Maximal models up to the first measurable in 
ZFC

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In this paper we prove in ZFC the existence of a complete sentence \(\phi\) of \(L_{\omega_1,\omega}\) such that \(\phi\) has maximal models in a set of cardinals \(\lambda\) that is cofinal in the first measurable \(\mu\) while \(\phi\) has no maximal models in any \(\chi \geq \mu\). In [BS17a], we proved a theorem with the same conclusion as the main result here; the earlier proof required that \(\lambda = \lambda^{<\lambda}\), and that there is an \(S \subseteq S_\lambda^{<\lambda}\) that is stationary non-reflecting, and \(\diamond S\) holds. Here, we show in ZFC that the sentence \(\phi\) defined in [BS17a] has maximal models cofinally in \(\mu\). The additional hypotheses in [BS17a] allow one to demand that if \(N\) is a submodel with cardinality \(< \lambda\) of the \(P_0\)-maximal model, \(N\) is \(K_1\)-free (Definition 2.2); that property fails for the example here. The existence of such a \(\phi\) which is not complete is well-known (e.g. [Mag16]).

This paper contributes to the study of Hanf numbers for infintary logics. Works such as [BKS09, BKS16, BS17b, KLH16] study the spectrum of maximal models in the context where the class has a bounded number of models. We list now some properties that are true in every cardinality for first order logic but are true only eventually for complete sentences of \(L_{\omega_1,\omega}\) or, more generally, for abstract elementary classes, and compare the cardinalities (the Hanf number) at which the eventual behavior must begin. Every model of a first order theory has a proper elementary extension and so each theory has arbitrarily large models. Moreover, the amalgamation property holds for every complete first order theory. Morley [Mor65] showed that every sentence of \(L_{\omega_1,\omega}\) that has models up to \(\beth_1\) has arbitrarily large models and provided counterexamples showing that cardinal was minimal. Thus he showed the Hanf number for existence of \(L_{\omega_1,\omega}\) is \(\beth_1\). Hjorth [Hjo02], by a much more complicated argument, showed there are complete sentences \(\phi_\alpha\) for \(\alpha < \omega_1\) such that \(\phi_\alpha\) has a model in \(\beth_\alpha\) and no larger so the Hanf number for complete sentences is \(\beth_\omega\). Boney and Unger [BU17], building on [She13] show that the Hanf number ‘for all AEC’s are tame’ is the first strongly compact cardinal. They also show the analogous property for various variants on tameness is equivalent to the existence of almost (weakly compact, measurable, strongly

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compact). The result here shows in ZFC that the Hanf number for extendability (every model of a complete sentence has a proper $L_{\omega_1,\omega}$-elementary extension) is the first measurable cardinal. However, [BB17] show that an upper bound on the Hanf number for amalgamation is the first strongly compact. The actual value remains open.

The first section of the paper defines the class of models $K_{-1}$ and explains the connections with [BS17a]. In Section 2 we construct in ZFC, for cofinally many $\lambda$ less than the first measurable, a $P_0$-maximal model $M_\ast \in K_{-1}$. Subsection 2.1 is a set theoretic argument for the existence of a Boolean algebra with certain specified properties in any cardinal $\lambda$ of the form $\lambda = 2^\mu$ that is less than the first measurable; this construction is completely independent of the model theoretic notation established in Section 1. Subsection 2.2 builds on this result to find a $P_0$-maximal model in $K_{-1}$ with cardinality $\lambda$ satisfying certain further restrictions. Finally in Section 3, this model is converted to the maximal model of $K_2$, the class of model of the complete sentence $(\phi)$ from [BS17a].

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1 Preliminaries

We include as needed definitions of the classes of model $K_{-1}, K_1, K_2$ introduced in [BS17a]. For each $i$, $K_{<\aleph_0}^i$ denotes the class of finitely generated members of $K_i$. Occasionally we fall into the notation $\hat{K}$ for the direct limits of a class $K$ of finitely generated models.

**Definition 1.1** $\tau$ is a vocabulary with unary predicates $P_0, P_1, P_2, P_4$, binary $R, E, \land, \lor, \leq$ unary functions $\neg, G_1, H_1$, $n$ unary functions $g_{n,i}$ for each $n$, constants $0, 1$ and unary functions $F_n$ for $n < \omega$. $\leq$ is a partial order on $P_1^M$ and the Boolean algebra can be defined from it.

We occasionally use the notations $(\forall^\infty n)$ and $(\exists^\infty n)$ to mean ‘for all but finitely many’ and ‘for infinitely many’ respectively. It is easy to see that $K_{-1}$ is $L_{\omega_1,\omega}$-axiomatizable but far from complete. We denote the power set of $X$ by $\mathcal{P}(X)$.

**Definition 1.2** $(K_{-1})$ $M \in K_{<\aleph_0}^{\neg 1}$ is the class of finitely generated structures $M$ satisfying the following conditions.

Note that $b_*$ is not a function symbol in $\tau$.

1. $P_0^M, P_1^M, P_2^M$ partition $M$.

2. $(P_1^M, 0, 1, \land, \lor, \leq, \neg)$ is a Boolean algebra ($\neg$ is complement). We may also write $B_M$ or $B[M]$ for $P_1^M$. We also consider ideals and restrictions to them of the relations/operations except for complement.

3. $R \subset P_0^M \times P_1^M$ with $R(M, b) = \{ a : R^M(a, b) \}$ and the set of $\{ R(M, b) : b \in P_1^M \}$ is a Boolean algebra. $f^M : P_1^M \to \mathcal{P}(P_0^M)$ by $f^M(b) = R(M, b)$ is a Boolean algebra homomorphism into $\mathcal{P}(P_0^M)$. 

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Note that $f$ is not in $\tau$; it is simply a convenient abbreviation for the relation between the Boolean algebra $P^M_1$ and the set algebra on $P_0$ by the map $b \mapsto R(M, b)$.

4. $P^M_{4,n}$ is the set containing each join of $n$ distinct atoms from $M$; $P^M_4$ is the union of the $P^M_{4,n}$ and so is an ideal. That is, $P^M_4$ is the set of all finite joins of atoms.

There is an element $b^* \in P^M_1$ and for each $n$, $P^M_{4,n} = \{c : c \leq^M b^*\}$.

5. $G^M_1$ is a bijection from $P^M_0$ onto $P^M_{4,1}$ such that $R(M, G^M_1(a)) = \{a\}$. (Note that $P^M_0 = \emptyset$ is allowed.)

6. $P^M_2$ is finite (and may be empty). Further, for each $c \in P^M_2$ the $F^M_n(c)$ are functions from $P^M_{4,n}$ into $P^M_1$. Note that it is allowed that for all but finitely many $n$, $F^M_n(c) = 0_{P^M_1}$.

7. If $a \in P^M_{4,1}$ and $c \in P^M_2$ then $(\forall^\infty n) a \neq^M F^M_n(c)$. This implies $\bigcap_n \{x : (G^M_1(x) \in F^M_n(c))\} = \emptyset$.

8. $P^M_4$ is generated as a Boolean algebra by $P^M_{4,1} \cup \{F^M_n(c) : c \in P^M_2, n \in \omega\} \cup X$ where $X$ is a finite subset of $P^M_1$.

\textbf{Definition 1.3} 1. $K_{-1}$ is the class of $\tau$ structures $M$ such that every finitely generated substructure of $M$ is in $K_{-1}$. $K^\mu_{-1}$ is the members of $K_{-1}$ with cardinality $\mu$.

