Shelah: many models with CH, $\aleph_1$-categorical under MA
   ii. Koerwien-Todorcevic: consistent to have many models under MA, $\aleph_1$-categorical from PFA.
b. Independence Open for $L_{\omega_1,\omega}$

The Paradigm Shift

The Paradigm Shift in Model Theory

The focal point of model theory

1. before 1950: LOGICS
2. 1950-70: properties of theories
   Many problems tied closely to axiomatic set theory.
3. post 1970: properties of classes of theories.

This led to:

1. a divorce of first-order model theory from axiomatic set theory
2. a fruitful interaction with many other areas of mathematics.