Iterated elementary embeddings and the model theory of infinitary logic

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Abstract

We use iterations of elementary embeddings derived from the nonstationary ideal on ω_1 to reprove some classical results about the number of models of cardinality $\aleph_1$ in various infinitary logics. We also consider Galois stability in light of Burgess's theorem on analytic equivalence relations and find variants of the earlier theorems for Abstract Elementary Classes.

In this paper we use iterated generic elementary embeddings to analyze the number of models in $\aleph_1$ in various infinitary logics and for Abstract Elementary Classes. The arguments presented here are very much in the spirit of [8, 9], in which these embeddings were used to prove forcing-absoluteness results. Those papers focused on the large cardinal context. Here we work primarily in ZFC, though we note several cases where our results can be extended assuming the existence of large cardinals. The technique here provides a uniform method for approaching and extending theorems that Keisler et al. proved in the 1970's.

We refer the reader to [1] for model-theoretic definitions such as *Abstract Elementary Class* and for background on the notions used here. For example, Theorem 0.2 is stated for atomic models of first order theories. The equivalence between this context and models of a complete sentence in $L_{\omega_1,\omega}$ is explained in Chapter 6 of [1]. Abstract Elementary Classes form a more general context unifying many of the properties of such infinitary logics as $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$, and $L_{\omega_1,\omega}(aa)$.

A fundamental result in the study of $\aleph_1$-categoricity for Abstract Elementary Classes is the following theorem of Shelah (see [1], Theorem 17.11).

Theorem 0.1 (Shelah)**.** *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

- *The Löwenheim-Skolem number,* $\mathrm{LS}(\mathbf{K})$*, is* $\aleph_0$*;*
- $\mathbf{K}$ *is* $\aleph_0$*-categorical;*
- *amalgamation fails for countable models in* $\mathbf{K}$[1]*.*

[1] Unlike first order logic, this is a strictly stronger statement than 'amalgamation fails over subsets of models of $\mathbf{K}$.'

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$. *Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$.

Theorem 0.1 is one of the two fundamental tools to develop the stability theory of $L_{\omega_1,\omega}$. The second is the following theorem of Keisler (see [1], Theorem 18.15).

Theorem 0.2 (Keisler)**.** *Suppose that* $\mathbf{K}$ *is the class of atomic models of a complete first order theory, and that uncountably many types over the empty set are realized in some uncountable model in* $\mathbf{K}$. *Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$.

The notion of ω-stability for sentences in $L_{\omega_1,\omega}$ is a bit subtle and is more easily formulated for the associated class $\mathbf{K}$ of atomic models of a first theory. For *countable* $A \subseteq M \in \mathbf{K}$, $S_{at}(A)$ denotes the set of first order types over A realized in atomic models[2]. $\mathbf{K}$ is ω-stable if for each countable $M \in \mathbf{K}$, $|S_{at}(M)| = \aleph_0$ [3].

Combining these two theorems, Shelah showed (under the assumption $2^{\aleph_0} < 2^{\aleph_1}$) that a complete sentence of $L_{\omega_1,\omega}$ which has less that $2^{\aleph_1}$ models in $\aleph_1$ has the amalgamation property in $\aleph_0$ and is ω-stable. Crucially, Shelah's argument relies on the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in two ways. It first uses a variation of the Devlin-Shelah weak diamond principle [5] for Theorem 0.1. Then using amalgamation, extending Keisler's theorem from types over the empty set to types over a countable model is a straightforward counting argument, as it is in this paper. We work on analogs of this analysis for arbitrary AEC in Section 4.

Using the iterated ultrapower approach we give a new proof of an extension of Theorem 0.2 to the logic $L_{\omega_1,\omega}(\mathrm{aa})$ (as claimed in [19]). Again, it suffices to consider the case where amalgamation holds. Theorem 0.3 follows from Theorem 2.4 below.

Theorem 0.3. *Suppose that* $\mathbf{K}$ *is the class of models of some fixed sentence of* $L_{\omega_1,\omega}(\mathrm{aa})$, *and that, for some countable fragment* F *of* $L_{\omega_1,\omega}(\mathrm{aa})$*-sentences, uncountably many* F*-types are realized over some countable model in* $\mathbf{K}$. *Suppose also that* $2^{\aleph_0} < 2^{\aleph_1}$. *Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* $\mathbf{K}$.

We can prove a partial extension of Keisler's Theorem for more general Abstract Elementary Classes, as follows. Hypothesis (3) below corresponds to one of the cases given by Burgess's theorem for analytic equivalence relations (see [12], Theorem 9.1.5). Theorem 0.4 follows from Theorem 4.7 below.

Theorem 0.4. *Suppose that* $\mathbf{K}$ *is an Abstract Elementary Class such that*

1. *the set of reals coding countable structures in* $\mathbf{K}$ *and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$ *are both analytic (we say analytically presented);*

[2]This definition does not extend to uncountable A, see page 138 of [1]

[3]This requirement that M is a model is essential; Example 3.17 of [1], covers of the multiplicative group of $\mathbb{C}$, is ω-stable but there are countable atomic A with $|S_{at}(A)| = 2^{\aleph_0}$

2. **K** *satisfies amalgamation for countable models;*

3. *there is a countable model in* **K** *over which there is a perfect set of reals coding inequivalent Galois types.*

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$*. Then there are* $2^{\aleph_1}$ *non-isomorphic models of cardinality* $\aleph_1$ *in* **K**.

Though the approach here can very likely be applied more generally, we restrict our attention in this paper to the contexts of Theorems 0.3 and 0.4.

In Section 1 we lay out the method of iterated ultrapowers of models of set theory; Section looaasec applies this method to classes defined syntactically in various infinitary logics. Section 3 discusses the descriptive set theory of analytic equivalence relation; Section 4 adapts these methods to study 'analytically presented' AEC, extending Keisler theorem to this context. In Section 5 we address the issue of absoluteness of $\aleph_1$-categoricity for AEC. Finally Section 6 raises some further problems.

1 Iterations

The main technical tool in this paper is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by NS_{ω_1}. We are using this as a device to reproduce Keisler's constructions for expanding a countable model of set theory in such a way that sets in the original model get new members in the extension if and only if they are uncountable from the point of view of the original model. Though this will not be relevant here, we note that this these iterated embeddings and their relatives play a fundamental role in Woodin's $\mathbb{P}_{\max}$ forcing [30]. Most of this section is a condensed version of Section 1 of [25].

The iterations constructed here could be developed using the construction of carefully specified extensions of models of set theory. See [20, 15, 7] for background on these methods. We illustrate this technique in [4].

Recall that NS_{ω_1} is closed under countable unions. Moreover, Fodor's Lemma (see, for instance, [17]) says that for any stationary $A \subseteq \omega_1$, if $f\colon A \to \omega_1$ is regressive (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A$), then f is constant on a stationary set. Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U). Given such M and U, we can form the generic ultrapower $\mathrm{Ult}(M,U)$, which consists of all functions in M with domain ω_1^M, where for any two such functions f, g, and any relation R in $\{=,\in\}$, fRg in $\mathrm{Ult}(M,U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$. By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M,U)$ with its Mostowski collapse. The corresponding elementary embedding $j\colon M \to \mathrm{Ult}(M,U)$ (where each element of M is mapped to the equivalence class of its corresponding constant function on ω_1^M) has critical point (i.e., first ordinal moved) ω_1^M (see Fact 1.2 and the discussion before). We say that such an embedding is *derived by* *forcing with* $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$ *over* M. Fodor's Lemma implies that the identity function represents the ordinal ω_1^M in the ultrapower. It follows then by the definition of $\mathrm{Ult}(M, U)$ that for each $A \in \mathcal{P}(\omega_1)^M$, $A \in U$ if and only if $\omega_1^M \in j(A)$. Each ordinal $\gamma \in \omega_2^M$ is represented in $\mathrm{Ult}(M, U)$ by a function of the form $f(\alpha) = o.t.(g[\alpha])$, where $g\colon \omega_1 \to \gamma$ is a surjection (and $o.t.$ stands for "ordertype"), so the ordinals of $\mathrm{Ult}(M, U)$ always contain an isomorphic copy of ω_2^M (which is less than or equal to $j(\omega_1^M)$, since each such f has range contained in ω_1^M) as an initial segment. We call such a function f a *canonical function* for γ. While it is possible to have well-founded ultrapowers of the form $\mathrm{Ult}(M, U)$ (at least assuming the existence of large cardinals), this does not always happen (see Lemma 1.8, for instance).