2. We say $M \in K_{-1}$ is atomic if $P^M_4$ is atomic as a Boolean algebra. That is, $P^M_4$ is dense in $B^M_M$.

\section{The first approximation}

In this section we construct in ZFC, for cofinally many $\lambda$ less than the first measurable, a $P_0$-maximal model $M_* \in K_{-1}$. In Section 3 we 'correct' that model to a model of the complete sentence $\phi$ of $L_{\omega \cdot 1, \omega}$ defined in [BS17a]. Subsection 2.1 can be read completely independently; it has no reliance on Section 1. In particular, there is no requirement here that the $A_n$ are independent.

\subsection{Set theoretic construction of a Boolean algebra}

The goal of this subsection is to prove the property $\boxplus$ in ZFC. The class $K_{-1}$ plays no role in section. The arguments here are similar to those around page 7 of [GS05]. In Subsection 2.2 we deduce Theorem 2.2.4 from $\boxplus$, showing there is a nicely free $P_0$-maximal (Definition 2.1) model in $K_{-1}$.

\textbf{Definition 2.1.1 ($\boxplus(\lambda)$)} denotes: There are a Boolean algebra $\mathbb{B} \subset P(\lambda)$ with $|\mathbb{B}| = \{\boxplus\}$ and a set $A \subseteq \omega \mathbb{B}$ such that:

\footnote{The subsets of $P^M_0$ are not elements of $M$.}
i) \( \mathcal{A} \) has cardinality \( \lambda \) and if \( \mathcal{A} = \{ A_n : n \in \omega \} \in \mathcal{A} \) then for \( \alpha < \lambda \) for all but finitely many \( n \), \( \alpha \notin A_n \).

ii) \( \mathcal{B} \) includes the finite subsets of \( \lambda \); but is such that for every non-principal ultrafilter \( D \) of \( \lambda \) (equivalently of \( \mathcal{B} \) and disjoint from \( \lambda^{<\omega} \)) for some sequence \( \langle A_n : n \in \omega \rangle \in \mathcal{A} \), there are infinitely many \( n \) with \( A_n \in D \).

We may say that \( (\mathcal{B}, \mathcal{A}) \) witness uniform \( \aleph_1 \)-incompleteness.

Lemma 2.1.2 (ZFC) Assume for some \( \mu \), \( \lambda = 2^\mu \) and \( \lambda < \) first measurable, then \( \Box (\lambda) \) from 2.1.1 holds.

We need the following structure.

Definition 2.1.3 1. Fix the vocabulary \( \sigma \) with unary predicates \( P,U \), a binary predicate \( C \), and a binary function \( F_2 \).

2. Let \( \langle C_\alpha : \alpha < \lambda \rangle \) list without repetitions \( \mathcal{P}(\mu) \) such that \( C_0 = \emptyset \) and also let \( \langle f_\alpha : \mu \leq \alpha < \lambda \rangle \) list \( \mu^{\omega} \).

3. Define the \( \sigma \)-structure \( M \) by:

   (a) The universe of \( M \) is \( \lambda \);

   (b) \( P^M = \omega; \ U^M = \mu \);

   (c) \( C(x,y) \) is binary relation on \( U \times M \) defined by \( C(x,\alpha) \) if and only \( x \in C_\alpha \). Note that \( C \) is extensional. I.e., elements of \( M \) uniquely code subsets of \( U^M \).

   (d) Let \( F_2^M(\alpha,\beta) \) map \( M \times U^M \rightarrow P^M \) by \( F_2^M(\alpha,\beta) = f_\alpha(\beta) \) for \( \alpha < \lambda \), \( \beta < \mu \);

   (e) \( F_2^M(\alpha,\beta) = 0 \) for \( \alpha < \lambda \) and \( \beta \in [\mu,\lambda) \).

We use the following, likely well-known, fact pointed out to us by Sherwood Hachtman.

Fact 2.1.4 Let \( D \subseteq \mathcal{P}(X) \) and suppose that for each partition \( Y \subseteq \mathcal{P}(X) \) of \( X \) into at most countably many sets, \( |D \cap Y| = 1 \). Then, \( D \) is a countably complete ultrafilter.

We need the following lemma about \( M \) before finding in \( M \) a representative of \( \exists \).

Lemma 2.1.5 If \( \lambda \) is less than the first measurable cardinal and \( \lambda = 2^\mu \) for some \( \mu \) there is a model \( M \), with \( |M| = \lambda \), and a countable vocabulary with \( P^M \) denoting the natural numbers such that every first order proper elementary extension \( N \) of \( M \) properly extends \( P^M \).

Proof. Fix \( M \) as in Definition 2.1.3. We first show that any proper elementary extension \( N \) of \( M \) extends \( U^M \). Suppose for contradiction there exists \( \alpha' \in N - M \) but \( U^N = U^N \).
\[ C(x, \beta) = \{ x : N \models C(x, \alpha') \}. \] This contradicts extensionality of the relation \( C \) in \( N \); but \( C \) is extensional in the elementary submodel \( M \).

Now we show that if \( U^M \subseteq U^N \) and \( P^M \neq P^N \), then there is a countably complete non-principal ultrafilter on \( \mu \), contradicting that \( \mu \) is not measurable. Note that the sequence \( \langle f_\alpha : \mu \leq \alpha < \lambda \rangle \) can be viewed as a list of all non-trivial partitions of \( \mu \) into at most countably many pieces. Let \( \nu^* \in U^N - U^M \). For \( \alpha \in N \), denote \( F_2^N(\alpha, \nu^*) \) by \( n_\alpha \). Since \( P^M = P^N \), \( n_\alpha \in M \). By elementarity, for \( \alpha \in M \), \( \eta \in U^M \), \( F_2^M(\alpha, \eta) = F_2^M(\alpha, \eta) = f_\alpha(\eta) \). Now, let

\[ D = \{ x \subseteq U^M : x \neq \emptyset \land (\exists \alpha \in M) x \cupgeq f_\alpha^{-1}(n_\alpha) \}. \]

We show \( D \) satisfies the conditions from Fact 2.1.4. Let \( W \) be a partition, indexed by \( f_\alpha \). Then \( f_\alpha^{-1}(n_\alpha) \neq \emptyset \) and is in \( D \). Suppose for contradiction there are \( x_0 \neq x_1 \) in \( W \) that are both in \( D \). Then, there are \( \alpha_i \in M \) such that \( x_i \in W \cap D \) contains \( f_\alpha^{-1}(n_\alpha) \) for \( i = 0, 1 \). So, \( N \models F_2(\alpha_i, \nu^*) = n_\alpha \) for \( i = 1, 2 \). Since \( \alpha_i \in M \) and \( M \prec N \), \( M \models \exists x (F_2(x, \alpha_0, x) = n_{a_0} \land F_2(x, \alpha_1, x) = n_{a_1}) \). So, by Definition 2.1.3 (d), for any witness \( a \) in \( M \) for this formula, \( a \in x_0 \cap x_1 \); but \( x_0 \cap x_1 = \emptyset \) since \( W \) is a partition.

Finally, \( D \) is non-principal on \( U^M \) since if it were generated by an \( a \in U^M \),

\[ D = \{ x \subseteq U : (\exists \alpha \in M) x \cupgeq f_\alpha^{-1}(n_\alpha) \} = \{ x \subseteq U : a \in x \}. \]

Since \( \{ a \} \in D \), for some \( \alpha_0 \in M \), \( \{ a \} = f_\alpha^{-1}(n_{\alpha_0}) \). Note that \( \alpha_0 \in M \), because the definition of \( D \) is about the model \( M \). That is, \( M \models \exists y F_2(\alpha_0, y) = n_{\alpha_0} \). But \( N \models F_2(\alpha_0, a) = n_{a_0} \land F_2(\alpha_0, \nu^*) = n_{a_0} \). This contradicts the assumption \( M \prec N \) and completes the proof. \( \square_{2.1.5} \)

The following claim completes the proof of Lemma 2.1.2

**Claim 2.1.6** If \( B \) is the boolean algebra of definable formulas in the \( M \) defined in Definition 2.1.3, there is an \( A \) such that \( (B, A) \) is uniformly \( \aleph_1 \)-incomplete so \( \equiv(\lambda) \) holds.

