# From Geometry to Algebra

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**Abstract**

Our aim is to see which practices of Greek geometry can be expressed in various logics. Thus we refine Detlefsen's notion of descriptive complexity by providing a scheme of increasing more descriptive formalizations of geometry

Following Hilbert we argue that defining a field structure on a line in 'Euclidean geometry' provides a foundation for both geometry and algebra. In particular we prove from first principles: $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, similar triangles have proportional sides, Euclid's 3rd axiom: circle intersection, the area of every triangle is measured by a segment. For these as Hilbert showed, no theory of limits is needed. Thus, the first order theory as described by Hilbert or Tarski is adequate for proportion and polygonal area.

We further consider the role of $\pi$ and determining the area and circumference of a circle. and the area of a circle of radius $r$ is $\pi r^2$. Here we extend the first order geometry by adding a constant for the length $\pi$. Here we will rely on the axiom of Archimedes but only in the metatheory and not at all on Dedekind completeness. The natural numbers are not definable in these geometries. Finally Dedekind completeness add a second order axiom to give the modern basics of modern analysis.

Very preliminary -not yet for general release. I expect to give a major reorganization to this material. But the present version explains the role of multiplication as similarity vrs as repeated addition and sketches the study of circles. This requires adding formal notions of area and arc length. This is partially carried out in Section 7. This version just clarifies a few points from the presentation at the Urbana meeting.

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# 1 Introduction

We aim here to lay out and compare some first order axiomatizations of (fragments of) Euclidean geometry. The first goal is to obtain a general treatment of proportion without resulting to either Eudoxus or any second order completeness axiom. Then we extend Hilbert's analysis to consider a first order axiomatization which includes the circumference and area of a circle and the assigns a measure to angles. In the process we consider Hilbert's axiomatization and argue that a more sympathetic attitude to Euclid provides a more natural axiomatization of the geometry of constructions. This is however an incomplete and undecidable theory. The natural completion is Tarski's decidable first order theory of the reals: the geometry of the plane over any real closed field. We employ Detlefsen's concept of descriptive completeness in Subsection 1.1 to assess the success of the axiomatization. In particular, our analysis uses a tool not available to Hilbert in 1899, a clear distinction between first and second order logic. Note in particular that the first order complete geometry described in Theorem 7.0.4 is biinterpretable with the theory of real closed fields and so can be proved consistent in systems of low proof theoretic strength[1]

A fundamental issue is the distinction between the arithmetic and geometric intuitions of multiplication. The first is as iterated addition; the second is as scaling or proportionality. The late 19th century developments provide a formal reduction of the second to the first but the reduction is only formal; the intuition is lost. In this paper we view both intuitions as fundamental and develop the second (Section 3): with the understanding that development of the first through the Dedekind-Peano treatment of arithmetic is in the background. See Remark 3.1.5 for the connection between the two.

This paper unites my interests in the foundations of mathematics and mathematics education. The mathematical sections were first written in [5] as notes for a workshop for teachers and are based on Hartshorne's undergraduate text [23]. But I realized this search for an understandable introduction to geometry (in contrast to Birkhoff-Moise axiomatization which destroyed high school geometry in the United States [6]) actually fits well into Hilbert's program.

## 1.1 The Goals of Axiomatization

{goalax}

Hilbert begins the Grundlagen [25] with

> The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of *independent axioms* and to deduce from them the most important geometrical theorems in such a manner as to bring

[1]The canonical reference is [24] pages 38-48. Tarski announces the result in [35]. Tomás gives a detailed proof in [36]. Harvey Friedman**??** proves the consistency of the theory of real closed fields in EFA (exponential or elementary) function arithmetic, a system which allows induction only over bounded quantifiers and has consistency strength $\omega^3$.

> out as clearly as possible the significance of the groups of axioms and the scope of the conclusions to be derived from the individual axioms.

In this paper, we adopt Hilbert's 'first order axioms' and attempt to disentangle the uses of arithmetic and second order logic 'to deduce from them the most important geometrical theorems'. That is, we argue that some uses of 'continuity' or more precisely Dedekind completeness in Hilbert's development were unnecessary.

For example, Dedekind (page 22 of [11]) writes ' ... in this way we arrive at real proofs of theorems (as, e.g. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before.' We will give an equally rigorous but much more simple proof of this in Theorem 4.0.1.

The meaning of Hilbert's introduction depends greatly on what Hilbert meant by 'complete'. While in the context of the preface it clearly means, 'decides relevant statements', in modern terminology, the completeness axiom in [25] (page 25) is semantic: it asserts that there is a unique maximal model of the five groups of axioms[2].

Hilbert's first formulation of completeness builds in categoricity. With Hilbert's later syntactic version, which just asserts the second order proposition that the line is Dedekind complete, one gets a formulation where categoricity is a metatheorem. We seek weaker axiomatizations which still fulfill the criteria of justifying the most important geometrical theorems. For this Detlefsen's criterion for descriptive completeness is relevant. He [14] quotes Huntington [28]:

> [A] miscellaneous collection of facts ... does not constitute a *science*. In order to reduce it to a science the first step is to do what *Euclid* did in geometry, namely, to *select a small number of the given facts as axioms and then to show that all other facts can be deduced from these axioms by the methods of formal logic*.

But what does 'fact' mean? Detlefsen says, 'a commonly accepted sentence pertaining to a given subject area'. This raises the issue of the meaning of sentence. A natural solution would be to specify a logic and a vocabulary and consider all sentences in that language. Detlefsen argues (pages 5-7 of [14]) that Gödel errs in seeing the problem as completeness of a first order theory. We examine here the case of plane geometry and see that questions arise that are not explicitly in our first order framework for geometry[3].

This problem arises in proving that similar triangles have proportional sides. Here the notion of proportional for Euclid requires going beyond the first order lan-

[2] In the 7th edition this was replaced by line completeness, which is a variant of Dedekind completeness. See [26].

[3] Thus in our case, even if we have a complete first order theory, we have to show that it correctly represents 'proportionality'. In this instance, we show that a small fragment of the complete theory suffices for this purpose.

guage of geometry – in particular using infinite sequences of approximations. This could lead to positing Dedekind completeness and separability of the coordinatizing field as in [9]. However, Hilbert[26] showed a that for a certain first order theory (Euclidian geometry, Notation 2.1.1) similar triangles have proportional sides. The key point is that for triangles in a model $M$ to be similar, the lengths of their sides must be in $M$.

We study several natural examples of problems that should be solved to get descriptive completeness and solve them within a first order context. We will present a first order complete set of axioms for plane geometry.

To meet Detlefsen's demand for *descriptive completeness*, we must show the consequences of these axioms include the 'commonly accepted sentences' pertaining to this subject area. We avoid Dedekind completeness while recovering a number of these commonly accepted proposition; their proofs are scattered in the paper.

Below, by 'measured by a segment' we mean that if the area of the triangle is $A$ square units, there is a segment with length $A$ units. Thus we adopt the modern view of number. See Remark 3.1.1.

Here is a summary of the geometrical theorems discussed and the type of logic necessary for the formalization. {goal}

**Theorem 1.1.1.** *The following hold in Euclidean geometry. (See Notation 2.1.1.)*

1. *first order: (Hilbert)*
   (a) $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$. *(Theorem 4.0.1)*
   (b) *the side-splitter theorem (Theorem 5.0.2)*
   (c) *Euclid 3: circle intersection (Theorem 4.0.4)*
   (d) *The area of every triangle is measured by a segment (Heron). Theorem 4.0.4*
2. *first order: (new)*
   (a) *Formulas for area and circumference of circle Theorem* **??**
   (b) *In some models all angles have measure.*

We will study in more detail in a later version of the paper.