Since we want to deal with structures whose existence can be proved in ZFC, we define the fragment ZFC° to be the theory ZFC − Powerset − Replacement + "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1). The theory ZFC° holds in every structure of the form $H(\kappa)$ or V_κ, where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$ (recall that $H(\kappa)$ is the collection of sets whose transitive closures have cardinality less than κ). For us, the importance of ZFC° is that it proves Fact 1.1 below, which implies that M is elementarily embedded in $\mathrm{Ult}(M, U)$ whenever M is a model of ZFC° and U is an M-ultrafilter on ω_1^M.[4] The proof of the fact is a direct application of the ω_1-Replacement-like scheme in ZFC°.

1.1 Fact (ZFC°). Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \Rightarrow \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

We let $j[x]$ denote $\{j(y) \mid y \in x\}$. One direction of Fact 1.2 below follows from the fact that every partition in M of ω_1^M into ω many pieces must have one piece in the ultrafilter U, so, if x is countable then every function from ω_1 to x in M (i.e., every representative of a member of $j(x)$) must be constant on a set in U and so must represent a member of $j[x]$). For the other direction, note that if x is uncountable then any injection from ω_1 to x represents an element of $j(x) \setminus j[x]$ in the ultrapower $\mathrm{Ult}(V, U)$.

1.2 Fact. Suppose that M is a model of ZFC°, and that $j\colon M \to \mathrm{Ult}(M, U)$ is an elementary embedding derived from forcing over M with $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then for all $x \in M$, $j(x) = j[x]$ if and only if x is countable in M.

If M is a countable model of ZFC° then there exist M-generic filters for the partial order $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Furthermore, if $j\colon M \to N$ is an ultrapower

[4] An M-ultrafilter on ω_1 is a maximal proper filter contained in $\mathcal{P}(\omega_1)^M$; in the cases we are interested in, the filter is not an element of M.

embedding of this form (where N may be ill-founded), then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable (recall that the ultrapower uses only functions from M), and there exist N-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^N$. We can continue choosing generic filters in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition.

1.3 Definition. Let M be a model of ZFC$^\circ$ and let γ be an ordinal less than or equal to ω_1. An *iteration* of M of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} \colon M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

- $M_0 = M$,
- each G_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system

  $$\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\},$$

  and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

The models M_α in Definition 1.3 are called *iterates* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j \colon M \to M^*$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \gamma, \beta < \gamma \rangle$$

of M, and that M^* is the final model of this iteration.

1.4 Remark. We emphasize that for any countable model M of ZFC$^\circ$ there are $2^{\aleph_0}$ many M-generic ultrafilters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. It follows that there are $2^{\aleph_1}$ many iterations of M of length ω_1.

1.5 Remark. As noted above, the ordinals of $\mathrm{Ult}(M, U)$ always contain an isomorphic copy of ω_2^M as an initial segment, whenever M is a countable (well-founded or illfounded) model of ZFC$^\circ$ and U is an M-normal ultrafilter. It follows from this that whenever

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \alpha \leq \delta \leq \omega_1, \beta < \omega_1 \rangle$$

is an iteration of M, $\omega_1^{M_{\omega_1}}$ contains a closed copy of ω_1 corresponding to the members of the set $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$. This set is called the *critical sequence* of the iteration.

Fact 1.6 below says that the final model of an iteration of length ω_1 is correct about uncountability. It is an immediate consequence of Fact 1.2 and the definition of iterations. This gives another proof of Corollary B on page 138 of [19]. Corollary A on page 137 can also be proved by considering ideals on other cardinals. The last sentence of Fact 1.6 follows from the remarks at the end of the second paragraph of this section. The second author observed that the absoluteness of the existence of a model in $\aleph_1$ of an arbitrary sentence is $L_{\omega_1,\omega}$ follows easily from Fact 1.6; it is shown in [6] that this argument can be carried out using Corollary A of [19].

1.6 Fact. Suppose that M is a model of ZFC$^\circ$, and that M_{ω_1} is the final model of an iteration of M of length ω_1. Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models$ "x is uncountable" if and only if $\{y \mid M_{\omega_1} \models x \in y\}$ is uncountable. Furthermore, ω_2^M is a proper initial segment of $\omega_1^{M_{\omega_1}}$.

Fact 1.7 records the fact that one can easily make M_{ω_1} correct about stationarity for subsets of its ω_1 (again, this is due to Woodin [30]). Note that the notion of stationarity makes sense for any uncountable set (so in particular, for $\omega_1^{M_{\omega_1}}$ as below, even if it is ill-founded) : $Y \subseteq [X]^{\aleph_0}$ is stationary if and only if every for every function $F\colon X^{<\omega} \to X$ there is a nonempty element of Y closed under F.

1.7 Fact. Suppose that M is a model of ZFC$^\circ$, $\{B_\xi : \xi < \omega_1\}$ is a partition of ω_1 into stationary sets and

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle \tag{1}$$

is an iteration of M of length ω_1. Suppose that for every $\alpha < \omega_1$ and every $A \in (\mathcal{P}(\omega_1) \setminus NS_{\omega_1})^{M_\alpha}$ there is a $\xi < \omega_1$ such that, for all $\beta \in \omega_1 \setminus \alpha$,

$$\beta \in B_\xi \Rightarrow j_{\alpha,\beta}(A) \in G_\beta.$$

Then for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary.

The following lemma gives a construction for building generic ultrapowers whose ω_1's are illfounded, though, as remarked above, they must be well-founded up to at least the ω_2 of the ground model. Given a function $f\colon \omega_1 \to \omega_1$, we let I_f be the normal ideal on ω_1 generated by sets of the form

$$\{\beta < \omega_1 \mid g(\beta) \geq f(\beta)\},$$

where g is a canonical function for an ordinal less than ω_2. Whenever $\gamma < \gamma' < \omega_2$, g is a canonical function for γ and γ' is a canonical function for γ', it follows that $\{\beta < \omega_1 \mid g(\beta) < g'(\beta)\}$ contains a club. It follows (using the regularity of ω_2) that for each $S \in \mathcal{P}(\omega_1)$, $S \in I_f$ if and only if $\{\beta \in S \mid f(\beta) \geq g(\beta)\}$ is nonstationary for some canonical function g for an element of ω_2. If $\langle \sigma_\beta : \beta < \omega_1\rangle$ is a $\diamondsuit$-sequence and $\pi\colon \omega_1 \to \omega_1 \times \omega_1$ is a bijection, then $\omega_1 \notin I_f$, where

$f\colon \omega_1 \to \omega_1$ is the function defined by letting $h(\beta)$ be $o.t.(\pi[\beta]) + 1$ whenever $\pi[\beta]$ is a wellordering (and 0 otherwise). We note that $\diamond$ is forced by the partial order which adds a subset of ω_1 by countable initial segments, and that this partial order does not add subsets of ω. Some hypothesis beyond ZFC$^\circ$ is needed for Lemma 1.8, as it is false for models in which the nonstationary ideal is saturated.

Lemma 1.8. *Suppose that M is a countable transitive model of* ZFC$^\circ$*, and that $f^*\colon \omega_1^M \to \omega_1^M$ is a function in M such that $\omega_1 \notin I_{f^*}$. Then there is an M-normal ultrafilter U such that the well-founded ordinals of* $\mathrm{Ult}(M, U)$ *are exactly ω_2^M.*

Proof. Applying the usual construction of an M-normal ultrafilter, it suffices to show that if

- S is a subset of ω_1^M in M,
- $f\colon S \to \omega_1^M$,
- $S \notin I_f$,
- $\{T_\alpha : \alpha \in \omega_1^M\}$ is a collection of stationary subsets of S in ω_1 whose diagonal union is S,

then there exist $\alpha < \omega_1^M$, $S' \subseteq T_\alpha$ and $f'\colon S' \to \omega_1$ in M such that

- for all $\beta \in S'$, $f'(\beta) < f(\beta)$,
- $S' \notin I_{f'}$.