Proof. We may assume \( \tau \) has Skolem functions for \( M \) and then define \( B \) and \( A \) as follows to satisfy \( \equiv(b) \). Let \( B \) be the Boolean algebra of definable subsets of \( M \). I.e.,

\[ B = \{ X \subseteq M : \text{for some } \tau \text{-formula } \phi(x, y) \text{ and } b \in \text{lg}(y)M, \phi(M, b) = X \} \]

Note \( B \) is a Boolean algebra of cardinality \( \lambda \) with the normal operations. We define the Skolem functions a little differently than usual; as maps \( \sigma_\phi \) from \( M^{n+1} \) to \( M \) for formulas \( \phi(x, w, y) \) such that \( \phi(\sigma_\phi(x, w, y), b, a) \). Then, we specialize the Skolem functions by considering the unary function arising from fixing the \( y \) entry of \( \sigma_\phi(w, y) \) to obtain \( \sigma_\phi(w, a) \).

\[ A_{n}^{\phi(x, w, a)} = \{ \alpha < \lambda : \phi(\sigma_\phi^M(\alpha, a), \alpha, a) \land P(\sigma_\phi^M(\alpha, a)) \land \sigma_\phi^M(\alpha, a) \neq n \} \]

\[ \cup \{ \alpha < \lambda : n = 0 \land \neg P(\sigma_\phi^M(\alpha, a)) \}. \]
and then let $\overline{A}_{\sigma_\phi(x,w,a)} = \{A_n^{\sigma_\phi(w,a)} : n < \omega\}$ and

\[(*) \quad A = \{\overline{A}_{\sigma_\phi(x,a)} : \text{for some } \tau_M - \text{term } \sigma_\phi(x,w,y) \text{ and } a \in \text{lg}(y)M.\}\]

Note $|A| = \lambda$ as for each $a \in M$ and each of the countably many terms $\sigma_\phi(x,w,a)$. $\overline{A}_{\sigma_\phi(x,w,a)}$ is a map from $\omega$ into $\mathbb{B}$.

For each $\alpha$, for each $0 < m < \omega$ and $\overline{A} = \overline{A}_{\sigma_\phi(x,a,b)}$, the set $\{m : \alpha \in A_m\}$ is finite, bounded by $\sigma_\phi(\alpha,a)$. Thus, clause i) of $\mathbb{H}$ is satisfied.

We now show Clause ii) of $\mathbb{H}$. Let $D$ be an arbitrary non-principal ultrafilter on $\lambda$ and where $\psi(v,y)$ is a first order $\tau$-formula such that $y$ and $a$ have the same length, define the type $p_D(x) = p(x)$ as:

$$p(x) = \{\psi(v,a) : \alpha \in M : M \models \phi(\alpha,a) \in D\}.$$

Since $D$ is an ultrafilter, $p$ is a complete type over $M$. So there is an elementary extension $N$ of $M$ where an element $d$ realizes $p$. Let $N$ be the Skolem hull of $M \cup \{d\}$.

Since $D$ is non-principal, so is $p_D$; thus, $N \neq M$. By Lemma 2.1.5, we can choose $c \in P^N - P^M$. Since, $N$ is the Skolem hull of $M \cup \{d\}$ there is a Skolem term $\sigma(u,y)$ and $a \in M$ such that $c = \sigma^N(d,a)$. Since $c \notin M$, for each $n \in P^M$, $N \models \bigwedge_{k<n} c \neq k$ so $N \models \bigwedge_{k<n} \sigma(d,a) \neq k$ so $\bigwedge_{k<n} \sigma(x,a) \neq k$ is in $p$. That is, for each $n$, $A_n^{\sigma_\phi(x,w,a)}$ is in $D$.

### 2.2 A $P_0$-maximal model in $K_{-1}$

In this section we prove Theorem 2.2.4, invoking Theorem 2.1.2. To even state the new result, we need some new definitions as well as recalling Definition 1.2.7.

**Definition 2.1** We say $M \in K_{-1}$ is $P_0$-maximal (in $K_{-1}$) if $M \subseteq N$ and $N \in K_{-1}$ implies $P_0^N = P_0$.

We now introduce the requirement that the Boolean algebras constructed will, when the atoms are factored out, be free. Moreover, different $c \in P_2^N$ generate disjoint collections of $F^N_n(c)$ as $c$ varies. This strong requirement is used inductively in this section to construct the first approximation. The correction in Section 3 loses this disjointness (and thus freeness).

**Definition 2.2 (Nicely Free)** We say $M \in K_{-1}$ is nicely free when $|P_1^M| = \lambda$ and there is a sequence $\mathbf{b} = \{b_\alpha : \alpha < \lambda\}$ such that

(a) $b_\alpha \in P_1^M - P_4^M$,

(b) $\{b_\alpha/P_4^M : \alpha < \lambda\}$ generate $P_1^M/P_4^M$ freely;

(c) there is a set $Y \subseteq P_2^M$ of cardinality $\lambda$ and a sequence $\{u_c : c \in Y\}$ of pairwise disjoint sets of distinct ordinals such that, with $u_c = \{F^M_n(c) : n < \omega\}$ the collection of $u_c$ partitions a subset of the basis $\{b_\alpha : \alpha < \lambda\}$.
Definition 2.2.1 (uf\((M)\)) Let \(M \in K_{-1}\), let uf\((M)\) be the set of ultrafilters \(D\) of the Boolean Algebra \(P^M_1\) such that \(D \cap P^M_{4,1} = \emptyset\) and for each \(c \in P^M_{2}\) only finitely many of the \(F^M_n(c)\) are in \(D\).

For applications we rephrase this notion with the following terminology. For any \(M \in K_{-1}\) and \(d \in P^M_{2}\), let \(S^M_d(D) = \{ n : F^M_n(d) \in D \}\). So uf\((M)\) = \(\emptyset\) if and only if for every ultrafilter \(D\) on \(P^M_{1}\), there exists a \(d \in P^M_{2}\) such that \(S^M_d(D)\) is infinite.

We use the following standard properties of a Boolean algebra \(B\) and ideal \(I\) in proving Lemma 2.2.3 and Claim 2.2.7 from Definition 2.2.6.

Fact 2.2.2
1. \(b \wedge c \in I\) implies \(b/I\) and \(c/I\) are disjoint.
2. \(b \wedge c \in I\) implies \(b/I = c/I\).
3. \(b \wedge c \in I\) implies \(b/I \leq c/I\).

For our collection of structures \(K_{-1}\), we can characterize \(P_0\)-maximality in terms of ultrafilters.

Lemma 2.2.3 An \(M \in K_{-1}\) is \(P_0\)-maximal if and only if uf\((M)\) = \(\emptyset\).

Proof. Suppose \(M \subset N\) with \(N \in K_{-1}\) and \(d \in P^N_0 - P^M_0\). Then \(\{ b \in M : R^N(d, b) \}\) is an ultrafilter \(D_0\) of the Boolean algebra \(P^M_1\). To see \(D_0\) is non-principal suppose there is a \(b_0 \in P^M_1\) such that \(D_0 = \{ b \in M : b_0 \leq b \}\). Note \(b_0 = G^M_1(a)\) for some \(a \in P^M_0\). But \(N = G^N_1(d) \not\supset b_0\), contradicting \(\{ d \} \in D\).