1. $L_{\omega_1,\omega}$: Archimedes axiom
   (a) In every model, every angle has a measure
2. Second order logic: Dedekind completeness
   (a) Categoricity and 'analysis'

## 1.2 Methodology

We identify two levels of formalization in mathematics. By the Euclid-Hilbert style we mean the axiomatic approach of Euclid along with the Hilbert insight that postulates are implicit definitions of classes of models. By the Hilbert-Gödel-Tarski-Vaught style, we mean that that syntax and semantics have been identified as mathematical objects; Gödel's completeness theorem is a standard tool[4]. In the development here, we want to have the best of both worlds. We will give our arguments in English; but we will be careful to specify the vocabulary and the postulates in a way that the translation to a first order theory is transparent. This will allow us to apply the insight that second order properties of models can strengthen the effect of first order assertions (as in Theorem 1.1.1.5. As part of our less formal approach, we make use of the analysis of Manders [30] to use diagrams to make proofs more understandable[5]. Properties that are *not* changed by minor variations in the diagram such as subsegment, inclusion of one figure in another, two lines intersect, betweenness are termed *inexact*. Properties that *can be* changed by minor variations in the diagram, such as whether a curve is a straight line, congruence, a point is on a line, are termed *exact*. We can rely on reading inexact properties from the diagram.We must write exact properties in the text. The difficulty in turning this insight into a formal deductive system is that, depending on the particular diagram drawn, after a construction, the diagram may have different inexact properties. The solution is case analysis but bounding the number of cases has proven difficult.

In this paper, we lay out formal axioms so that our work can be grounded in modern logic and even support some arguments that invoke the completeness theorem but discuss the connections between the axioms as laid out by Hilbert and more construction oriented versions which are truer to Euclid's practice.

## 1.3 Setting

Adrian Mathias [31] quotes Sylvester on the three divisions of mathematics.

> There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order.

This is a slightly unfamiliar trio. We are all accustomed to opposition between arithmetic and geometry. While Newton famously founded the calculus on geometry

[4]See [7] for further explication

[5]This approach is probably best described as the normal proof mode of a mathematician. But we are aware that no completeness theorem has been yet proved for Euclidean geometry with diagrams at the strength we employ.

(see e.g. [15]) the 'arithmetization of analysis' in the late 19th century reversed the priority. From the natural numbers the rational numbers are built by taking quotients and the reals by some notion of completion. And this remains the normal approach today. We want here to consider reversing the direction again: building a firm grounding for geometry and then finding first the field and then some completion and considering incidentially the role of the natural numbers. In this process, Sylvester's third cardinal notion order will play an ambiguous role. On the one hand in the first section, the notion that one point lies between two others will be fundamental and an order relation will naturally follow. In a sequel, we will eschew order and consider the development of projective or complex geometry.

Bolzano discusses the 'dissimilar objects' found in Euclid[6] and finds Euclid's approach fundamentally flawed.

> Firstly triangles, that are already accompanied by circles which intersect in certain points, then angles, adjacent and vertically opposite angles, then the equality of triangles, and only much later their similarity, which however, is derived by an atrocious detour [ungeheuern Umweg], from the consideration of parallel lines, and even of the area of triangles, etc.! (1810, Preface)

Much of this paper is an exploration of the relations between different themes in Euclid: congruence and parallelism, area, similarity. We will use segment arithmetic to analyze these connections.

By 'their similarity', I take Bolzano to be referring to the proof that that two triangles are similar (corresponding angles are equal) if and only the sides are proportional. (Euclid VI.4 and VI.5, which follow Euclid VI.1 and VI.2 which heavily involve parallelism and area. As outlined in [2, 8], Hilbert both carries out Euclid's argument and shows that while the parallel postulate is not essential; the theorem of Desargues, which follows using both parallel postulate and congruence, is. However, Hilbert continues to rely on the use of area in particular (implicitly ???) on De Zolt's axiom that a non-trivial triangle has a positive area. However, as we describe below, Hartshorne [23] has shown (using a variant on Hilbert construction of the field) that this reliance is *not* necessary. In a later paper we will discuss the role of the Desargues theorem in finite and complex geometry; but here we introduce order and use the parallel postulate.

[6] This is taken from [19].

# 2 The geometry of Euclid/Hilbert

{euclid}

## 2.1 Introduction

This section pertains to our fundamental goal of establishing a *simple* set of axioms for geometry. But for the principal goal of this paper, using the coordinatization of the field as a tool to define proportionality, the exact axioms are not important. {HP5}

**Notation 2.1.1.** We follow [23] in the following nomenclature.

A *Hilbert plane* is any model of Hilbert's incidence, betweenness, and congruence axioms. We abbreviate these axioms by HP. We will write HP5 for these axioms plus the parallel postulate. A *Euclidean plane* is one that satisfies HP5 and in addition the circle-circle intersection postulate 2.3.5. Our official axiom list will be an alternative axiomatization of a Euclidean plane.

Notations such as Eax1,HC4, HaC4 in the axiom list show the corresponding axiom in Euclid, Hilbert, or Hartshorne. Axioms without an author code such as C1 or B2 denote the official axioms of this exposition, coded by C for construction, B for betweenness etc.

This section is primarily concerned with the interplay between the axiom systems of Euclid, Hilbert, and Hartshorne and our official variant which we hope is more appropriate for high school instruction. The definition of the field in Section 3 relies only on HP5.

## 2.2 Foundational Issues

{fund}

In this section we expand on some perhaps idiosyncratic interpretations of Euclid's axioms (as opposed to the postulates) that serve to give a more unified account of the foundations of geometry.

The foundations here are an ahistorical reformulation and variant of Euclid, taking great care with some of the corrections of Hilbert and less with others. First we distinguish between axioms and postulates. Axioms are general mathematical assumptions. Postulates are subject specific; we study postulates for geometry.

Axiom 1. Things which equal the same thing also equal one another.

Axiom 2. If equals are added to equals, then the wholes are equal.

Axiom 3. If equals are subtracted from equals, then the remainders are equal.

Axiom 4. Things which coincide with one another equal one another.

Axiom 5. The whole is greater than the part.

Euclid used 'equal' in a number of ways: to describe congruence of segments and figures, to describe that figures had the same measure. Numbers in the modern sense do not appear in Euclid, but we introduce them in Section 3. *Thus, we regard the common notions as properties that first and foremost describe congruence (between segments and between angles).* To be precise in the modern sense we explicitly remark properties of equality that were tacit in Euclid.

**Remark 2.2.1.** We take the following to be intended by the notion of equality in the common notions:

Things which equal the same thing also equal one another. Every thing is equal to itself (reflexivity) and if one thing is equal to another then the second is equal to the first (symmetry).

These are 'logical axioms'. They hold in all contexts. Exactly how they are interpreted in discussed in specific cases below. In particular 'equality' in the axioms can be interpreted as either identity, congruence, or same area. Thus the first conclusion from the axioms is that both equality of points and congruence of segments are equivalence relations. But of course Euclid means more by equality in various contexts and we explore those meanings below.

We follow Hilbert in regarding the notions of point and line and the relations of lie-on and between as undefined but determined by the postulates. As we now understand it, Euclid's definitions of such concepts are only explanations motivating the implicit definition given by the postulates. We fix now the fundamental notions as a list of relation symbols giving vocabulary (similarity type) for our study. We will be giving first order axioms for this vocabulary. {geovoc}

**Notation 2.2.2.** *The fundamental relations of plane geometry are:*

1. *two-sorted universe: points ($P$) and lines ($L$).*
2. *Binary relation $I(A, \ell)$:*

   *Read: a point is incident on a line;*
3. *Ternary relation $B(A, B, C)$:*

   *Read: $B$ is between $A$ and $C$ (and $A, B, C$ are collinear).*
4. *quaternary relation, $C(A, B, C, D)$: Read: two segments are congruent, in symbols $\overline{AB} \approx \overline{CD}$.*
5. *6-ary relation $C'(A, B, C, A', B', C')$: Read: the two angles $\angle ABC$ and $\angle A'B'C'$ are congruent, in symbols $\langle ABC \approx \langle A'B'C'$.*

*$\tau$ is the vocabulary containing these symbols.*

Note that I freely used defined terms such as collinear, segment, and angle in giving the reading.