This implication gives a recipe for building an M-normal filter with the property that every function in M from ω_1^M to the ordinals either represents an ordinal below ω_2^M or dominates on a set in the filter another function which does not represent an ordinal below ω_2^M. The recipe uses an enumeration $\{h_n : n \in \omega\}$ of $(\omega_1^{\omega_1})^M$. In each step, starting with $f = f^*$ and $S = \omega_1$, it applies the implication above to $\min\{f, h_n\}$ (for the next n, considered in order) if $S \notin I_{\min f, h_n}$, and to f otherwise.

To see that the implication holds, fix f and S as given. Since I_f is normal and $S \notin I_f$, there is an α such that $S \cap T_\alpha \notin I_f$. Let S_0 be the set of $\beta \in S \cap T_\alpha$ for which $f(\beta)$ is a successor ordinal. If S_0 is not in I_f, then let $S' = S_0$ and let $f'(\beta) = f(\beta) - 1$ for $\beta \in S'$. Then since adding 1 to the values of any canonical function for any $\gamma < \omega_2$ gives a canonical function for $\gamma + 1$, we have that $S' \notin I_{f'}$.

If $S_0 \in I_f$, there is an I_f-positive $S_1 \subseteq S \cap T_\alpha$ such that $f(\beta)$ is a limit ordinal for all $\beta \in S_1$. Let $f_n\colon S_1 \to \omega_1$ $(n \in \omega)$ be functions such that for each $\beta \in S_1$, $\langle f_n(\beta) : n < \omega\rangle$ is an increasing sequence with supremum $f(\beta)$. It suffices to see that $S_1 \notin I_{f_n}$ for some $n \in \omega$. Supposing towards a contradiction that $S_1 \in I_{f_n}$ for each $n \in \omega$, fix, for each n a canonical function g_n (for some ordinal $\gamma_n < \omega_2^M$) such that $\{\beta \in S_1 \mid f_n(\beta) \geq g_n(\beta)\}$ is nonstationary. Let γ

be an element of ω_2^M greater than all the g_n's, and fix a canonical function g for γ. Then for each $n \in \omega$ the set $\{\beta \in S_1 \mid f_n(\beta) > g(\beta)\}$ is nonstationary, which means that the set $\{\beta \in S_1 \mid f(\beta) > g(\beta)\}$ is nonstationary, which means that $S_1 \in I_f$, giving a contradiction. $\square$

The following consequence of large cardinals (due to Woodin, but see [8]) will be used in Remark 4.9 and Section 5. Given an ordinal δ, $Col(\omega_1, {<}\delta)$ is the partial order which consists of countable partial functions $f\colon \omega_1 \times \delta \to \delta$, with the stipulation that $f(\alpha, \beta) < \beta$ for all $(\alpha, \beta) \in \mathrm{dom}(f)$, ordered by inclusion. This partial order preserves stationary subsets of ω_1 and does not add countable sets of ordinals. If δ is a regular cardinal, then δ is the ω_2 of any forcing extension by $Col(\omega_1, {<}\delta)$.

A model whose iterates are all well-founded is said to be *iterable*.

Theorem 1.9. *Suppose that κ is a regular cardinal, $\lambda < \kappa$ is a measurable cardinal and $\delta < \lambda$ is a Woodin cardinal. Let X be a countable elementary submodel of either V_κ or $H(\kappa)$, with δ and λ in X. Let M be the transitive collapse of X, and let $\bar{\delta}$ be the image of δ under this collapse. Let $g \subseteq Col(\omega_1, {<}\bar{\delta})$ be an M-generic filter. Then $M[g]$ is iterable.*

2 $L_{\omega_1,\omega}(\mathrm{aa})$

Briefly, the logic $L_{\omega_1,\omega}$ is the extension of first order logic where one allows conjunctions and disjunctions of countable sets of formulas so that only finitely many free variables appear in the union of the set of formulas. Each formula in $L_{\omega_1,\omega}$ has a *rank*, the number (less than ω_1) of steps it takes to construct the formula from atomic formulas (see the appendix to [2]). More explicitly, we may think of sentences of $L_{\omega_1,\omega}$ as well-founded trees of height of at most ω; then the rank of a sentence is just the rank of the corresponding tree in the sense of Section 3. An ill-founded model of ZFC$^\circ$ can contain objects which it thinks are sentences of $L_{\omega_1,\omega}$ which are really not, i.e., if the rank of the sentence as computed in the model is an ill-founded ordinal of the model. On the other hand, if a (real) sentence ϕ of $L_{\omega_1,\omega}$ exists in an ω-model M of ZFC$^\circ$, then M computes the rank correctly, and is therefore well-founded at least up the rank of ϕ. Furthermore, M correctly verifies whether the models that it sees satisfy ϕ. In both cases, the computation of the rank and the verification of the truth value, M runs exactly the same process that is carried out in V.

The logic $L_{\omega_1,\omega}(\mathrm{aa})$ extends $L_{\omega_1,\omega}$ by adding the quantifier aa, where $\mathrm{aa}x \in [X]^{\aleph_0}\,\phi$ means "for stationarily many countable $x \subseteq X$, ϕ holds", i.e., for any function $f\colon X^{<\omega} \to X$, there is a countable $x \subseteq X$ closed under f such that x satisfies ϕ. Note that "there exist uncountably many $x \in X$ such that ϕ holds" can be expressed using aa. Note also that if M is a model of ZFC$^\circ$ as in conclusion of Fact 1.7, i.e., such that for all $A \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$, $M_{\omega_1} \models$ "A is stationary" if and only if A is stationary, then if X is a set in M of cardinality $\aleph_1$ (in M) and Y is a subset of $[X]^{\aleph_0}$ in M, then $M_{\omega_1} \models$ "Y is stationary" if and only if Y is stationary

The second parts of the equivalences in the following theorems are $\underset{\sim}{\Sigma}^1_1$, and therefore absolute. The forward directions simply involve taking the transitive collapse of a countable elementary submodel of suitable initial segment of the universe. The reverse directions involve building iterations as in the previous section (using Fact 1.7 for correctness about stationarity). Since the final models of these iterations are well-founded up to at least the ω_2 of the corresponding original models, they verify correctly truth for ϕ and for members of the set F for the models that they see.

Theorem 2.1. *Given a sentence ϕ of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of ϕ of size $\aleph_1$ is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $\{\phi,\omega\}$ which thinks there is a model of ϕ of size $\aleph_1$.*

Theorem 2.2. *Given a countable fragment F of $L_{\omega_1,\omega}(\mathrm{aa})$, the existence of a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types is equivalent to the existence of a countable model of* ZFC$^\circ$ *containing $F\cup\{F,\omega\}$ which thinks there is a model of size $\aleph_1$ satisfying $\aleph_1$-many F-types.*

We prove in Theorem 2.4 below that the second part of the equivalence in the previous theorem implies that there are $2^{\aleph_1}$ many models of size $\aleph_1$, pairwise satisfying only countably many F-types in common. First we present an easier argument for getting $\aleph_1$ many such models.

Suppose that M is an ω-model of ZFC$^\circ$ and $\bar{x} = \langle x_\alpha : \alpha < \omega_1^M\rangle$ is a sequence of distinct subsets of ω in M. Then given any iteration of M as above, $\bar{x}$ will be an initial segment of $j_{0,\omega_1}(\bar{x}) = \langle x_\alpha : \alpha < \omega_1^{M_{\omega_1}}\rangle$, and $x_\alpha \notin M_\beta$ whenever $\alpha \geq \omega_1^{M_\beta}$ (by the remarks before Fact 1.2).