For each \(c \in P^M_2\), since \(N \in K_{-1}\), by clause 7 of Definition 1.2, for all \(a \in P^N_0\) and all but finitely many \(n\), \(G^N_1(a) \subseteq F^N_n(c)\). Since \(F^N_n(c) = F^M_n(c)\), only finitely many of the \(F^M_n(c)\) can be in \(D_0\), which implies \(D_0 \in \text{uf}(M)\). By contraposition we have the right to left.

Conversely, if \(D \in \text{uf}(M)\), we can construct an extension by adding an element \(d \in P^N_0\) satisfying \(R^N(d, b)\) iff \(b \in D\). Let \(P^N_1\) be the Boolean algebra generated by \(P^M_1 \cup \{ G^N_1(d) \}\) modulo the ideal generated by \(\{ G^N_1(d) - b : b \in D\}\); this implies that in the quotient \(G^1_1(d) \leq b\). (Compare Fact 2.2.2). Let \(P^2_N = P^M_2\) and \(F^n_n(c) = F^n_M(c)\). Since \(D \in \text{uf}(M)\), it is easy to check that \(N \in K_{-1}\). \(\square_{2.2.3}\)

Here is the main theorem of Section 2. The hypotheses \(\lambda = 2^\mu\) and \(\lambda\) is less than the first measurable cardinal were used essentially as the hypotheses for proving \(\exists\), the existence of a uniformly \(\aleph_1\)-incomplete boolean algebra. But here we use \(\exists\) and don’t rely again on these cardinal hypotheses. The argument here depends on \(\lambda = \lambda^{\aleph_0}\), which follows from \(\lambda = 2^\mu\). Recall Definition 2.1 of \(P_0\)-maximal. By constructing a nicely free model, we introduce at this stage the independence requirements, needed in Section 3 to satisfy Definition 3.1.6, on the \(F^M_n(c)\).

Theorem 2.2.4 If for some \(\mu, \lambda = 2^\mu\) and \(\lambda\) is less than the first measurable cardinal then there is a \(P_0\)-maximal model \(M\) in \(K_{-1}\) such that \(|P^M_i| = \lambda\) (for \(i = 0, 1, 2\)), \(P^M_1\) is an atomic Boolean algebra, uf\((M)\) = \(\emptyset\), and \(M\) is nicely free.
Proof. We first construct by induction a model in $K^{-1}$. The hypothesis $\Xi$ appears in the construction in Specification f) and in the proof that the construction works in considering possibility 2. We choose $M$, $D$, and other auxiliaries by induction for $\epsilon \leq \omega + 1$ to satisfy the following specifications of the construction.

**Construction 2.2.5 (Specifications)**

(a) For $\epsilon \leq \omega + 1$, $M_\epsilon$ is a continuous increasing chain of members of $K^{-1}$ with each $P^M_1$ atomic and $P^M_{\omega + 1} = P^M_1$.

(b) For all $\epsilon \leq \omega$, $|P^M_1| = \lambda$ and $P^M_\omega = P^M_{\omega + 1}$ for $i = 0, 1$.

(c) For all $\epsilon \leq \omega$, $P^M_1 / P^M_4$ is a free Boolean algebra.

(d) (i) If $\epsilon < \omega$, $D_\epsilon \in \text{uf}(M_\epsilon)$.

(ii) If $\epsilon = 0$, then $\langle b_{-1, \alpha}: \alpha < \lambda \rangle$ is a free basis of $P^M_0 / P^M_4$, listed without repetition, and $\langle F^M_0(c): n < \omega, c \in P^M_0 \rangle$ lists $\langle b_{-1, \alpha}: \alpha < \lambda \rangle$ without repetition.

(iii) if $\epsilon = \zeta + 1 < \omega$ then there is a free basis $\langle b_{\zeta, \alpha}/P^M_4: \alpha < \lambda \rangle$ of $P^M_1 / P^M_4$ over $P^M_1 / P^M_4$. Note $b_{\zeta, \alpha} \in P^M_1 - P^M_4$. 

(e) if $\epsilon = \omega + 1$, for each $d \in \omega(P^M_{\omega + 1} - P^M_4)$ such that for each $a \in P^M_0$ for all but finitely many $n, a \not\in R(M_\epsilon, d_n)$, then for some $c \in P^M_{\omega + 1}, F^M_\omega(c) = d_n$. (We will in fact have that $P^M_{\omega + 1} = P^M_1$ and $P^M_4 = P^M_2$.)

(f) $\epsilon = \zeta + 1 < \omega$.

Let $B$ and $A$ be as in Definition 2.1.1. There is a $1$-$1$ function $f_\epsilon$ from $\lambda$ onto $P^M_4$ such that:

i) for every $X \in B$ (from $\Xi$) there is a $b = b_X \in P^M_1$ such that
$$\{ \alpha < \lambda: f_\epsilon(\alpha) \leq_M b_X \} = \{ \alpha < \lambda: \alpha \in X \};$$

ii) for each $A = \langle A_n: n < \omega \rangle \in A$ there is a $c \in P^M_2$ such that for each $n$:
$$A_n = \{ \alpha < \lambda: f_\epsilon(\alpha) \leq_{P^M_1} F^M_n(c) \}.$$

**Carrying out the construction.**

Below, the element $b_{\zeta, \alpha}$ is the $b_{A_n}$ from Specification 2.2.5.f.(i).

**Case 1:** When $\epsilon = 0$, take $P^M_0$ as the Boolean algebra generated by a set $P^M_0$ of cardinality $\lambda$ along with a set $\{ b_{-1, \alpha}: \alpha < \lambda \}$ of independent subsets of $P(\lambda)$. Let $G_1$ be a bijection between a set $P^M_0$ and $P^M_1$. Set $P^M_4$ as the ideal generated by the image of $G_1$. For $a \in P^M_0$ and $b \in P^M_1$, define $R(a, b)$ to hold if $G_1(a) < b$. Set $P^M_2 = \emptyset$ and so there are no $F^M_n$ to define. Thus, any non-principal ultrafilter on $P^M_0$ is in $\text{uf}(M_0)$.

**Case 2:** For $\epsilon = \omega$, $M_\omega = \bigcup_{n<\omega} M_n$.

**Case 3:** If $\epsilon = \zeta + 1 < \omega$, the main effort is to verify clauses (c), (d), and (f) of Specification 2.2.5.

Now, to construct $M_\epsilon$: 


i Recall that $D_\zeta \in \text{uf}(M_\zeta)$.

ii choose a set $B_\zeta \subseteq P(\lambda)$; with $B_\zeta \cap M_\zeta = \emptyset$ and $|B_\zeta| = \lambda$ as the new atoms introduced at this stage.

iii Let $f_\zeta$ be a one-to-one function from $\lambda$ onto $B_\zeta \cup P_{4,1}^{M_\zeta}$.

iv Let $(X_\gamma : \gamma < \lambda)$ list the elements of $B$ from $\mathbb{B}$. With $B_\zeta \cap M_\zeta = \emptyset$ and $|B_\zeta| = \lambda$ as the new atoms introduced at this stage.

v Fix a sequence $\{b_{\zeta,\alpha} : \alpha < \lambda\}$, which are distinct and not in $M_\zeta \cup B_\zeta$, and let $\mathbb{B}_\zeta'$ be the Boolean Algebra generated freely by

\[ P_{1}^{M_\zeta} \cup \{b_{\zeta,\alpha} : \alpha < \lambda\} \cup \{f_\zeta(\alpha) : \alpha < \lambda\}. \]

Using Lemma 2.2.2, we apply the following definition at the successor stage.

**Definition 2.2.6 (Ideal)** Let $I_\zeta$ be the ideal of $\mathbb{B}_\zeta'$ generated by:

(i) $\sigma(a_0, \ldots, a_m)$ when $\sigma(x_0, \ldots, x_m)$ is a Boolean term, $a_0, \ldots, a_m \in P_{1}^{M_\zeta}$, and $P_{1}^{M_\zeta} \models \sigma(a_0, \ldots, a_m) = 0$.