**Postulate 2.2.3. Eax1, HC4, HaC4** *Congruence of segments (angles) is an equivalence relation.*

## 2.3 Construction Postulates

{constpost}

Following Euclid we consider first construction postulates. Hilbert chose to view many of the same principles from a slightly different perspective as incidence or congruence postulates. The following list also provides a correspondence between the formulations of Euclid [18], Hilbert [26], and Hilbert as numbered by Hartshorne [23] (these we label with Ha).

**Postulate 2.3.1.** *Construction postulates* {conp}

**C1: E1, HI1, HaI1** *Given points $A$ and $B$, a unique line segment may be drawn between them.*

**C2: (Eax4, E2), HIII3, HaC3** *the segment $a \cong a'$ and $b \cong b'$ then the result $a \bigoplus b$ and $a' \bigoplus b'$ obtained by applying 1) to copy $b$ ($b'$) after $a$ ($a'$) are congruent.*

Hilbert and Hartshorne (HaC3) give an additional postulate corresponding to Euclid's, equals added to equals are equal for segments. We consider this as a consequence of Axiom 2.

**Dicta** Note that each construction postulate is given data and constructs further points, line segments, or circles.

There are three further construction postulates. The first two give the second two incidence postulates in Hartshorne (page 66) or Hilbert I3. {E2}

**Theorem 2.3.2** (Euclid's 2nd postulate)**.** *A line segment may be extended indefinitely in either direction.*

This is not explicit in Hilbert but evidently follows from the 3rd construction postulate above. That is, why we labeled C2 as ((Eax4,E2), HIII3, HaC3).

Euclid[7] defines a circle as: A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another. In this we see his greater generality than modern treatments in regarding curves as lines. We will use line as an abbreviation for straight line. Then we define.

[7] All references to Euclid are to Heath's translation, usually taken from Clarke's website `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html#posts`

**Definition 2.3.3** (Circle)**.** *A circle with center $A$ and radius $AB$ is the collection $X$ of points such that for each $P \in X$, $AX \cong AB$.* {E3}

**Postulate 2.3.4. C3: Euclid's 3rd postulate** *Given a point $P$ and any segment $AB$ there is a circle with $P$ as center whose radius is congruent to $AB$.*

Theorem 2.3.2 immediately implies every line contains at least two points and Postulate2.3.4 that there are 3 non-collinear points. These conditions are additional axioms in Hilbert and Hartshorne.

Euclid renders Postulate 3 as: To describe a circle with any center and radius.

There is no postulate in Hilbert or Hartshorne corresponding precisely to Euclid's third; they just regard a circle as the locus of points equidistant from a fixed center. The following postulate is not explicit in either Euclid or Hilbert so the first two entries of the coding is blank. {ccp}

**Postulate 2.3.5. C4: (_, _, HaE)Circle Intersection Postulate** *If from points $A$ and $B$, circles with radius $AC$ and $BD$ are drawn such that one circle contains points both in the interior and in the exterior of the other, then they intersect in two points, on opposite sides of $AB$.*

While some regard the absence of this axiom as a gap in Euclid, Manders (page 66 of [30] asserts: 'Already the simplest observation on what the texts do infer from diagrams and do not suffices to show the intersection of two circles is completely safe.'

We give Postulate 2.3.5 a special status beyond the postulates for a Hilbert plane as it has exact algebraic content which we state in (Theorem 4.0.4).

Note that two of Hilbert's congruence axioms (HC1- segment copy) and HC3- segment addition well-defined) follow easily from Circle intersection (for HC1 by Euclid I.1 and I.2) and Common notion 2 (for HC3). So if we read Circle intersection as a consequence of a proper reading of diagrams, we are showing an equivalence of Euclid's axioms and postulates ('properly read') and Hilbert's system.

Hartshorne (Exercise 39.31 of [23]) observes that equilateral triangles can fail to exist in a Hilbert plane [8] and so circle intersection fails by Euclid 1.2. But this model also fails the parallel postulate.

## 2.4 Betweenness {betp}

Euclid allowed the diagrams to carry information about betweenness. We follow Hilbert and (specifically) Hartshorne and give explicit betweenness axioms but let the

[8] A specific Poincaire model

vocabulary carry more of the information. Note that betweenness is a way to impose order; in conjunction with axioms that imply the definability of a field, this will imply the line is infinite (and densely ordered); for simplicity we posit this ordering first.

The statement '$A$ is between $B$ and $C$' cannot be made unless $A, B, C$ are collinear.

**Postulate 2.4.1.** *Betweenness*

*Postulate B1 $A$ is between $B$ and $C$ implies $A$ is between $C$ and $B$.*

*Postulate B2 For any pair of points $A, B$ there is a point $C$ with $B$ between $A$ and $C$*

*Postulate B3 For any three distinct points, exactly one is between the other two.*

*Postulate B4 Pasch's postulate: A line that intersects one side of triangle (not in a vertex) must intersect one of the other two.*

We omit the careful development of the plane and line separation theorems. Note however, the betweeneess postulates imply.

**Theorem 2.4.2.** *Each line is given a dense linear order, defined by $A < C$ if and only if $(\exists y)B(A, C)$.*

This linear order imposes the order topology on the plane. When the theory is completed by making the field real closed we access the power of o-minimality [12].

## 2.5 Congruence

Hilbert introduces 4 congruence postulates. The first three concern congruence of segments: segment copy, segment addition is well-defined, segment congruence is an equivalence relation; we have discussed the first two in Subsection2.3 and the 3rd in Subsection 2.2.

We choose to take SSS as our fundamental congruence postulate because of the ease of deducing the ability to copy angles. {sss}

**Postulate 2.5.1** (The triangle congruence postulate: SSS)**.** *Let $ABC$ and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$*

In Euclid this result, SSS, is proved from SAS. The proof is 4 steps: Euclid Propositions 1.5 to 1.8. These 4 steps are not hard and are correct. But his proof of SAS depends on the unstated notion of superposition, so we have to add **one** congruence axiom; we choose to add SSS. One reason for this choice is that just as the second proposition in Euclid allows one to copy a line segment, SSS gives an immediate method to copy angles. All the other criteria for congruence (SAS, ASA, HL ...) are theorems.

Hartshorne points out in the exercises on page 103 of [23]:

**Fact 2.5.2.** *In any Hilbert plane, one can construct angle bisectors, midpoints, perpendicular to a line $\ell$ through a point $A$ (on or off $\ell$) and more.*

In particular, this allows the construction of an incenter.

# 3 From geometry to segment arithmetic to numbers

{num}

In this section we will explore several mechanisms for moving from geometric constructions to operations on numbers. We assume what we called HP5 in Notation 2.1.1.

## 3.1 Segment arithmetic defined

For this we introduce *segment arithmetic*. This topic appears in Euclid, gets a different interpretation in Descartes and still another in the 19th century arithmetic of real numbers.