Furthermore, if A is any countable set of reals not in M, one can easily build an iteration of M such that $A \cap M_{\omega_1} = \emptyset$. Now let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$, and let M be a ω-model of ZFC$^\circ$ in which F is countable, which thinks there exists a model N of size $\aleph_1$ realizing uncountably many F-types. Then there are uncountably many iterations $\{j^\xi : \xi < \omega_1\}$ of M producing models $\{M^\xi_{\omega_1} : \xi < \omega_1\}$ such that the models $M^\xi_{\omega_1}$ pairwise have only the reals from M in common, and thus the models $j^\xi(N)$ pairwise realize just countably many F-types in common.

To get $2^{\aleph_1}$ many uncountable iterates pairwise having just countably many reals in common, we use Theorem 2.3 below. Note that one can force $\mathrm{MA}_{\aleph_1}$ (the restriction of Martin's Axiom which asserts the existence of a filter meeting any $\aleph_1$ many maximal antichains from a c.c.c. partial order) to hold over any countable model of ZFC$^\circ$. By "distinct iterations" we mean literally iterations that are not the same set, formally speaking. In particular, this means (using the notation from Theorem 2.3) that there is some β such that $G_\beta \neq G'_\beta$. When β is minimal with this property, $M_\beta = M'_\beta$ and there is a set $A \in \mathcal{P}(\omega_1)^{M_\beta}$ such that $A \in G_\beta$ and $\omega_1^{M_\beta} \setminus A \in G'_\beta$, since G_β and G'_β are distinct M_β-ultrafilters.

Theorem 2.3 (Larson [24])**.** *If M is a countable model of* ZFC$^\circ$ + $\mathrm{MA}_{\aleph_1}$ *and*

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle$$

and

$$\langle M'_\alpha, G'_\beta, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \beta < \omega_1\rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} = \mathcal{P}(\omega)^{M_\beta},$$

where β is least such that $G_\beta \neq G'_\beta$.

For the reader's convenience, we sketch the proof of the version of Theorem 2.3 for iterations of length 1 (which appears in [10]). Suppose that M is a countable model of $\mathrm{ZFC}^\circ + \mathrm{MA}_{\aleph_1}$ and let G and G' be two distinct M-generic filters for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^M$. Then there exist disjoint sets A, A' in $(\mathcal{P}(\omega_1 \setminus NS_{\omega_1})^M$ such that $A \in G$ and $A' \in G'$. Let $N = \mathrm{Ult}(M, G)$ and $N' = \mathrm{Ult}(M, G')$, and fix $x \in \mathcal{P}(\omega)^N \setminus M$ and $x' \in \mathcal{P}(\omega)^{N'} \setminus M$. Then there exist functions $f\colon A \to \mathcal{P}(\omega)^M$ and $f'\colon A' \to \mathcal{P}(\omega)^M$ representing x in N and x' in N' respectively. Applying Fodor's Lemma we see that, since x and x' are not in M, there exist $B \subseteq A$ and $B' \subseteq A'$ in G and G' respectively on which f and f' (respectively) are injective. Applying Fodor's Lemma again we can thin B and B' to sets C and C' on which the ranges of f and f' are disjoint and contain only infinite, co-infinite sets, by subtracting nonstationary sets. Finally, it is a consequence of $\mathrm{MA}_{\aleph_1}$ (see [18], for instance) that for any two disjoint sets of infinite, co-infinite subsets of ω, there is a subset of ω which intersects each member of the first set infinitely, and no member of the second set infinitely. Thus if M satisfies $\mathrm{MA}_{\aleph_1}$ there is such a $z \subseteq \omega$ in M with respect to the ranges of $f \restriction C$ and $f \restriction C'$, which means that $x \cap z$ is infinite and $x' \cap z$ is not.

Using this, one gets the following version of Keisler's theorem (see Fact 18.15 of [1]), for $L_{\omega_1,\omega}(\mathrm{aa})$.

Theorem 2.4. *Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.*

Proof. Let N be a model of cardinality $\aleph_1$ realizing uncountably many F-types, let X be a countable elementary submodel of $H((2^{2^{\aleph_1}})^+)$ containing $\{N\}$ and the transitive closure of $\{F\}$. Let M be the transitive collapse of X, and let N_0 be the image of N under this collapse. Let M' be a c.c.c. forcing extension of M satisfying Martin's Axiom. By choosing a pair of distinct generic ultrafilters for each model we can build a tree of iterates of M' giving rise to $2^{\aleph_1}$ many distinct iterations of M' of length ω_1 (as in Remark 1.4). Since F-types can be coded by reals using an enumeration of F in M, the images of N_0 under these iterations will pairwise realize just countably many F-types in common, by Theorem 2.3. $\square$

If one assumes in addition that $2^{\aleph_0} < 2^{\aleph_1}$, then, as in Theorem 18.16 of [1], one gets that if there exists a model of cardinality $\aleph_1$ realizing uncountably many types over some countable subset, then there exists a $2^{\aleph_1}$-sized family of nonisomorphic models. That is, if there is an uncountable model N with a countable subset A over which uncountably many types are realized, then there are models N_f $(f \in 2^{\aleph_1})$ all containing the same countable set A and all realizing different sets of types over A, so that any isomorphisms of any two N_{f_1} and N_{f_2} into a third N_{f_3} must map A pointwise to different sets (which is impossible if $2^{\aleph_1} > 2^{\aleph_0}$).

We conclude this section by showing that a strengthening of Lemma 5.1.8 of [1] can be proved using Lemma 1.8.

Lemma 2.5. *Suppose that ϕ is a sentence of $L_{\omega_1,\omega}(\mathrm{aa})$ in a language with a binary predicate $<$, and suppose that there is a model M of ϕ for which the ordertype of $(M,<)$ is ω_1. Then there is a model M' of ϕ of cardinality $\aleph_1$ such that $(M',<)$ embeds $\mathbb{Q}$. Furthermore, if θ is a regular cardinal greater than $2^{2^{\aleph_1}}$, then M' can be taken to be an element of a model N of* ZFC$^\circ$ *such that $(M',<)$ is isomorphic to ω_1^N and M' satisfies every formula in N that M does in $H(\theta)$.*

Proof. Let θ' be a regular cardinal greater than θ and let X be a countable elementary submodel of $H(\theta')$ with $\theta, M \in X$. Let N_0 be the transitive collapse of $X \cap H(\theta)$, let N_1 be the transitive collapse of X and let M_0 be the image of M under these collapses (it is the same under each). Let N_2 be a forcing extension of N_1 (with the same reals) satisfying $\diamondsuit$. Applying Lemma 1.8 (for the first step of the iteration) and Fact 1.7 (for the rest), we can find an iteration $j\colon N_2 \to N$ of length ω_1 such that that well-founded ordinals of N are exactly $\omega_2^{N_2}$. Letting M' be the image of M_0 under this iteration, we have that $(M',<)$ is isomorphic to ω_1^N, which embeds $\mathbb{Q}$ as it is illfounded. Furthermore, $M' \in j(N_0)$, M' satisfies the same sentences in $j(N_0)$ that M does in $H(\theta)$ and $\omega_1^{j(N_0)} = \omega_1^N$. $\square$

3 Analytic equivalence relations

In this section we prove two lemmas about analytic equivalence relations on the reals in ω-models of set theory. The second of these, Lemma 3.3, will be applied in the next section to an equivalence relation corresponding to the notion of Galois type. After writing this section we noticed that for our purposes one could replace Lemma 3.3 with the classical fact that every partial function from ω^ω to itself with analytic graph has a Borel extension. While we have not found this fact stated in the literature, it is an easy consequence of the First Separation Theorem, via the argument for Theorem 4.5.2 in [29] or Exercise 35.13 of [22]. We retain our original argument for completeness, and note at the end of this section how one might use the Borel extension fact instead.

In this paper, a *tree* is a set of finite sequences closed under initial segments. If $T \subseteq X^{<\omega}$ is a tree, for some set X, then $[T]$ is the set of $x \in X^\omega$ such that $x \restriction n \in T$ for all $n \in \omega$. If $T \subseteq (X \times Y)^{<\omega}$, for some sets X and Y, then the *projection* of T, $p[T]$ is the set of $f \in X^\omega$ such that for some $g \in Y^\omega$, $(f,g) \in [T]$ (this definition involves a standard identification of pairs of sequences with sequences of pairs). For any positive $n \in \omega$, a subset of $(\omega^\omega)^n$ is *analytic* if it has the form $p[T]$ for some tree $T \subseteq (\omega^n \times \omega)^{<\omega}$.