The next two clauses aim to show that in $M_\zeta/I_\zeta$, the element $b_{\zeta,\gamma}$ is the $b_{X_\gamma}$ from Specification 2.2.5 f.i). That is, $\{\alpha < \lambda : f_\zeta(\alpha) \leq M_\zeta X_\gamma\} = \{\alpha < \lambda : \alpha \in X_\gamma\}$.

Recall (Definition 2.1.1) that the $X_\gamma$ enumerate $\mathbb{B}$ and are subsets of $\lambda$.

(ii) $f_\zeta(\gamma) - b_{\zeta,\gamma}$ when $\alpha \in X_\gamma$ and $\alpha, \gamma < \lambda$.

(iii) $b_{\zeta,\gamma} \land f_\zeta(\alpha)$ when $\alpha \in \lambda - X_\gamma$ and $\alpha, \gamma < \lambda$.

To show the $f_\zeta(\gamma)$ are disjoint atoms we add:

(iv) For any $f_\zeta(\gamma)$ and any $b \in \mathbb{B}_\zeta'$ either $(f_\zeta(\gamma) \land b)$ \in $I_\zeta$ or $(f_\zeta(\gamma) - b)$ \in $I_\zeta$.

(v) $f_\zeta(\gamma_1) \land f_\zeta(\gamma_2)$ when $\gamma_1 < \gamma_2 < \lambda$;

(vi) $f_\zeta(\alpha) - b$ when $\alpha < \lambda$, $f_\zeta(\alpha) \notin P_{4,1}^{M_\zeta}$ and $b \in D_\zeta$.

This asserts: Every new atom is below each $b \in D_\zeta$ and is used at the end of case 3 of the construction.

Let $\mathbb{B}_\zeta'' = \mathbb{B}_\zeta'/I_\zeta$. Applying Fact 2.2.2, we see from Definition 2.2.6:

**Claim 2.2.7** The structure $P_{1}^{M_\zeta}$ is embedded as a Boolean algebra into $\mathbb{B}_\zeta''$ by the map $b \mapsto b/I_\zeta$ and

1. For $\gamma < \lambda$, $f_\zeta(\gamma)/I_\zeta$ is an atom of $\mathbb{B}_\zeta''$;

2. If $b \in P_{1}^{M_\zeta}$ is non-zero, then $bI_\zeta \supseteq \mathbb{B}_\zeta'' f_\zeta(\gamma)$ for some $\gamma < \lambda$. 

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We take a further quotient of $P_1^M$. Let

$$J_\zeta = \{ b \in P_1^M : b/I_\zeta \cap \mathbb{B}_\zeta', f_\zeta(\gamma) = 0 \text{ for every } \gamma < \lambda \}.$$ 

Then $J_\zeta$ is an ideal of $P_1^M$ extending $I_\zeta$ so $b \mapsto b/J_\zeta$ is a homomorphism. Further, $f_\zeta(\gamma)$ is an atom of $P_1^M/J_\zeta$ for $\gamma < \lambda$. These atoms are distinct and dense in $P_1^M/J_\zeta$.

**Notation 2.2.8** Let $\mathbb{B}_\zeta$ be $P_1^M/J_\zeta$.

Now we define $M_\epsilon$ by setting $P_1^M_\epsilon = \mathbb{B}_\zeta$ which contains $P_4^M$. $P_4^M$ is the injective image in $P_1^M$ of $P_4^M \cup B_\epsilon$. For $a \in P_4^M$ and $b \in P_1^M$ set $R^M(a,b)$ if some $\gamma$, $a = f_\epsilon(\gamma)/J_\epsilon$ and $f_\epsilon(\gamma)/J_\epsilon \leq_{\mathbb{B}_\zeta} b/J_\zeta$. Finally, let $D_\epsilon$ be the ultrafilter on $P_1^M$, generated by

$$D_\zeta \cup \{ j_\epsilon(-b_{\zeta,\gamma}) : \gamma < \lambda \} \cup \{ j_\epsilon(-f_\epsilon(\gamma)) : \gamma < \lambda \}.$$

By Claim 2.2.7, we have the cardinality and atomicity conditions of Specification 2.2.5 (a) and (b); the definition of $I$ guarantees, (c) and (d). (ii), (d). (iii). We verify $M_\epsilon \in \mathcal{K}_- \zeta$ below. The first set of new elements in $D_\epsilon$ show along with (our later) definition of $F_n^M(c)$ show $D_\epsilon \in \text{uf}(M_\epsilon)$ (as no new $F_n(c)$ is in $D_\epsilon$); the second set show $D_\epsilon$ is non-principal. Note that Specification 2.2.5 (e) does not apply except in the $\omega + 1$st stage of the construction.

For Specification 2.2.5 (f) (i), let $X$ be a set of atoms of $M_\epsilon$ and note that we can choose $b_X$ by conditions ii) and iii) in Definition 2.2.6 of $I_\zeta$.

We can choose $P_2^M$ and $F_n^M$ to satisfy Specification 2.2.5 (f) (ii). Fix an $\overline{A} \in \mathcal{A}$ (as given by $\mathbb{B}$). Fix a $c = c^\overline{A}$ and define, using the last paragraph, the $F_n^M(c)$ as $b_{A_n}$, so that for each $n$, $A_n = \{ \alpha < \lambda : f_\epsilon(\alpha) \leq_{P_1^M} F_n^M(c) \}$. These are the only new $c \in P_2^M$.

Thus, it remains only to show that $M_\epsilon \in \mathcal{K}_- \zeta$. I.e., that $M_\epsilon$ satisfies Definition 1.2.7:

(♦) If $a \in P_4^M$ and $c \in P_2^M$ then $(\forall \infty n) a \not\in^*_M F_n^M(c)$.

If $c \in P_2^M$, $F_n^M(c) = F^M(c) \subseteq P_1^M$ and we know by induction that ♦ holds for $a \in P_4^M$.

For $a \in P_4^M - P_2^M$, Definition 1.2.5. and condition (v) on $I_\zeta$ (from Definition 2.2.6) imply $a \leq_{M_\zeta} b$ for every $b \in D_\zeta$. As $c \in P_2^M$ and $D_\zeta \in \text{uf}(M_\zeta)$, all but finitely many $e_n = F_n^\zeta(c)$, are not in $D_\zeta$. So $e_n^\zeta \not\in D_\zeta$. That is, $a \leq_{M_\zeta} e_n$; so $a \land_{M_\zeta} e_n = \emptyset$ as required.

If $c \in P_2^M - P_4^M$ then by our choice of $P_2^M$ and the $F_n^M$, there is an $\overline{A}_c$ that is enumerated by the $F_2^M(c)$ and satisfies ♦ by (i) of $\mathbb{B}$ (Definition 2.1.1. (i)). This completes the verification of ♦ at stage $\epsilon$ and the $M_\epsilon$ satisfies all the specifications of the induction.

**case 4.** $\epsilon = \omega + 1$:

Only clauses (b) and (e) of Specification 2.2.5 are relevant. Define $P_2^M$ and $F_n^M$ to satisfy clause (e). Since $P_i^M = P_1^M$ for $i = 0, 1$, specification c) is immediate. This completes the construction.
The construction suffices.

Having completed the induction, let \( M = M_{\omega+1} \). Using specifications c) and d) of 2.2.5, it is straightforward to verify that \( M \in K_{-1} \) and the Boolean algebra is atomic. By (b), \( M_i^{\lambda{\omega}} \) for \( i = 0, 1 \) have cardinality \( \lambda \). And by (f), the same holds for \( M_2^{\lambda{\omega}+1} \).