We want to define the multiplication of 'lengths' to give another length. This differs from the treatment of multiplication in Book II of Euclid (and everywhere else in antiquity) as giving an area. Identify the collection of all congruent line segments as having a common 'length' and choose a representative segment $OA$ for this class. There are then three distinct historical steps. (See in particular [21] and Heath's notes to Euclid VI.12 (`http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html`.) In Greek mathematics numbers (i.e. 1, 2, 3 ...) and magnitudes (what we would call length of line segments) were distinct kinds of entities and areas were still another kind.

**Remark 3.1.1.** From geometry to numbers {geonum}

1. Euclid shows that the area of a parallelogram is jointly proportional to its base and height. [9]

2. Descartes defines the multiplication of line segments to give another segment[10]. Hilbert shows the multiplication on segments satisfies the field axioms[11].

[9] In modern terms this means the area is proportional to the base times the height. But Euclid never discusses the multiplication of magnitudes.

[10] He refers to the fourth proportional ('ce qui est meme que la multiplication'[13])

[11] In [26], the axioms for a semiring (no requirement of an additive inverse) are verified.

Descartes for good reason does not fix the unit segment. We do by naming points $0, 1$.

3. Identify the points of the line with (a subfield) of the real numbers. Now, fixing $0, 1$, addition and multiplication can be defined on the points of the line through $0, 1$[12].

The standard treatment in contemporary U.S. secondary school geometry books is to begin with stage 3, taking the operations on the real numbers as basic. We will pass rather from geometry to number, concentrating on stage 2. Thus, not all real numbers may be represented by points on the line in some planes. {segeq}

**Notation 3.1.2.** *Note that congruence forms an equivalence relation on line segments. We fix a ray $\ell$ with one end point $0$ on $\ell$. For each equivalence class of segments, we consider the unique segment $0A$ on $\ell$ in that class as the representative of that class. We will often denote the class (i.e. the segment $0A$ by $a$. We say a segment (on any line) $CD$ has length $a$ if $CD \cong 0A$.*

We first introduce an addition and multiplication on line segments. Then we will prove the geometric theorems to show that these operations satisfy the field axioms except for the existence of an additive inverse. We note after Definition 3.2.5 how to remedy this difficulty by the passing to points as in stage 3. {segadddef}

**Definition 3.1.3** (Segment Addition)**.** *Consider two segment classes $a$ and $b$. Fix representatives of $a$ and $b$ as $OA$ and $OB$ in this manner: Extend $OB$ to a straight line, and choose $C$ on $OB$ extended (on the other side of B from A) so that so that $BC \cong OA$. $OC$ is the sum of $OA$ and $OB$.*

**Diagram for adding segments**


It is easy to see that this addition is associative and commutative.

Of course there is no additive inverse if our 'numbers' are the lengths of segments which must be positive. We discuss finding an additive inverse after Definition 3.2.5. Following Hartshorne [23], here is our official definition of segment multiplication[13]

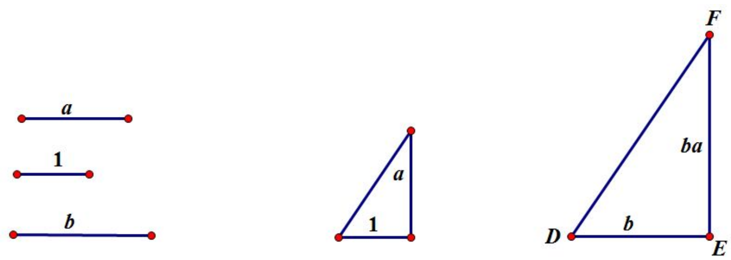
[12] And thus all axioms for a field are obtained. Hilbert had done this in lecture notes in 1894[27].

[13] Hilbert's definition goes directly via similar triangles. The clear association of angle with right muliplication by $a$ recommends Hartshorne's version.

{segmultdef}

**Definition 3.1.4.** *[Multiplication] Fix a unit segment class* $1$*. Consider two segment classes* $a$ *and* $b$*. To define their product, define a right triangle*[14] *with legs of length* $1$ *and* $a$*. Denote the angle between the hypoteneuse and the side of length* $a$ *by* $\alpha$*.*

*Now construct another right triangle with base of length* $b$ *with the angle between the hypoteneuse and the side of length* $b$ *congruent to* $\alpha$*. The length of the vertical leg of the triangle is* $ab$*.*




Note that we must appeal to the parallel postulate to guarantee the existence of the point $F$. {twomult}

**Remark 3.1.5.** We now have two ways in which we can think of the product $3a$. On the one hand, we can think of laying 3 segments of length $a$ end to end. On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length $1$ laid end to end) by the segment of length $a$. It is an easy exercise to show these are the same. But it makes an important point. The (inductive) definition of multiplication by a natural number is indeed 'multiplication as repeated addition'. But the multiplication by an other field element is based on similarity, implies the existence of multiplicative inverses, and so is a very different object.

We access the first notion of multiplication by natural numbers and derivatively positive rationals converting the set of positive points on the line into a $\mathbb{Q}$-module from the outside; there is no uniform definition of this *scalar* multiplication within the field; multiplication by $\frac{17}{27}$ is defined in the geometry but not multiplication by $\frac{17}{27}$. However, we now have the multiplication uniformly defined and this intrinsic geometrical multiplication restricts to that imposed by counting where it is defined.

## 3.2 Verifying the field properties

Before we can prove the field laws hold for these operations we introduce a few more geometric facts. This proof from multiplication defined from congruence obtains the

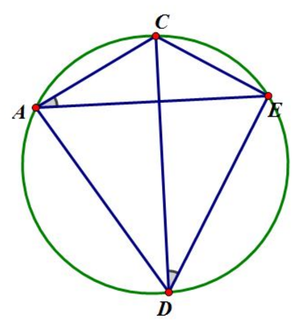
[14] The right triangle is just for simplicity; we really just need to make the two triangles similar.

commutative law before the associative. Hilbert (Section 31 of [26]) shows in the absence of congruence but with the Archimedean property commutativity holds. {ceninsang}

**Fact 3.2.1.** *[Euclid III.20]* **CCSS G-C.2** *If a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.*

We need proposition 5.8 of [23], which is a routine (if sufficiently scaffolded) high school problem. {cquad}

**Corollary 3.2.2. CCSS G-C.3** *Let $ACED$ be a quadrilateral. The vertices of $A$ lie on a circle (the ordering of the name of the quadrilateral implies $A$ and $E$ are on the opposite sides of $CD$) if and only if $\angle EAC \cong \angle CDE$.*




Proof. Given the conditions on the angle draw the circle determined by $ABC$. Observe from Lemma 3.2.1 that $D$ must lie on it. Conversely, given the circle, apply Lemma 3.2.1 to get the equality of angles. $\square_{3.2.2}$

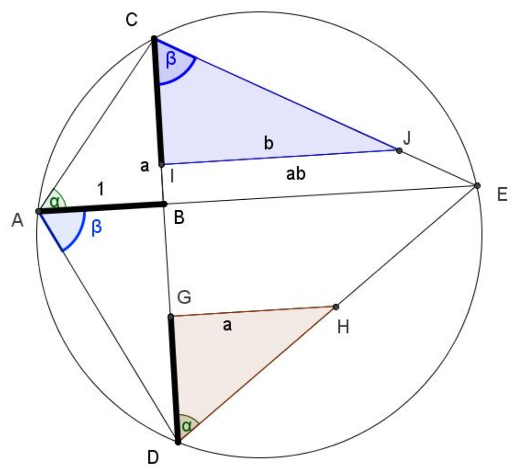
Now we get the main result. In our exposition here we give full details only for part 2 with an indication of how to extend to part 3. The others are easy. {mult2works}

**Theorem 3.2.3.** *The multiplication defined in Definition 3.1.4 satisfies.*

1. *For any $a$, $a \cdot 1 = 1$*
2. *For any $a$, $b$*
$$ab = ba.$$
3. *For any $a, b, c$*
$$(ab)c = a(bc).$$
4. *For any $a$ there is a $b$ with $ab = 1$.*
5. *$a(b + c) = ab + ac$.*

Proof. 2) Given $a, b$, first make a right triangle $\triangle ABC$ with legs 1 for $AB$ and $a$ for $BC$. Let $\alpha$ denote $\angle BAC$. Extend $BC$ to $D$ so that $BD$ has length $b$. Construct $DE$ so that $\angle BDE \cong \angle BAC$ and $E$ lies on $AB$ extended on the other side of $B$ from $A$. The segment $BE$ has length $ab$ by the definition of multiplication.