Recall that for a tree $T \subseteq X^{<\omega}$ for some set X, the ranking function $rank_T \colon T \to Ord \cup \{\infty\}$ is defined in such a way that for all $t \in T$, $rank_T(t)$ is the smallest ordinal α such that $\alpha > rank_T(s)$ for all proper extensions s of t in T, and $rank_T(t) = \infty$ if no such α exists (which happens if and only if $rank_T(s) = \infty$ for some proper extension s of t). We write $rank(T)$ for $rank_T(\langle\rangle)$. Then $rank(T) = \infty$ if and only if T has an infinite branch.

Now suppose that M is an ω-model of ZFC$^\circ$, and $T \subseteq X^{<\omega}$ is a tree in M, for some X in M. If $rank(T)^M = \infty$, then there is an infinite branch through T in M. If $rank(T)^M$ is in the well-founded part of M, then there is no infinite branch through T (in V). It follows easily from the definition of $rank(T)$ that if $rank(T)^M$ is an ill-founded ordinal of M, then T has an infinite branch in V but no infinite branch in M. This happens, for instance, in the case where t is an illfounded ordinal of M and T is the tree of descending sequences from t.

Given sets X, Y, a tree $T \subseteq (X \times Y)^{<\omega}$ and $s^* \in X^{<\omega}$, T_{s^*} is the set of $(s, t) \in T$ such that s is compatible with s^* (i.e., one of them extends the other).

Lemma 3.1. *Suppose that M is a (possibly ill-founded) ω-model of* ZFC$^\circ$*, and that $T \subseteq (X \times Y)^{<\omega}$ is a tree in M, for some sets X and Y. Suppose that x is the unique element of $p[T]$. Then $x \in M$.*

Proof. Since $p[T]$ is nonempty, $rank(T)^M$ cannot be in the well-founded part of M. If $rank(T)^M = \infty$, then $[T] \cap M$ is nonempty, which means that $p[T] \cap M$ is nonempty. Suppose then that $rank(T)^M$ is an ill-founded ordinal of M. Then, starting with with $\langle\rangle$, M can find all the initial segments of x by the following process. Suppose that $s \in X^{<\omega}$ is an initial segment of x. Then $rank(T_s)^M$ is an ill-founded ordinal of M. Since s is an initial segment of the unique element of $p[T]$, the unique integer n such that $s^\frown\langle n\rangle$ is an initial segment of x is also the unique integer n such that

$$\sup\{rank_T^M(s^\frown\langle n\rangle, t) : (s^\frown\langle n\rangle, t) \in T\}$$

is greater than

$$\sup\{rank_T^M(s^\frown\langle m\rangle, t) : (s^\frown\langle m\rangle, t) \in T\}$$

for all $m \in \omega \setminus \{n\}$, since the former set contains ill-founded ordinals of M and the latter contains only well-founded ordinals. $\square$

The proof just given cannot in general give an element of $[T]$ in M. Consider, for instance a tree of the form $\{(x{\restriction}n, t) : n \in \omega, |t| = n, t \in T\}$, for x an element of $\omega^\omega \cap M$ and T a tree in M whose rank is an illfounded ordinal of M.

The proof of Lemma 3.1 gives the following result, which can also be used to prove Lemma 3.3.

Theorem 3.2. *If $f \colon \omega^\omega \to \omega^\omega$ is partial function which is analytic as a subset of $\omega^\omega \times \omega^\omega$, then f extends to a Borel partial function $f' \colon \omega^\omega \to \omega^\omega$.*

Proof. Let $T \subseteq (\omega \times \omega \times \omega)^{<\omega}$ be a tree projecting to the graph of f. For each $y \in \omega^\omega$, let T_y be the tree consisting of those pairs (a, c) for which $(a, y{\restriction}|a|, c) \in T$. Then T_y exists in any model of ZFC$^\circ$ containing T and y, and, it projects to

$\{f(y)\}$ if y is in the domain of f, and to $\emptyset$ otherwise. The corresponding search for x (i.e., $f(y)$) outlined in the proof of Lemma 3.1 (using T_y in place of T, and in each step finding the unique n such that $\sup\{rank^M_{T_y}(s^\frown\langle n\rangle, t) : (s^\frown\langle n\rangle, t) \in T_y\}$ is greater than $\sup\{rank^M_{T_y}(s^\frown\langle m\rangle, t) : (s^\frown\langle m\rangle, t) \in T_y\}$ for all $m \in \omega \setminus \{n\}$, if such an n exists) returns the same value x in any model M containing y and T if it returns a value in any such model. The set of y for which a value x is returned is then analytic and co-analytic, and thus Borel, and the corresponding function is likewise Borel. □

In the proof above, domain of f' may include some y's not in the domain of f, i.e., where an x not in $p[T_y]$ is found. In these cases the values $rank^M_{T_y}(s)$ in the construction from Lemma 3.1 are all necessarily well-founded.

Now suppose that E is an analytic equivalence relation on an analytic set $X \subseteq \omega^\omega$. By the Burgess Trichotomy Theorem (Theorem 9.1.5 of [12]), either E has at most $\aleph_1$ many equivalence classes, or there is a perfect set P consisting of E-inequivalent members of X. The following lemma shows that in this second case, if M is an ω-model containing codes for E and P, and $x \in \omega^\omega \cap M$ is E-equivalent to a member of P, then this member of P is also in M. The lemma follows from Lemma 3.1 plus the fact that the set of members of P which are E-equivalent to x is an analytic set with a unique member.

Lemma 3.3. *Suppose that M is a (possibly ill-founded) ω-model of* ZFC°*, and E is an analytic equivalence relation on ω^ω which is the projection of a tree T on $\omega \times \omega \times \omega$ in M. Suppose that P is a perfect set of E-inequivalent members of ω^ω such that $P = [S]$ for a tree $S \subseteq \omega^{<\omega}$ in M. Let $x \in M \cap \omega^\omega$ be such that xEy for some $y \in P$. Then $y \in M$.*

As noted above, the classical fact that every partial function from ω^ω to itself with analytic graph has a Borel extension can be used in place of Lemma 3.3 in the next section. We briefly sketch the argument for this. Suppose that E is an analytic equivalence relation on ω^ω, and A is an analytic set of E-inequivalent reals. Then the set of pairs (x, y) from ω^ω for which xEy and $y \in A$ is a partial function with analytic graph. Let f be a Borel extension of this function. Then if M is an ω-model of ZFC° containing a suitable code for f, and $x \in \omega^\omega \cap M$ is E-equivalent to a member of A, then $f(x)$ is this member, and $f(x) \in M$.

4 Abstract Elementary Classes

In this section we work with an abstract elementary class $\mathbf{K}$ in a countable vocabulary τ with Löwenheim number $\aleph_0$. To study the countable members of $\mathbf{K}$ in descriptive set theoretic structure we regard them as collections of relations (indexed by τ) on ω. The class of countable structures is a Polish spaces and, for any given $L_{\omega_1,\omega}$ sentence ϕ, the set of models of ϕ is Borel.

We study classes satisfying the following additional condition.

Definition 4.1. *An abstract elementary class* $\mathbf{K}$ *is* analytically presented *if the set of countable models in* $\mathbf{K}$*, and the corresponding strong submodel relation* $\prec_{\mathbf{K}}$*, are both analytic.*

This requirement is not as *ad hoc* as it might seem. Shelah's presentation theorem (Theorem 4.15 of [1]) asserts that any AEC of τ-structures with countable Löwenheim-Skolem can be presented as the reducts to τ of models of a first order theory in a countable language τ' which omit a family of at most $2^{\aleph_0}$-types. If the collection of omitted types is countable, in [**?**] we called these $PC\Gamma(\aleph_0, \aleph_0)$ classes[5]. Sentences in $L_{\omega_1,\omega}(Q)$ are $PC\Gamma(\aleph_0, \aleph_0)$. To say an AEC is a $PC\Gamma(\aleph_0, \aleph_0)$ class implies as well that the strong substructure relation is $PC\Gamma(\aleph_0, \aleph_0)$. We now note that 'analytically presented' is another *nom de plume* for such a class.