We now show \( M \) is nicely free. Let \( b = \langle b_n : \beta < \lambda \rangle \) enumerate \( \langle b_{n,n} : n < \omega, \alpha < \lambda \rangle \) without repetition. We show \( b \) satisfies the requirements in Definition 2.2 of nicely free. By Specifications 2.2.5. (c), (d) and since \( P_1^{M} \) is constructed as the union of the \( P_1^{M{\omega}}, P_1^{M}/P_1^M \) is generated freely by \( b/P_1^M \). Finally, clause c) of Definition 2.2 holds by clause (d).ii) of Specification 2.2.5.

The crux is to show \( M = M_{\omega+1} \) is \( P_0 \)-maximal. For this, assume for a contradiction:

\((*)\) \( P_0^M \) is not maximal; by Lemma 2.2.3, there is a \( D \in \text{uf}(M_{\omega+1}) = \text{uf}(M_\omega) \). For every \( n < \omega \), : is there a \( d \in D \) such that \( R(M_n, d) \cap M_n = \emptyset \)?

Possibility 1 : For every \( n < \omega \), the answer is yes, exemplified by \( d_n \in D \). Now for each \( a \in P_0^{M_n} \), \( a \notin R(M_n, d_m) \) for all \( m \geq n \). So the sequence \( d = \langle d_n : n < \omega \rangle \) satisfies the hypothesis of Specification 2.2.5.(e) and so there is a \( c \in P_2^M \) such that for each \( n < \omega \), \( F_n^M (c) = d_n \). Thus, recalling Definition 2.2.1, \( D \notin \text{uf}(M) \).

Possibility 2 : For some \( n < \omega \), there is no such \( d_n \); without loss of generality, assume \( n > 0 \). We apply specification f) with \( \epsilon = n \). Recall that \( f_n \) is a 1-1 map from \( \lambda \) onto \( P_4^{M_n} \). Let \( g_1 \) be the following homomorphism from the Boolean algebra \( P_1^{M_{\omega+1}} = P_1^{M_{\omega}} \) into \( P(\lambda) : g_1(b) = \{ \alpha < \lambda : f_n(\alpha) \leq B \} \). By Specification f.i) of 2.2.5, the Boolean algebra \( B \) provided by \( \text{uf} \) is contained in the range of \( g_1 \).

Let \( I_n \) denote the ideal of \( P_1^{M} \) generated by \( P_4^{M}-P_4^{M_1} \). Since \( D \) is non-principal, \( I_n \cap D = \emptyset \). Now, \( g_1 \) maps any \( b \in P_1^{M_{\omega}} - P_4^{M_{\omega}} \) (and, thus, any \( b \in P_1^{M_{\omega}} - I_n \)) to a nonempty subset of \( \lambda \). Recalling \( I_n \cap D = \emptyset \), \( g_1 \) embeds the quotient algebra \( P_1^{M_{\omega}} / I_n \) into the Boolean algebra \( P(\lambda) \). Hence, \( D_1 = g_1^n(D) \) is an ultrafilter of the Boolean algebra \( \text{rg}(g_1) \) and so \( D_2 = D_1 \cap B \) is an ultrafilter of the Boolean algebra \( B \). We show, for any \( \alpha < \lambda \), \( \{ \alpha \} \notin D_2 \). As, \( f_n(\alpha) \in P_4^{M_n} \) and so \( \{ f_n(\alpha) \} \) is not in \( D \). So \( \{ \alpha \} \notin D_1 \). Thus, \( \lambda - \{ \alpha \} \notin D_1 \) and so \( \lambda - \{ \alpha \} \notin D_2 \). So \( \{ \alpha \} \notin D_2 \) as promised.

Now we apply the second clause of \( \text{uf} \) to the ultrafilter \( D_2 \). Since we satisfied specification f.ii) in the construction, we can conclude there is \( A = \langle A_n : n < \omega \rangle \in A \) such that for infinitely many \( k \), \( A_k \) is in \( D_2 \). Thus, \( u = \{ k : A_k \in D \} \) is infinite. We will finish the proof by showing there is a \( c \) such that \( u = u_c \) (Definition 2.2) is the set of images of the \( F_n^M (c) \).

Since each \( A_k \in B \), \( A_k \in \text{rg}(g_1) \). So we can choose \( d_k \in P_1^{M_{\omega}} \) with \( g_1(d_k) = A_k \). As \( A_k \in D_2 \), by the choice of \( D_1, D_2 \) we have \( d_k \) is in the ultrafilter \( D \) from the hypothesis for contradiction: \((*)\).

We show the sequence \( d = \langle d_k : k < \omega \rangle \) satisfies the hypothesis of clause e of Specification 2.2.5. First, \( d_k \in P_1^{M_{\omega}} - P_4^{M_{\omega}} \) as \( D \) is a non-principal ultrafilter on \( P_1^{M_{\omega}} \) so the first hypothesis is satisfied. Further, for every \( a \in P_0^{M_{\omega}} \) all but finitely many \( k, G_1^{M_{\omega}}(a) \notin M_{\omega} \) \( d_k \) because \( A \in A \), which implies by \( \text{uf} \) ii) that for every \( \alpha < \lambda \) for some \( k_{o} \), we have \( k \geq k_{o} \) implies \( \alpha \notin A_k \). Now by the definition of \( g_1 \), recalling \( g_1(d_k) = A_k \), we have \( k \geq k_{o} \) implies \( f_k(\alpha) \notin d_k \) (in \( P_1^{M_{\omega}} \)). So by
3 Correcting \( M_\ast \) to a model of \( K_2 \)

We now ‘correct’ the structure \( M_\ast \) constructed in Section 2 of a \( P_0 \)-maximal model of \( K_{-1} \) to obtain a \( P_0 \)-maximal model \( M \) (Definition 2.1) of the complete sentence constructed in [BS17a], i.e. \( M \in K_2 \). In Theorem 3.18 we modify \( M_\ast \), to construct a model \( M \in K_2 \) with \( P_0^M \subseteq P_{M_\ast}^M \) and redefining the \( F_n^M \), but retaining \( M \upharpoonright (P_0^M \cup P_1^M) = M_\ast \upharpoonright (P_0^M \cup P_1^M) \). The old values of \( F_n^M \) will be used to divide the work of ensuring each ultrafilter \( D \) is not in \( \text{uf}(M) \) by for each \( D \) attending to only those \( c \) with infinitely many \( F_n(c) \) in \( D \).

For this we need to introduce some terminology from [BS17a]. We first describe the finitely generated models.

**Definition 3.1 (\( K_{<\aleph_0} \), Defined)** \( M \) is in the class of structures \( K_{<\aleph_0} \) if \( M \in K_{\aleph_0}^{-1} \) and there is a witness \( \langle n_\ast, B, b_\ast \rangle \) such that:

1. \( b_\ast \in P_1^M \) is the supremum of the finite joins of atoms in \( P_1^M \). Further, for some \( k \), \( \bigcup_{j \leq k} P_{4,j}^M = \{ c : c \leq b_\ast \} \) and for all \( n > k \), \( P_{4,n}^M = 0 \).
2. \( B = \langle B_n : n \geq n_\ast \rangle \) is an increasing sequence of finite Boolean subalgebras of \( P_1^M \).
3. \( B_n, \supseteq \{ c \in P_1^M : c \leq b_\ast \} = P_4^M \); it is generated by the subset \( P_4^M \cup \{ F_n^M(c) : n < n_\ast, c \in P_2^M \} \).