Since $\angle CAB \cong \angle EDB$ by Corollary 3.2.2, $ACED$ lie on a circle. Now apply the other direction of Corollary 3.2.2 to conclude $\angle DAE \cong \angle DCA$ (as they both cut off arc $AD$. Now consider the multiplication beginning with triangle $\triangle DAE$ with one leg of length 1 and the other of length $b$. Then since $\angle DAE \cong \angle DCA$ and one leg opposite $\angle DCA$ has length $a$, the length of $BE$ is $ba$. Thus, $ab = ba$.




3) To prove associativity note that the following diagram encodes right multiplication by $a$ and $c$.

Now the following diagram suffices to prove the associative law using arguments similar to those for commutativity.

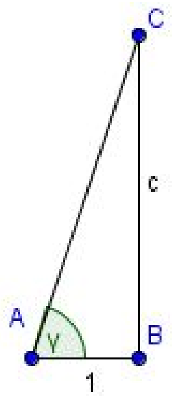



Figure 1: multiply by $a$

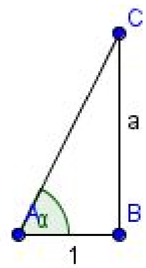
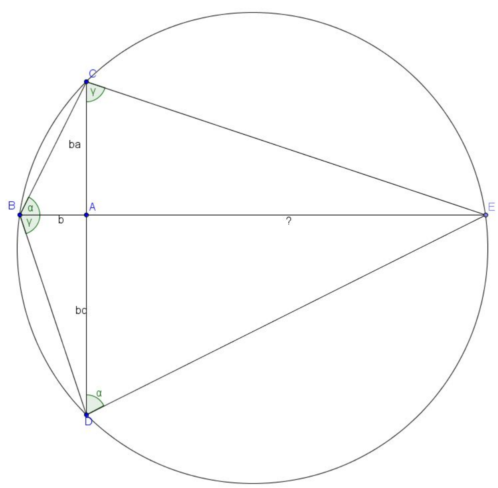



Figure 2: multiply by $c$




Note that $AE$ can be represented as either $(ba)c$ or $(bc)a$ use the commutative law twice to complete the proof. $\square_{3.2.3}$

It is now fairly straightforward to show:

**Corollary 3.2.4.** *Moreover the addition and multiplication respect the order.*

The remainder of this section is a modification to identify points on the line with numbers and so have additive inverses. Thus we obtain the full field multiplication. This step can be done intrinsically in the style of Artin [4] or as in Hartshorne [23], one can extend abstractly from the multiplication on the positive semiring to the full field.

{pointadd}

**Definition 3.2.5** (Adding points)**.** *Recall that a line is a set of points. Fix a line $\ell$ and a point $0$ on $\ell$. We define an operations $+$ on $\ell$. Recall that we identify $a$ with the (directed length of) the segment 0a.*

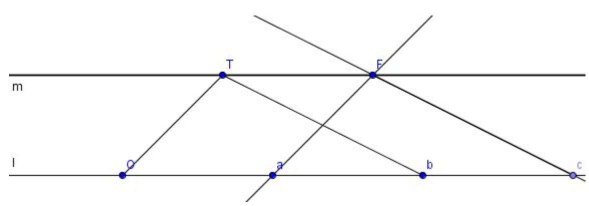
*For any points $a, b$ on $\ell$, we define the operation $+$ on $\ell$:*

$$a + b = c$$

*if $c$ is constructed as follows.*

1. *Choose $T$ not on $\ell$ and $m$ parallel to $\ell$ through $T$.*
2. *Draw $0T$ and $BT$.*
3. *Draw a line parallel to $0T$ through $a$ and let it intersect $m$ in $F$.*
4. *Draw a line parallel to $bT$ through $a$ and let it intersect $\ell$ in $c$.*

**Diagram for point addition**




$$0b \cong ac$$

It is straightforward to verify:

**Lemma 3.2.6.** *The addition of points is associative and commutative with identity element $0$. The additive inverse of $a$ is $a'$ provided that $a'0 \cong 0a$ where $a'$ is on $\ell$ but on the opposite side of $0$ from $a$.*

To summarize (details in section 21 of [23]):

**Theorem 3.2.7.** *The theory of Hilbert fields satisfying the parallel postulate is biinterpretable with theory of ordered pythagorean*[15] *planes. The interpreting formulas are first order with constants naming two points.*

[15] A field is Pythagorean if for every $a$, $1 + a^2$ has a square root.

# 4 The role of algebra

{sqrt}

We established in Section 3 that one could define an ordered field in any plane satisfying $HP5$. The converse is routine, the ordinary notions of lines and incidence in $F^2$ creates a geometry over any ordered field which is easily seen to satisfy $HP5$. We now exploit this equivalence.

We will prove some algebraic facts using our defined operations, thus basing them on geometry. On the other hand we reduce the completeness of the theory of our geometry to the well-known completeness of real closed fields. Dedekind proves $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ by a detour through the approximations of irrationals by cuts in the rationals. In contrast the following triviality holds in all ordered fields. {dedprob}

**Theorem 4.0.1.** *In an ordered field, for any positive $a$, if there is an element $b > 0$ with $b^2 = a$, then $b$ is unique (and denoted $\sqrt{a}$. Moreover, for any positive $a, c$ with square roots, $\sqrt{a} \cdot \sqrt{c} = \sqrt{ac}$.*

*In particular it holds for any field coordinatizing a plane satisfying HP5.*

*Thus the algebra of square roots in the real field is established without any appeal to limits.* The usual (e.g. [33, 1]) developments of the theory of complete ordered fields invoke the least upper bound principle to obtain the existence of the roots although the multiplication rule is obtained by the same algebraic argument as here. More appropriate to Dedekind's concerns, our treatment is based on the fundamental concept of congruence [16] which we regard as an equally fundamental intuition as 'number'. The justification for the existence of roots does not invoke limits.

It is well-known that the Pythagorean Theorem is equivalent for Hilbert planes to the parallel postulate.

Euclid's proof of Pythagoras I.47 uses an area function as we will justified in Section 6. Another standard proof uses the theory of similar triangles that we will develop Section 5. In any case, we have

**Theorem 4.0.2.** *The Pythagorean theorem holds in any Hilbert plane with the parallel postulate (HP5).* {exfields}

**Example 4.0.3.** Hartshorne [23] introduces two instructive examples.

1. A pythagorean field is one closed addition, subtraction, multiplication, division and for every $a$, $\sqrt{(1 + a^2)}$. However, the Cartesian plane over a Pythagorean field may fail to be closed under square root and the Poincaire model over a such a field may fail to have equilateral triangles and thus the circle-circle intersection postulate. (Exercise 39.30, 30.31 of [23])

[16]Given Dedekind's animus to the notion of 'measureable magnitudes' (page 15 of [11]).

2. On page 146, Hartshorne observes that the smallest ordered field closed under addition, subtraction, multiplication, division and square roots of positive numbers satisfies the circle-circle intersection postulate.