Note first than any $PC\Gamma(\aleph_0, \aleph_0)$-presented AEC is analytically presented, as omission of a countable family of types in τ' is Borel, and taking the reduct to τ makes the class of countable models analytic.

The assumption that $\mathbf{K}$ is analytically presented implies that the countable models of $\mathbf{K}$ are the countable models of a $PC\Gamma(\aleph_0, \aleph_0)$ class. (Just add a predicate for the natural numbers, symbols for the arithmetic operations, and omit the type of a non-standard natural number; then the Σ_1 definition of the members of $\mathbf{K}$ gives an extended language τ' and a first order in it.) But any AEC with Löwenheim number $\aleph_0$ that is $PC\Gamma(\aleph_0, \aleph_0)$ on its countable models is $PC\Gamma(\aleph_0, \aleph_0)$ witnessed by the same theory and collection of types. For, if $\mathbf{K}_{\aleph_0}$ is the class of countable reducts of a τ'-theory T which omit a collection Γ' of types, each uncountable τ'-model of T is a direct limit of finitely generated submodels and the reduct to τ of each such finitely generated submodel is in $\mathbf{K}$.

Following [1] we define for $\mathbf{K}$ a reflexive and symmetric relation $\sim_0$ on the set of triples of the form (M, a, N), where M and N are countable structures in $\mathbf{K}$ with $M \prec_{\mathbf{K}} N$, and $a \in N \setminus M$. We say that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$ if $M_0 = M_1$ and there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 \colon N_0 \to N$ and $f_1 \colon N_1 \to N$ such that $f_0 \restriction M_0 = f_1 \restriction M_1$ and $f_0(a_0) = f_1(a_1)$. We let $\sim$ be the transitive closure of $\sim_0$. The equivalence classes of $\sim$ are called *Galois types*.

If an abstract elementary class is given syntactically the Galois types over a countable M refine the syntactic types and in general there may be more Galois types than syntactic types (e.g. [3]).

There is a natural coding of triples (M, a, N) as above by elements of ω^ω, and for analytically presented AEC the set B consisting of those $x \in \omega^\omega$ coding such a triple is an analytic set. We let E be the equivalence relation on B where xEy if and only if x and y code (respectively) triples (M_0, a_0, N_0) and (M_1, a_1, N_1) such that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$. Then E is analytic. Furthermore, if we let E be the equivalence relation on B such that xEy if and only if there exist $\sim$-equivalent triples coded by x and y, then E is analytic. For a given model

[5]Shelah writes $PC_{\aleph_0}$ or $PC(\aleph_0, \aleph_0)$, suppressing the type omission, and Keisler writes PC_δ over $L_{\omega_1,\omega}$ for this notion. Technically, Keisler describes only the class of models, not the AEC.

M, we let E_M be the equivalence relation E restricted to the set B_M consisting of codes for triples whose first element is M. Then E_M is also analytic. Note, however, that a real codes a countable structure only up to isomorphism, and the established definition of Galois type requires that the first coordinates of the triples be literally the same. Given a real $x \in B$ and a $\mathbf{K}$ structure N^*, we say that N^* *realizes the Galois types coded by* x if there is a triple (M, a, N) coded by x such that $N \prec_{\mathbf{K}} N^*$. If in addition N_0 is countable and $N_0 \prec N^*$, we say that N^* *realizes the Galois type coded by* x *over* M_0 if there is a triple (M_0, a, N) coded by x such that $N \prec_{\mathbf{K}} N^*$.

By Burgess's Trichotomy, for each such M there are either at most $\aleph_1$ many E_M-equivalence classes, or a perfect set of E_M-inequivalent reals. For the syntactic types discussed in the earlier sections the intermediate possibility of $\aleph_1$-types without there being a perfect set of types is impossible, as for each countable fragment of $(L_{\omega_1,\omega},\ L_{\omega_1,\omega}(Q),\ L_{\omega_1,\omega}(aa))$ the set of types is Borel (See 4.4.13 in [26].) Note that this intermediate possibility is obscured in the presence of the CH if this notion is described in terms of the number of classes.

But even for analytically presented AEC all three parts of the trichotomy can occur (see Example 4.4 below) and Theorem 0.2 does not generalize in full. Following [28], we use the following definitions.

4.2 Definition. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be *Galois ω-stable* if for every countable $M \in \mathbf{K}$, E_M has countably many equivalence classes, and *almost Galois ω-stable* if for each countable $M \in \mathbf{K}$, E_M does not have a perfect set of equivalence classes.[6]

The analog for Galois types of the first order theorem that ω-stability implies stability in all powers fails except under very restrictive conditions. Baldwin and Kolesnikov [3] exhibit complete sentences that are ω-Galois stable but not Galois stable in $\aleph_1$.

4.3 Example. Consider the abstract elementary class $(\mathbf{K}, \prec)$ where $\mathbf{K}$ is the class of well-order types of length $\leq \omega_1$ and $\prec$ is initial segment. $(\mathbf{K}, \prec)$ has amalgamation and joint embedding in $\aleph_0$, is almost Galois ω-stable, but not Galois ω-stable despite being $\aleph_1$-categorical.

In view of Example 4.3, there is no hope of a direct generalization of Theorem 0.2 to arbitrary Abstract Elementary Classes. The existence of almost Galois ω-stable but not Galois ω-stable classes is one obstruction. This example seems extreme as there are no models beyond $\aleph_1$ and no nice syntactic description of the class. In particular it is not analytically presented. But, we can find apparently more tractable examples of almost ω-Galois stability.

A linear order L is 1-transitive (equivalently, groupable, i.e admits a compatible group structure) if for any a, b in L, there is an automorphism of L taking a to b. The class of groupable linear orders has exactly $\aleph_1$ countable models.

[6] We make the definition this way to avoid the awkwardness that if almost Galois ω-stable is defined as having only $\aleph_1$ classes, then under CH every AEC is almost Galois ω-stable. It is not clear to us which notion is more natural for larger κ.

(See Corollary 8.6 of [27].) The following example is a variant by Jarden of a somewhat less natural version in Chapter 1 of [28].

4.4 Example. Let $(\mathbf{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended. Since there are only $\aleph_1$-isomorphism types of components this class is almost Galois ω-stable. This AEC is analytically presented and definable as a reduct of a class in $L(Q)$. But it has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.

We sketch an argument (told to us by Kesälä) that implies every almost ω-Galois stable sentence of $L_{\omega_1,\omega}$ with the amalgamation property and jep is ω-Galois stable. Hyttinen and Kesälä introduced the important notions: finite character and weak Galois type. An AEC $\mathbf{K}$ has *finite character* if for $M \subseteq N$ with $M, N \in \mathbf{K}$: if for every finite $\boldsymbol{a} \in M$ there is a $\mathbf{K}$-embedding of M into N fixing $\boldsymbol{a}$, then $M \prec_{\mathbf{K}} N$. The key point is that any sentence of $L_{\omega_1,\omega}$ has finite character and any such AEC is very close to $L_{\omega_1,\omega}$. Generally speaking, sentences of $L_{\omega_1,\omega}(Q)$ do not have finite character. Two points have the same *weak Galois type* over a model M if they have the same Galois type over every finite subset of M.