Moreover, the Boolean algebra \( B_n, \) is free over the ideal \( P_4^M \) (equivalently, \( B_n, / P_4^M \) is a free Boolean algebra\(^2\)).
4. \( \bigcup_{n \geq n_\ast} B_n = P_1^M \).
5. \( P_2^M \) is finite and not empty. Further, for each \( c \in P_2^M \) the \( F_n^M(c) \) for \( n < \omega \) are independent over \( \{0\} \).
6. The set \( \{ F_m(c) : m \geq n_\ast, c \in P_2^M \} \) (the enumeration is without repetition) is free from \( B_{n_\ast} \) over \( \{0\} \). \( B_{n_\ast} \supseteq P_4^M \) and \( F_m(c) \land b_\ast = 0 \) for \( m \geq n_\ast \). (In this definition, \( 0 = 0_{P_4^M} \).

In detail, let \( \sigma(\ldots x_{c_i} \ldots) \) be a Boolean algebra term in the variables \( x_{c_i} \) (where the \( c_i \) are in \( P_2^M \) which is not identically \( 0 \). Then, for finitely many \( n_i \geq n_\ast \) and a finite sequence of \( c_i \in P_2^M \):

\[
\sigma(\ldots F_{n_i}(c_i) \ldots) > 0
\]

and some \( n < \omega \). Further, for any non-zero \( d \in B_{n_\ast} \) with \( d \land b_\ast = 0 \), (i.e. \( d \in B_{n_\ast} - P_4^M \)),

\[
\sigma(\ldots F_{n_i}(c_i) \ldots) \land d > 0.
\]

\(^2\)A further equivalence: \( |\text{Atom}(B_{n_\ast})| - |P_4^M| \) is a power of two.
7. For every \( n \geq n_* \), \( B_n \) is generated by \( B_{n_*} \cup \{ F_m(c) : n > m \geq n_*, c \in P_2^M \} \).

Thus \( P_1^M \) and so \( M \) is generated by \( B_{n_*} \cup P_2^M \).

Note that the free generation in item 6 of Definition 3.1 is not preserved by arbitrary direct limits and so is not a property of each model in \( K_1 \). In particular, as \( M_\ast \) is corrected to a model of \( K_1 \), we check this property only for finitely generated submodels as it will be false in general.

Recall some terminology from [BS17a].

**Definition 3.2 (\( K'_1 \) Defined)**
1. \( K_1 \) denotes the collection of all direct limits of models in \( K_{<\aleph_0}^1 \).
2. We say a model \( M \) in \( K_1 \) is rich if for any \( N_1, N_2 \in K_{<\aleph_0}^1 \) with \( N_1 \subseteq N_2 \) and \( N_1 \subseteq M \), there is an embedding of \( N_2 \) into \( M \) over \( N_1 \).
3. \( K_2 \) is the class of rich models.

Since \( K_{<\aleph_0}^1 \) has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model, thus \( K_2 \) is not empty.

**Fact 3.3** There is a countable generic model \( M \) for \( K_0 \) (Corollary 3.2.19 of [BS17a]). We denote its Scott sentence by \( \phi \). \( K_2 \) is the class of models of this \( \phi \); it can also be thought of as the class of rich models in \( K_1 \).

We now describe some of the salient properties of the model \( M \) obtained by ‘correcting’ the \( M_\ast \) of Section 2.

**Remark 3.4 (The Corrections)**
1. The structures constructed in this Section are subsets of \( M_\ast \); the \( F_n \) are redefined so the new structures are substructures only of the reduct of \( M_\ast \) to \( \tau - \{ F_n : n < \omega \} \).
2. In particular, for all the \( M \) considered here \( P_1^M = P_1^{M'} \) and these Boolean algebras have the same set of ultrafilters. However, \( uf(M) \neq uf(M') \) as the definition of uf depends on properties of the \( F_n \).
3. The set \( \{ F_n^M(c) : c \in P_2^M \} \) is not required to be an independent subset in \( K_{<\aleph_0}^1 \); the final constructed model is not nicely free.
4. Claim 3.15 demands a sequence of finite Boolean algebras \( B_n \) to witness membership in \( K_1 \) (not required for \( K_{<\aleph_0}^1 \)) in Section 2 and [BS17a].
5. In [BS17a], the proof that a non-maximal model in \( \lambda \) makes \( \lambda \) measurable depends on \( \diamond \).

The main task of this section is to prove:

**Theorem 3.5** If \( \lambda \) is less that the first measurable cardinal and for some \( \mu \), \( 2^\mu = \lambda \) and \( 2^{\aleph_0} < \lambda \), then there is a \( P_0 \)-maximal model in \( K_2 \) of cardinality \( \lambda \).
Context 3.6, summarises the results of the construction in Theorem 2.2.4, specifically to fix our assumptions for this section. The requirement that for some \( µ \) is needed only to guarantee (by Theorem 2.2.4) there is a model \( M_\ast \) in \( \lambda \) satisfying Context 3.6.

Context 3.6

1. \( P^M_1 \) is an atomic Boolean algebra and \( M_\ast \) is \( P_0 \)-maximal. Further, \( |P^M_1| = \check{\lambda} \) for \( i = 0, 1 \).
2. \( P^M_{4,1} \) is the set of atoms of \( M_\ast \).
3. \( M_\ast \) is nicely free (Definition 2.2); in particular, \( P^M_1/P^M_1 \) is a free Boolean algebra of cardinality \( \lambda \).

In order to ‘correct’ \( M_\ast \) to a model in \( K_2 \), we lay out some notation for the generating set of \( P^M_1 \), the free basis of the boolean algebra \( P^M_1/P^M_1 \), and the indexing of the tasks performed in the construction.

Notation 3.7 We define a family of trees of sequences:

1. Let \( T_\alpha = \{ \{ \} \} \cup \{ \langle \alpha \rangle ; \eta \in {}^<\omega 3 \} \) and \( T = \bigcup_{\alpha < \lambda} T_\alpha \).
2. \( \lim(T_\alpha) \) is the collection of paths through \( T_\alpha \).

Combining the specifications for constructing \( M_\ast \) (Specification 2.2.5) and the Definition 2.2 of nicely free, we introduce the collection of models which is the starting point for the following construction.

Claim 3.8 (Fixing Notation) Without loss of generality, we may assume:

1. The universe of \( M_\ast \) is \( \lambda \) and the \( 0 \) of \( P^M_1 \) is the ordinal 0.
2. We can choose sequences of elements of \( P^M_1 \), \( b = \langle b_\eta ; \eta \in T \rangle \) so that their images in the natural projection of \( P^M_1 \) on \( P^M_1/P^M_1 \) freely generate \( P^M_1/P^M_1 \).
3. For every \( a \in P^M_{4,1} \) and the even ordinals \( \alpha < \lambda \), for some \( n \), for any \( \nu, \rho \in T_\alpha \) with \( \lg(\eta) \geq n \) and \( \lg(\rho) \geq n \), \( a \leq_{P^M_1} b_\nu \) if and only if \( a \leq_{P^M_1} b_\rho \).

Proof. The only difficulty is deducing from c) of Definition 2.2 (nicely free) that 3) holds. For that, for even \( \alpha \), let \( \{ b'_{\omega \alpha + n} ; n < \omega \} \) enumerate \( u_\alpha = \{ F^M_0(c) : n < \omega \} \) (from Definition 2.2.c) for the \( \alpha \)th \( c \) in some enumeration of \( P^M_2 \). Now for \( \alpha > 0 \), let \( \{ b_\eta ; \eta \in T_\alpha \setminus \{ {} \} \} \) list \( \{ b'_{\omega \alpha + n} ; n < \omega \} \) without repetition and \( \{ b_\eta ; \eta \in T_0 \} \) list \( \{ b'_n ; n < \omega \} \).