Recall that we distinguished a Hilbert plane for a Euclidean plane in Notation 2.1.1. As in [23], we have: {ccstrength}

**Theorem 4.0.4.** *A Hilbert plane satisfies the circle-circle intersection postulate, 2.3.5 if and only every positive element of the coordinatizing plane has a square root.*

*Similarly, in every Euclidean plane such that every positive element of the coordinatizing plane has a square root, Heron's formula computes the area of a triangle from it side lengths.*

Immediately, from Tarski [34] we get {tarskiax}

**Theorem 4.0.5.** *The following set of axioms is first order complete for the vocabulary in Notation 2.2.2.*

1. *Euclidean geometry*
2. *An infinite set of axioms declaring that every polynomial of odd-degree has a root.*

The field over real numbers is clearly a Euclidean plane. Moreover, we can prove the existence of such a model by realizing the type of each cut in the rationals. However, we can't guarantee separability in such 'a first order way'.

In [20] the completeness is imposed by an axiom Schema of Continuity - a definable version of Dedekind cuts:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(axy)] \rightarrow (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$$

where $\alpha, \beta$ are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x.

In Detlefsen's terminology we have found a Gödel complete axiomatization of plane geometry. But in accord with his analysis, we must wonder whether it is descriptively complete. In particular, is the notion of proportional included in our analysis. The test question is the similar triangle theorem. We turn to this issue now.

# 5 Multiplication is not repeated addition

{similar}

In the natural numbers, addition can be defined as iterated succession and multiplication as iterated addition. But the resulting structure is essentially undecidable. However, this structure does not illuminate the essential aspect of multiplication as similarity; many elements have no multiplicative inverse.

**Definition 5.0.1.** *Two triangles* $\triangle ABC$ *and* $\triangle A'B'C$ *are similar if under some correspondence of angles, corresponding angles are congruent; e.g.* $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$.

Various texts define 'similar' as we did, or as corresponding sides are proportional or require both (Euclid). We now prove our principal result, which shows the choice doesn't matter. {simtri}

**Theorem 5.0.2.** *Two triangles are similar if and only if corresponding sides are proportional.*

We need to define proportional; this is of course easy for commensurable segments. Euclid appeals in chapter VI to the theory of Eudoxus to ground the notion for incommensurable segments. Euclid defines commensurable in terms of the multiplication by natural numbers as we described just after Definition 3.1.4.

We simply define proportionality as it is now understood.

**Definition 5.0.3.** *Proportionality*

$$CD : CA :: CE : CB$$

*is defined as*

$$CD \times CB = CE \times CA.$$

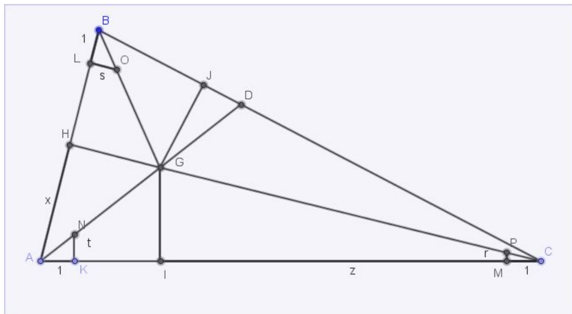
*where* $\times$ *is taken in the sense of segment multiplication defined in Definition 3.1.4.*

Here is Hartshorne's proof of the fundamental result.

Proof of Theorem 5.0.2: If $ABC$ and $A'B'C'$ are similar triangles then **using the segment multiplication we have defined**

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Consider the triangle $ABC$ below with incenter $G$.




Proof. The point $G$ is the incenter so $HG \cong GI \cong GJ$. Call this segment length $a$.

Now construct $AK \cong BL \cong MC$ all with standard unit length. Let the lengths of $BL$ be $s$, $NK$ be $t$ and $PM$ be $r$.

Let the lengths of $AI \cong AH$ be $x$, $BH \cong BJ$ be $y$, and $CI \cong AJ$ be $z$.

By the definition of multiplication $t \cdot x = s \cdot z = a$. Therefore the length of $AC$ is $\frac{a}{t} + \frac{a}{r} = \frac{a(r+t)}{rt}$.

Duplicate on the second triangle $A'B'C'$ to get the length of $A'C'$ is $\frac{a'}{t} + \frac{a'}{r} = \frac{a'(r+t)}{rt}$. *The crucial point is that because the angles are congruent $r, s, t$ are the same for both triangles.*

But then $\frac{A'C'}{AC} = \frac{a'}{a}$. Now note the same is true for the other two pairs of sides so the sides of the triangle are proportional.

The same ideas allow one to reverse the argument and show triangles with proportional sides are similar. $\square_{5.0.2}$

**Remark 5.0.4.** *Conversely if the sides of similar triangles are proportional, we can divide so any subgeometry satisfying this proposition is coordinatized by a subfield.*

**Remark 5.0.5.** Note the following.

For any model $M$ of HP5: similar triangles have proportional sides.

There is no assumption that the field is Archimedean or satisfies any sort of completeness axiom.

There is no appeal to approximation or limits.

It is easy[17] to check that the multiplication defined on the reals by this procedure is exactly the usual multiplication on the reals because they agree on the rationals.

[17] One has to verify that segment multiplication is continuous but this follows from the density of the order since the addition respects order.

# 6 Area of polygonal figures

{area}

In Section 5 we saw Bolzano's challenge is answered by a proof that similar triangles have proportional sides without resorting to the concept of area. But area is itself a vital geometric notion. We show now that using segment multiplication we ground the familiar methods of calculating the area of polygons.

As we discussed in Section 2.2, Euclid treats the equality of areas as a special case of his common notions. Hilbert specified more particular properties; we use the formulation from a high school text [16]. Here is an initial definition of those configurations that have area.

**Definition 6.0.1.** *A* figure *is a subset of the plane that can be represented as a finite union of disjoint triangles.*

Intuitively, two figures have *equal content* if we can transform one into the other by adding and subtracting congruent triangles.

There are serious issues concerning the formalization in first order logic of the notions in this section. Notions such as polygon and limit involve quantification over integers; this is strictly forbidden within the first order system. We can approach these notions with axiom schemes. We want to argue that we can give a uniform metatheoretic definition of the relevant concepts and prove that the theorems hold in all models of the axioms. {areaax}

**Axiom 6.0.2** ( Area Axioms)**.** *The following properties*[18] *of area are used in Euclid I.35 and I.38.*

1. *Congruent figures have the same area.*
2. *The area of two 'disjoint' figures (i.e. meet only in a point or along an edge) is the sum of the two areas of the polygons.*
3. *Two figures that have equal content* [19] *have the same area.*
4. *If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.*

Observe that while these axioms involve notions that are not uniformly definable in first order geometry, we can replace 'figures' by $n$-gons for each $n$. For the

[18] We combine the versions in [22] and from pages 198-199 of [16]; the high school version restricted to polygons.

[19] The high school reads 'scissor-congruent' rather than 'equal content'; this relies on the assumption about the real numbers just before the statement of Postulates 3.3 and 3.4. That is, on Hilbert's argument [26] that for geometries over Archimedean fields, scissors-congruent and equal content are the same. Also for simplicity the text restricts to polygons.

crucial area of a triangle is proportional to the base and the height, we need only 'triangles or quadrilaterals'. In general we could formalize formalize these notions with either equi-area predicate symbols[20] or by a schema, or by a function mapping into the line as in Definition 6.0.6. Here is the basic step.