It follows easily from work of Kueker [23] and Hyttinen-Kesälä [14] that *for countable models* of an AEC with finite character satisfying the amalgamation and joint embedding properties, almost Galois ω-stability implies Galois ω-stability. Here is the argument. Hyttinen and Kesala call an AEC satisfying these conditions *weakly Galois ω-stable* if there are only countably many weak types over each countable model. For such classes, Hyttinen and Kesala show, if two elements have the same weak Galois type over a *countable* model M they have the same Galois type over M. Kueker proves (Corollary 4.9 of [23]) that for finitary AEC (with ap) points a and b have the same weak-Galois type over a countable model M if and only if $\mathrm{tp}_{\infty,\omega}(a/M) = \mathrm{tp}_{\infty,\omega}(a/M)$[7]. Thus for countable models of such sentences, syntactic ω-stability implies Galois ω-stability. Since we noted above that almost Galois ω-stability implies syntactic ω-stability (If there were a model M with uncountably many syntactic types, it would have a perfect set of syntactic types and thus there would be a perfect set of Galois types over M.), we get the following.

4.5 Fact. If a sentence in $L_{\omega_1,\omega}$-sentence, satisfying amalgamation and joint embedding, is almost Galois ω-stable then it is Galois ω-stable.

Baldwin, Larson, and Shelah [4] have shown a related fact, which we apply below:

Theorem 4.6. *If an almost Galois ω-stable $PC\Gamma(\aleph_0, \aleph_0)$ class satisfying amalgamation* has only countably many models in $\aleph_1$*, then it is Galois ω-stable.*

We deal here with the case that there is a perfect set of E_M-inequivalent reals, for some M (i.e., the case where almost Galois ω-stability fails). This

[7] Note this type is evaluated in a fixed Galois-saturated monster model.

perfect set plays roughly the role that the syntactic types played in Theorem 2.4. Since a Galois type is not a real but a set of reals, we cannot reproduce the same argument from an uncountable set of Galois types, but rather use this perfect set to identify a sufficiently large set of Galois types with reals.

In the following generalization of Keisler's Theorem 2.4, we do not assume that $\mathbf{K}$ satisfies amalgamation or the joint embedding property. Note that this gives a uniform proof of the results for various logics. However, one would typically use amalgamation to obtain hypothesis (4) of the theorem.

Theorem 4.7. *Suppose that*

1. $\mathbf{K}$ *is an analytically presented abstract elementary class;*
2. N *is a* $\mathbf{K}$*-structure of cardinality* $\aleph_1$*, and* N_0 *is a countable structure with* $N_0 \prec_{\mathbf{K}} N$*;*
3. P *is a perfect set of* E_{N_0}*-inequivalent members of* ω^ω*;*
4. N *realizes the Galois types of uncountably many members of* P *over* N_0*.*

Then there exists a family $\{N^\alpha : \alpha \in 2^{\aleph_1}\}$ *of* $\mathbf{K}$*-structures of cardinality* $\aleph_1$ *such that*

- *for each* $\alpha \in 2^{\aleph_1}$*,* $N_0 \prec_{\mathbf{K}} N^\alpha$*;*
- *for each* $\alpha \in 2^{\aleph_1}$*,* N^α *realizes the Galois types of uncountably many members of* P *over* N_0*.*
- *for each distinct pair* α*,* α' *from* $2^{\aleph_1}$*, the set of* $x \in P$ *for which both* N^α *and* $N^{\alpha'}$ *realize the Galois type of* x *over* N_0 *is countable.*

Proof. Fix a regular $\kappa > 2^{2^{\aleph_1}}$, and let Y be a countable elementary submodel of $H(\kappa)$ with $\mathbf{K} \cap H(\aleph_1)$, N_0, N and P in Y. Let M^* be the transitive collapse of Y, and let N^* be the image of N under this transitive collapse. There is a tree $S \subseteq \omega^{<\omega}$ in M^* such that $P = [S]$. Let M_0 be a forcing extension of M^* satisfying $\mathrm{MA}_{\aleph_1}$. Let X be the set of reals of $M_0 \cap P$ coding triples which are $\sim$-equivalent to triples (N_0, a, N') with $N' \prec_{\mathbf{K}} N^*$. Then $X \in M_0$, since M_0 is well-founded and thus computes Σ^1_1-truth correctly, and X is uncountable in M_0 by elementarity. By Theorem 2.3, there are $2^{\aleph_1}$ many iterates $\{M^\alpha : \alpha \in 2^{\aleph_1}\}$ of M_0 pairwise having just countably many reals in common. Let M^α be such an iterate via an iteration j^α, and let N^α be the corresponding image of N^*. Then in M^α, N^α realizes the Galois types of uncountably many members of $j(P)$ over N_0. Since $j(P) = [j^\alpha(S)]^{M^\alpha} = [S]^{M^\alpha} = [S] \cap M^\alpha$, $j(P) \subseteq P$, so N^α realizes the Galois types of uncountably many members of P over N_0. For each countable $N' \prec_{\mathbf{K}} N^\alpha$, there is a countable $N'' \prec_{\mathbf{K}} N^\alpha$ with $N' \prec_{\mathbf{K}} N''$. In this case, if $N_0 \prec_{\mathbf{K}} N'$ and $a \in N' \setminus N_0$, then $(N_0, a, N') \sim_0 (N_0, a, N'')$ via the identity map on N''. It follows then, by Lemma 3.3, that for each $y \in P$ coding a Galois type realized by N^α over N_0, $y \in M^\alpha$. It follows then that for any distinct pair α, α' in $2^{\aleph_1}$, since M^α and $M^{\alpha'}$ have just countably many reals in common, the set of $x \in P$ for which both N^α and $N^{\alpha'}$ realize the Galois type of x over N_0 is countable. $\square$

4.8 Remark. The proof above gives a slightly stronger conclusion, that the set of $x \in P$ for which there exist $N_1 \in M^\alpha$ and $N_2 \in M^{\alpha'}$ such that N^α realizes the Galois type of x over N_1 and $N^{\alpha'}$ realizes the Galois type of x over N_2 is countable.

4.9 Remark. The assumption in Theorem 4.7 that the set of reals coding countable structures in $\mathbf{K}$ be analytic can be can be relaxed to the requirement this set of codes be universally Baire (see [11]), if one is willing to assume the existence of a Woodin cardinal with a measurable cardinal above it (see Theorem 1.9 and [8, 9]). However, the corresponding versions of Burgess's Theorem are weaker (see [13]), which means that the range of applications should be narrower.

5 Absoluteness of $\aleph_1$-categoricity

In first order logic, the Baldwin-Lachlan equivalence between '$\aleph_1$-categorical' and 'ω-stable with no two-cardinal models' makes the notion of $\aleph_1$-categoricity Π^1_1 and hence absolute. Shelah provided an example of an AEC, definable in $L(Q)$, which is $\aleph_1$-categorical under MA and has $2^{\aleph_1}$ models in $\aleph_1$ under $2^{\aleph_0} < 2^{\aleph_1}$. It is an open question whether there is such a non-absolute example in $L_{\omega_1,\omega}$. [2] shows that $\aleph_1$-categoricity is absolute for a sentence of $L_{\omega_1,\omega}$ which satisfies amalgamation and jep in $\aleph_0$ and is ω-stable. In fact since ω-stability implies amalgamation in $\aleph_0$ (Corollary 19.14.3 of [1]), we have absoluteness of $\aleph_1$-categoricity for ω-stable sentences of $L_{\omega_1,\omega}$. The notion of ω-stability in that analysis is a syntactic one. Here we generalize this analysis to AEC and (almost) Galois ω-stability. However, the study of ω-stability in AEC is not sufficiently advanced as to deduce $\aleph_0$-amalgamation from ω-stablity.

Shelah's $L(Q)$-example fails amalgamation in $\aleph_0$ and is not ω-stable. We focus here on the question of whether assuming amalgamation is enough to make $\aleph_1$-categoricity absolute for analytically presented AEC. Amalgamation for countable models in an analytically presented AEC is Π^1_2 and therefore absolute. The argument for Theorem 2.1 shows that the existence of an uncountable model is Σ^1_1 in a real parameter and therefore also absolute. This claim should not extend to AEC which are not analytically presented: if membership in $\mathbf{K}$ were Π^1_1 or more complicated, amalgamation would not automatically satisfy Shoenfield absoluteness.

Let us consider for a moment the case where $\mathbf{K}$ is almost Galois ω-stable and satisfies amalgamation and the joint embedding property. In this case, there is a model in $\mathbf{K}$ of size $\aleph_1$ which realizes every Galois type over every one of its countable substructures (i.e., it is $\aleph_1$-Galois saturated). Furthermore, all such saturated models are isomorphic. The question of $\aleph_1$-categoricity for $\mathbf{K}$ then just depends on whether $\mathbf{K}$ has a model of size $\aleph_1$ omitting some Galois type over some countable substructure.

The second part of the following statement is $\underset{\sim}{\Sigma}^1_2$ and thus absolute. The relation $\sim_0$ was defined near the beginning of Section 4, and the projection of a tree was defined at the beginning of Section 3.

Theorem 5.1. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC. Then the following statements are equivalent.*

1. *There exist a countable* $M \in \mathbf{K}$ *and an* $N \in \mathbf{K}$ *of cardinality* $\aleph_1$ *such that*
   - $M \prec_{\mathbf{K}} N$*;*
   - *the set of Galois types over* M *realized in* N *is countable;*
   - *some Galois type over* M *is not realized in* N*.*
2. *There is a countable model* P *of* ZFC° *such that*
   - ω_1^P *is well-founded;*
   - P *contains trees on* ω *projecting to the set of codes for countable elements of* $\mathbf{K}$*, and to the relations on reals corresponding to* $\prec_{\mathbf{K}}$ *and* $\sim_0$*;*
   - P *satisfies statement (1).*

Proof. The implication from (1) to (2) just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix M and N witnessing (1) in P. In P, there exists a countable set S containing a member of each Galois type over M realized in N, and a member t of a Galois type over M not realized in N. Fixing elements of $\omega^\omega \cap P$ coding t and the elements of S, the statement that t is not Galois-equivalent to any member of S is Π^1_1 in these codes. Since P believes this statement, and since ω_1^P is well-founded, it is true in V also that t is not Galois-equivalent to any member of S.

Then, if P' is an iterate of P by an iteration of length ω_1, and N' is the corresponding image of N, P' thinks that every element of $N' \setminus M$ satisfies a Galois type corresponding to a member of S, which, being countable in P, was fixed by this iteration. It follows that no member of N' satisfies the Galois type corresponding to t. $\square$

A similar argument shows that almost Galois ω-stability is $\underset{\sim}{\Pi}^1_2$, and therefore absolute, for analytically presented AEC.

Theorem 5.2. *Suppose that* $\mathbf{K}$ *is an analytically presented AEC. Then the following statements are equivalent.*

1. $\mathbf{K}$ *is not Galois* ω*-stable.*
2. *There is a countable model* P *of* ZFC° *such that*
   - ω_1^P *is well-founded;*
   - P *contains trees on* ω *projecting to the set of codes for countable elements of* $\mathbf{K}$*, and to the relations on reals corresponding to* $\prec_{\mathbf{K}}$ *and* $\sim_0$*;*
   - P *thinks that* $\mathbf{K}$ *is not Galois* ω*-stable.*

Proof. Again, the forward direction just involves taking the transitive collapse of an elementary submodel. For the reverse direction, fix P, a countable $M \in \mathbf{K} \cap P$ and a tree $T \subseteq \omega^{<\omega}$ in P without terminal nodes, such that the paths through T code distinct Galois types over M. Since ω_1^P is well-founded, P correctly witnesses the fact that no two paths through T give E_M-equivalent reals. Furthermore, every path through T in a $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^P$-ultrapower of P codes a Galois type over M, as witnessed in the ultrapower. The set of such paths is an uncountable analytic set, and therefore contains a perfect set. $\square$

In the case that the set of codes for countable models in $\mathbf{K}$ is Borel (for instance, the set of countable models of a sentence of $L_{\omega_1,\omega}$), almost Galois ω-stability of $\mathbf{K}$ is more easily seen to be $\undertilde{\Pi}^1_2$, as it asserts that for any countable $M \in \mathbf{K}$ and any subtree of $\omega^{<\omega}$ without terminal nodes, either there is a path through the tree not coding a Galois type over M, or there exist distinct E_M-equivalent paths through the tree.

On its surface, Galois ω-stability for an analytically presented AEC $\mathbf{K}$ is $\undertilde{\Pi}^1_4$, as it says that for every M, if $M \in \mathbf{K}$ then there are countably many reals such that every suitable real is E_M-equivalent to one of them. Statements of this type are also forcing-absolute in the presence of suitable large cardinals, though not in ZFC. For all we know, an analytically presented almost Galois ω-stable $\mathbf{K}$ can be sometimes Galois ω-stable and sometimes not.

The proof of Theorem 4.6, as given in [4], however, shows that if an analytically presented $\mathbf{K}$ satisfies amalgamation and is almost Galois ω-stable but not Galois ω-stable, then $\mathbf{K}$ contains uncountable small models of uncountably many distinct Scott ranks. The existence of two (or any number up to ω many) uncountable small models in $\mathbf{K}$ of distinct Scott ranks is equivalent, by the iteration construction of this paper, to the existence of a countable model of ZFC$^\circ$ which is well-founded up to the supremum of these ranks and thinks there are such models. Again, this latter statement is Σ^1_2 and therefore absolute. Summarizing, we have the following.

Theorem 5.3. *Let $\mathbf{K}$ be an analytically presented almost Galois ω-stable AEC satisfying amalgamation in $\aleph_0$, and having an uncountable model. Then the $\aleph_1$-categoricity of $\mathbf{K}$ is equivalent to a Π^1_2-sentence, and therefore absolute.*

Proof. In this situation, $\aleph_1$ categoricity is equivalent to the conjunction of three conditions:

1. Joint embedding of any two countable models that have uncountable extensions.

2. All uncountable small models in $\mathbf{K}$ have the same Scott rank.

3. Part (1) of Theorem 5.1 fails.

Clearly $\aleph_1$-categoricity implies the first two conditions. Since the second condition holds, $\mathbf{K}$ is Galois ω-stable by Theorem 4.6 and the argument of the last paragraph before this theorem. $\aleph_1$-categoricity also implies the third condition because the Galois ω-stability and amalgamation in $\aleph_0$ imply the existence of a Galois-saturated model in $\aleph_1$ which realizes only countably many Galois types over each countable submodel.

Conversely, if condition 2) holds then as in the first paragraph $\mathbf{K}$ is Galois ω-stable. Condition 1), Galois ω-stability and amalgamation imply the existence of Galois-saturated model in $\aleph_1$ which realizes only countably many Galois types over each countable submodel. Condition 3) asserts that each model in $\aleph_1$ has this property.

Finally, these conditions are absolute. The negation of Condition 1) is Σ^1_2, using the idea behind Theorem 2.1 to verify the uncountable extensions. As remarked in the preceding paragraph, the negation of Condition 2) is equivalent to the existence of two small models in $\mathbf{K}$ of distinct Scott ranks and thus is absolute. Finally, by Theorem 5.1 condition is equivalent to 2 of Theorem 5.1, which is absolute. $\square$

5.4 Remark. We should point out that our absoluteness results in this section and the previous one relied only on the fact that the Galois types are induced by an analytic equivalence relation. In the same way, the results of Section 2 were analyzing Borel equivalence relations. Each approach then can be applied much more generally, though we have no applications for this degree of generality at this time.

6 Questions

The following questions have been left unresolved.

6.1 Question. Is there (consistently) an example like Example 4.3 (Example 4.14 of [1]) (i.e., $\aleph_1$-categorical, satisfying amalgamation and joint embedding almost Galois-stable but not ω-Galois stable) which is analytically presentable? By Theorem 0.4 and the main theorem of [4], there cannot be a ZFC example.

6.2 Question. Can there be an almost Galois ω-stable analytically presented AEC whose Galois ω-stability (or lack thereof) is not absolute? Consider Fact 4.5 and the succeeding paragraph.

6.3 Question. Is there a $PC\Gamma(\aleph_0, \aleph_0)$ AEC, which is almost Galois ω-stable, not Galois ω-stable and with κ models in $\aleph_1$, where $\kappa < 2^{\aleph_1}$? (We have shown $\kappa \geq \aleph_1$.

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