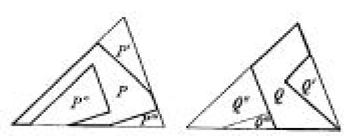
**Definition 6.0.3.** *Two figures $\alpha$ and $\beta$ (e.g. two triangles or two parallelograms) have equal content in one step there exist figures $\alpha'$ and $\beta'$ such that the disjoint union of $\alpha$ and $\alpha'$ is congruent to the disjoint union of $\beta$ and $\beta'$ and $\beta \cong \beta'$.*

Now, Euclid's I.35 and I.36 become. {areaprop}

**Theorem 6.0.4.** *[Euclid/Hilbert] If two parallelograms (triangles) are on the same base and between parallels they have equal content in 1 step.*

As Hilbert showed (and this is why[21] 'equal content' replaces 'decomposable') this theorem holds without the Archimedean axiom. Now for arbitrary figures:

**Definition 6.0.5** (Equal content)**.** *Two figures $P, Q$ have* equal content *in $n$ steps* [22] *if there are figures $P'_1 \dots P'_n$, $Q'_1 \dots Q'_n$ such that none of the figures overlap, each $P'_i$ and $Q'_i$ are scissors congruent and $P \cup P'_1 \dots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \dots \cup Q'_n$.*



Varying Hilbert, Hartshorne (Sections 19-23 of [23]) shows that these axioms for area are satisfied in the first order axiom system we label Euclidean geometry (Notation2.1.1). The key tool is: {areafn}

**Definition 6.0.6.** *An* area function *is a map $\alpha$ from the set of figures, $\mathcal{P}$, into an ordered additive abelian group with* 0 *such that*

1. *For any nontrivial triangle $T$, $\alpha(T) > 0$.*
2. *Congruent triangles have the same area.*
3. *If $P$ and $Q$ are disjoint figures $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.*

[20] For example, we could have 8-ary relation for quadrilaterals have the same area, 6-ary relation for triangles have the same area and 7-ary for a quadrilateral and a triangle have the same area.
[21] After Theorem 27 in [26].
[22] The diagram is taken from [26].

This formulation hides the quantification over arbitrary $n$-gons. We clarify the method of translating to first order in Definition 7.0.5.

It is evident that if a plane admits an area function then the area axioms hold. This obviates the need for positing separately De Zolt's axiom that one figure properly included in another has smaller area[23] In particular this implies Common Notion 4 for 'area'. Using the segment multiplication, Hilbert (compare the exposition in Hartshorne) establishes the existence of an area function for any plane satisfying HP5. The key point is to show that formula $A = \frac{bh}{2}$ does not depend on the choice of the base and height.

[23] Hartshorne notes that (page 210 of [22]) that he knows no 'purely geometric' (without segment arithmetic and similar triangles) proof for this.

# 7 $\pi$

{pi}

Dedekind (page 37-38) observes that what we would now call the real closed field based on the field of real algebraic numbers is 'discontinuous everywhere' but 'all constructions that occur in Euclid's elements can ... be just as accurately effected as in a perfectly continuous space'. Strictly speaking, for *constructions* this is correct. But the proportionality constant between a circle and its circumference $\pi$ is absent, so even more not both the circumference and the diameter are in the model. This absence emphasizes the awareness of the authors of the Common Core State Standards for Mathematics[24] (that students should be required to understand as primitive terms both 'length' and 'arc length' [25]). We want to find countable models where 'circles behave properly'.

Euclid's construction axiom (Postulate 2.3.4) implies that a circle is uniquely determined by its radius and center. And thus two segments of a circle are congruent if they cut the same central angle. As the example of geometry over the real algebraic numbers shows, there is no guarantee that there is a straight line segment whose 'length' is $\pi$. We remedy this with the following extension of the system.

To avoid complications, we restrict our discussion of 'arc length' to circles and straight lines with the following notation.

**Notation 7.0.1.** *$L(A, B, C, D)$ is a predicate of two pairs of points. Each of $A, B$ and $C, D$ must be either collinear or two points on a circle. We read $L(A, B, C, D)$ as $AB$ and $CD$ are same length.*

*We will use the capitalized word* Line *to mean either a straight line or a circle.*

Without writing out the formalities we use the notions of greater and lesser length[26]. We adapt the following postulates from Archimedes article, On the sphere and the circle I [3]. We have restricted from his more general axioms covering surfaces and more general curves since we aim only at the circle.

**Postulate 7.0.2** (Length).

1. *If two Line segments are congruent they have same length.*
2. *A straight line is the shortest distance between two points.*
3. *If $A$ and $B$ lie on a circle and $C$ is exterior to the circle then the sum of $BC$ and $AC$ is greater than $AB$.*

[24] G-CO.1 [29]: Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

[25] Thus we reject the Birkhoff-Moise 'ruler' and 'protractor' axioms which insert the second order theory of the reals as a subset of the axioms of high school geometry.

[26] The key idea is that one circular segment is less than another if it has the same length as a straight line segment that is properly contained a segment the same length as the other.

{defpiax}

**Definition 7.0.3** (Axioms for $\pi$). *Add to the vocabulary a new constant symbol $\pi$. Let $i_n$ ($c_n$) be length of a side of a regular $n$-gon inscribed (circumscribed) in a circle of radius $1$. Add for each $n$,*

$$i_n < 2\pi < c_n$$

*to give a collection of sentences $\Sigma(\pi)$.*

At this point we need some modern model theory. A first order theory for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent to a Boolean combination of equalities and inequalities [12]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [34]. {piax}

**Theorem 7.0.4.** *The following set $T_\pi$ of axioms is first order complete for the vocabulary $\tau$ in Notation 2.2.2 along with the constant symbols $0, 1, \pi$.*

1. *the axioms of a Euclidean plane.*
2. *A family of sentences declaring every odd-degree polynomial has a root.*
3. $\Sigma(\pi)$

Proof. We have established that there is a well-defined field multiplication on any line. By Tarski, the theory of this multiplication is complete. The field is bi-interpetable with the plane so the theory of the geometry $T$ is complete as well. Further by Tarski, the field is o-minimal. Thus the type of any point over the empty set is determined by its position in the ordered subfield of rational numbers. This position is given for $\pi$ by $\Sigma$. Thus $T \cup \Sigma$ is complete. $\square_{7.0.4}$

While we have not used the axiom of Archimedes in the object language, the assertion that $\Sigma$ is a complete type invokes the fact that $\lim i_n = \lim c_n$ which holds because the reals are archimedean.

Now we want to argue that $\pi$ as implicitly defined by the theory $T_\pi$ serves its geometric purpose. For this, we add a new unary function symbol $C$ mapping our fixed line to itself and satisfying the following scheme asserting that for each $n$, $C(r)$ is between the perimeter of a regular inscribed $n$-gon and a regular circumscribed $n$-gon. {circfn}

**Definition 7.0.5.** *Consider the following properties of a unary function $C(r)$.*

$\iota_n$ *For any points $P, X_1, \ldots X_n$ such that all the segments $PX_i$ are congruent with length $r$, and all the segments $X_iX_{i+1}$ (including $X_nX_1$) are congruent, the sum $i_n(r)$ of the lengths of the segments $X_iX_{i+1}$ (including $X_nX_1$) is less than $C(r)$.*

$\gamma_n$ *For any points $P, X_1, \dots X_n, Y_1, \dots X_n$ such that all $PX_i$ are congruent with length $r$, $Y_i$ is in the exterior of the circle, $X_i$ is the midpoint of $Y_iY_{i+1}$, and all $Y_iY_{i+1}$ (including $Y_nY_1$) are congruent the sum $c_n(r)$ of the lengths of the segments $Y_iY_{i+1}$ (including $Y_nY_1$) is greater than $C(r)$.*

*Any function $C(r)$ satisfying these axioms is called* a circumference function*, we call $C(r)$ the* circumference *of a circle with radius $r$.* {circumfn}

**Definition 7.0.6.** *The theory $T_{\pi,C}$ is the extension by definitions of the $\tau \cup \{0, 1, \pi\}$-theory $T_\pi$ obtained by the explicit definition $C(r) = 2\pi r$.*

As an extension by explicit definition, $T_{\pi,C}$ is complete. Since by similarity $i_n(r) = ic_n$ and $c_n(r) = rc_n$, the approximations of $\pi$ by Archimedes and our definition of $T_\pi$ make the following metatheorem immediate. {circform}

**Theorem 7.0.7.** *In $T_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function.*

In an Archimedean field there is a unique interpretation of $\pi$ and thus a unique choice for a circumference function with respect to the vocabulary without the constant $\pi$. By adding the constant $\pi$ to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad of $2\pi r$ would equally well fit our condition for being the circumference.

In order to consider the area of a circle we extend the impact of the Area Axioms in Axiom 6.0.2 by expanding a notion of *figure*. We called the expanded class with a capital $F$.

**Definition 7.0.8.** *A* Figure *is a figure or a sector of a circle. That is, either a sector of circle or a subset of the plane that can be represented as a finite union of disjoint triangles.*

In analogy with Definition 7.0.3 define $I_n$ and $C_n$ as the area of the regular n-gon inscribed or circumscribing the unit circle.

Now, as in the segment case, by formalizing a notion of equal area, including a schema for approximation by finite polygons, we can externally get a formal area function $A(r)$. {circarea}

**Theorem 7.0.9.** *The area of a circle is $A(r) = \pi r^2$.*

Note that if we carry out the procedure independently for circumference and area in non-Archimedean models we can get different values of $\pi$. We can fix this by combining the types or by working in infinitary logic to allow Archimedes.

We omit the tedious inductive argument that any figure contained in a circle is contained in a polygon with vertices the center and points on the circle. Now Archimedes (Proposition 1 of The measurement of a circle) shows us that under our definition.

**Theorem 7.0.10.** *If the supremum of the areas of the figures that are contained in a circle $C$ is equal to the infinum of the areas of the figures that contain a circle $C$ then that common number is the area of $C$.*

The next task is to assign a measure to each angle. Recall that we have a field structure on the line through $01$.

**Definition 7.0.11.** *A* measurement of angles *function is a map $\mu$ from congruence classes of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side $BC$, $\mu(\angle ABD) = \angle ABC + \angle CBD$*

If we omitted the additivity property this would be trivial: Given an angle $\angle ABC$ less than a straight angle, let $C'$ be the intersection of a perpendicular to $AC$ through $B$ with $AC$ and let $\mu(\angle ABC) = \frac{BC'}{AB}$. (It is easy to extend to the rest of the angles.) To obtain the additivity, we proceed as follows.

**Definition 7.0.12.** *Normalize by making $B$ the center and $A, C$ points on a unit circle. Define $\mu(\angle ABC)$ to be twice the area of the sector of the circle $ABC$.*

(For motivation, note that the area of a quarter circle is $\pi/4$ and in normal usage a right angle is $\pi/2$ radians.)

The additivity of area gives the additivity of $\mu$ and Theorem 7.0.9 tells us the range is in $[0, 2\pi)$.

But, *the definition may be essentially vacuous*; we know for example that in the plane over the real algebraic numbers a right angle does not have a measure.

**Theorem 7.0.13.** *For every countable model $M$ of $T_{\pi,C}$, there is a countable model $M'$ containing $M$ such that $M' \models$ '$\mu$ is onto'.*

Proof. We can adapt Archimedes proof finding the area of a circle to that of a sector by building polygons inscribed and circumscribing the sector (also using the two radii as two sides, but choosing new points to refine the approximation by bisecting each central angle at each stage). By Axiom 6.0.2.4, the common limit of the areas of the exterior and interior polygons is the area of the sector. As in the proof of Theorem 7.0.4, we obtain the area as a type over the emptyset in $T_\pi$. Now any choice for the area must realize that cut.

Now proceed inductively, Given a model $N$, let $N'$ be a countable elementary extension of $N$ realizing all the, countably many, angle measure cuts in $N$. Now proceed inductively, let $M_0 = M$ and $M_{n+1} = M'_n$. Then $M_\omega$ is required model where $\mu$ is onto.

Here is a sketch. Define a function $AL(A, B, r, P)$ which is intended to be the arc length for between $A$ and $B$ on the circle with center $P$ and radius $r$. Demand that for each $n$ the value is between that of n-segments from $A$ to $B$ all inside the circle and

$n$-segments external to the circle. Then the measure of the angle $BPA$ is $\frac{AL(A,B,r,P)}{2\pi r}$. This is unsatisfactory in that I am realizing many types but no more than the number of points in the model so it seems I could get a countable model that is sufficiently saturated.

Euclid XII.2 argued (cf. Proposition 25.1 of [23]) that the ratio of the area of a circle to square of its radius is bounded above and below by the $i_n$ and the $c_n$. Our theory has guaranteed the existence of a number satisfying these conditions. But since the field is Archimedean, there is at most one such number.

If we want to ground *right angle trigonometry* we replace $\Sigma$ by $\Sigma_1 = \mathrm{Th}(\Re, \sin\restriction(-\pi,\pi], \pi)$ which is o-minimal by Van Den Dries [12]. The axioms here are extremely complicated (involving the Weierstrauss preparation theorem)[27]. So meeting the goal of an 'elementary' in the informal sense foundation is dubious. Note that we can't define $\sin$ globally by this technique as $\mathrm{Th}(\Re, \sin)$ is patently not o-minimal.

# 8 Conclusion

We have expounded a procedure [22] to define the field operations in an arbitrary Euclidean plane. In the steps of Hilbert, we show related algebraic arguments show that with this definition of multiplication the square root function behaves properly on the field defined in any such plane. More important this notion of multiplication gives a meaning to 'proportion' that with no appeal to approximations implies that the sides of similar triangles have proportional length. In particular, this result applies to the real numbers. However, the existence of the reals in the usual sense requires some analog of the Dedekind construction to show that each cut in the rationals is realized exactly once. This argument also avoids Euclid's notorious 'detour' through area. Hilbert's construction also establishes the theory of area for polygons in any Euclidean field and in strong form by avoiding DeZolt's axiom.

By adding axioms to require the field is real closed we obtain a complete first order theory. The plane over the real algebraic numbers satisfies this theory; thus, there is no guarantee that there is a line segment of length $\pi$. Using the o-minimality of real closed fields, we can guarantee there is such a segment by adding a constant for $\pi$ and requiring it to realize the proper cut in the rationals. However, guaranteeing the uniqueness of such a realization requires second order axioms. The exploration of analysis is a further project. Note that by the categoricity of $ACF_0$, any model of $ACF_0$ with cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}$ and so admits a topological structure isomorphic to $\mathbb{C}$. Thus in a sense we have a discrete axiomatization of the continuous structure. There is no $2^{\aleph_0}$-categorical axiomatization of $\Re$ so some extension is necessary. However, it may be that something weaker than a categorical axiomatization will suffice. Tarski

[27]See [37]; the theorem is proved for restricted sin and restricted exponentiation. Similar methods and in particular later work by Gabrielov allow the reduct to just restricted $\sin$.

[20] suggests a 'continuity schema' analogous to that for first order Peano. Peterzil and Starchenko approach the complex case through o-minimality of the real part [32]. Finally, D'aquino, Knight, and Starchenko [10] connect nonstandard models of first order Peano with appropriate models of $RCF$.

In a sense, our development is the opposite of that in [17]. Rather than trying to unify all numbers great and small, we are interested in the minimal collection of numbers that allow the development of a geometry according with our fundamental intuitions.

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