By Definition 1.2.7 we have. For every \( a \in P^M_{4,1} \) for all but finitely many \( n \), \( a \cap b'_{\omega \alpha + n} = 0_{P^M_1} \); whence for even \( \alpha \) all but finitely many of the \( \nu \in T_\alpha \) with \( \nu(\alpha) \neq 0 \), \( a \cap b_\nu = 0_{P^M_1} \). Since for each \( n \), the intersection of the \( F_n(c) \) is empty, clause (3) follows as for all sufficiently large \( n \), \( a \not\in F_n(c) \).

Note that Claim 3.8 provides a 1-1 map from \( P^M_2 \) to ordinals less than \( \lambda \). We introduce the collection of models which is the starting point for the following construction.
Definition 3.9 (M_1 Defined) Let M_1 = \{M \mid M \in K_{\lambda-1}\} be the set of M \in K_{\lambda-1} such that the universe of M is contained in \lambda, the universe of M_\lambda, and for i < 2, P_i^M = P_i^{M_\lambda}, M \rceil (P_0^M \cup P_1^M) = M_\lambda \rceil (P_0^{M_\lambda} \cup P_1^{M_\lambda}) while P_2^M \subseteq P_2^{M_\lambda}.

The posited M_\lambda differs from any M \in M_1 only in that P_2^M may be a proper subset of P_2^{M_\lambda} and the F_n^{M_\lambda}(c) need not equal the F_n^M(c).

We now spell out the tasks which must be completed to correct M_\lambda to the required member of K_2. The F_n^{M_\lambda}(c) are used as oracles.

Definition 3.10
1. Let T_1, the set of 1-tasks, be the set of pairs (N_1, N_2) such that:
   (a) N_1 \subseteq N_2 \subseteq \lambda
   (b) N_1, N_2 \in K_{<\aleph_0}
   (c) N_1 \subseteq M for some M \in M_1.
2. Let T_2, the set of 2-tasks, be the set of c \in P_2^{M_\lambda}.
3. T = T_1 \cup T_2.
4. Let \{t_\alpha : \alpha < \lambda\} enumerate T.

Note |T_1| = |T_2| = |T|.

Definition 3.11 The task t is relevant to the structure M if M \in M_1 and i) if t is 1-task (N_1, N_2) then N_1 \subseteq M or ii) if t is a 2-task \{c\} and c \in P_2^{M_\lambda}.

We say M \in M_1 satisfies the task t if either:
A) t = (N_1, N_2) \in T_1 (so N_1 \subseteq M) and there exists an embedding of N_2 into M over N_1.
B) t = \{c\}, where c \in P_2^{M_\lambda}, is in T_2 and for every ultrafilter D on P_2^M, such that for infinitely many n, F_n^{M_\lambda}(c) \in D, there is a d \in P_2^M such that for infinitely many n, F_n^M(d) \in D.

Recall Definition 2.2.1 of uf(M) and Lemma 2.2.3 connecting uf(M) with P_0-maximality of M.

Claim 3.12 If M \in M_1 satisfies all tasks in T and is in K_1 then M is P_0-maximal and, in particular, satisfying the tasks in T_1 guarantees it is in K_2.

Proof. For P_0-maximality of M, it suffices, by Lemma 2.2.3, to show uf(M) = \emptyset. But, since uf(M_\lambda) = \emptyset, for every ultrafilter D on P_1^{M_\lambda}, there is c \in P_2^{M_\lambda} with S^{M_\lambda}_c(D) infinite and satisfying task c means there is d \in P_2^M such that S^M_d(D) is infinite and so not in uf(M). Since M and M_\lambda have the same ultrafilters, this implies uf(M) = \emptyset, as required. The second assertion follows by realizing that satisfying all the tasks in T_1 establishes the model is rich, which suffices by Fact 3.3. 

Thus our job is reduced to showing each ultrafilter D is countably incomplete. Definition 3.13 lays out the use of the generating elements b_\eta in correcting the F_n^{M_\lambda} to
require independence while maintaining that infinite intersections of members of the
ultrafilter under consideration are empty. The infinite sequence \( \eta_d \) will guide the choice of
possibilities for \( F_n(d) \).

We define a class \( \mathcal{M}_2 \subseteq \mathcal{M}_1 \) such that for each \( d \in P_2^M \in \mathcal{M}_2 \) there is an ordinal
\( \alpha_d \), a tree of elements of \( P_1^M \), indexed by sequences in \( T_{\alpha_d} \subseteq \omega^3 \), a target path
\( \eta_d \) through that tree and a sequence \( a_{d,n} \), whose indices are not in \( T_{\alpha_d} \), but which
satisfy that each \( a \in P_{1,1}^M = P_{4,1}^M \) is in at most finitely many \( a_{d,n} \). Further, elements
indexed by \( T_{\alpha} \) are combined with the \( a_{d,n} \) to get values of the \( F_n^M(d) \) which are both
independent and satisfy \( \bigcap_{n<\omega} F_n^M(d) = \emptyset \).

**Definition 3.13 (\( \mathcal{M}_2 \) Defined)** Let \( \mathcal{M}_2 \) be the set of \( M \in \mathcal{M}_1 \) such that there is a
sequence \( w = (a_{d,n}) : d \in P_2^M, n < \omega \) witnessing the membership, which
means:

A. (a) For each \( d \in P_2^M \), \( \alpha_d < \lambda \) is even and \( d_1 \neq d_2 \) implies \( \eta_{d_1} \neq \eta_{d_2} \). (Note
that it is possible that \( d_1 \neq d_2 \) while \( \alpha_{d_1} = \alpha_{d_2} \).)

(b) \( \langle \alpha_d \rangle < \eta_d \in \lim(T_{\alpha_d}) \).

B. The \( a_{d,n} \) are in \( P_1^M \) and for each \( d \in P_2^M \) and \( n < \omega \), there are\(^3\) \( \nu_1[d,n] \neq \nu_2[d,n] \) in \( n+1 \beta \) such that:

(a) For a fixed function \( n_M : P_2^M \to \omega \), we have, for every \( n \geq n_M(d) \):

\[
F_n^M(d) = (b_{\nu_1[d,n]} \triangle b_{\nu_2[d,n]}) \triangle a_{d,n}.
\]

For \( n < n_M(d) \), \( F_n^M(d) = F_{n_M}^M(d) \).

(b) \( \eta_d \upharpoonright n \leq \nu_1[d,n] \) and \( \eta_d \upharpoonright n \leq \nu_2[d,n] \);

(c) for each \( a \in P_{4,1}^M \) and each \( d \in P_2^M \), there are only finitely many \( n \) with

\( a \leq n_{M}^* : a_{d,n} \).

C. For each \( Y \subseteq P_2^M \) there is a list \( \langle d_\ell : \ell < |Y| \rangle \) of \( Y \) such that: (*) for every
\( \ell < |Y| \), letting \( \alpha_\ell \) abbreviate \( \alpha_{d_\ell} \), we have

\[
W_\ell = \{a_{d_\ell,n} : k \leq \ell \land n < \omega \} \cup \{b_\nu : \nu(0) \neq \alpha_\ell, \alpha_k \neq \alpha_\ell \}
\cup \{F_\ell^M(d_k) : i < n_M(d_k), k \leq \ell, d_k \neq d_\ell \}
\]

is included in the subalgebra \( \mathcal{B}_\ell \) of \( P_{1,1}^M \) generated by

\[
\{b_\nu : \nu(0) \neq \alpha_\ell \land \nu \in T \} \cup \{b_0 \} \cup P_{4,1}^M.
\]

(a) The \( d_\ell \) list \( Y \) without repetition.

(b) If \( i_1 < i_2 < i_3 < |Y| \) and \( \alpha_{i_1} = \alpha_{i_3} \) then \( \alpha_{i_2} = \alpha_{i_1} \).

The following facts about the relation of symmetric difference and ultrafilters are
central for calculations below.

---

\(^3\)I.e., \( \nu_1[d,n] \) depends on \( d \) and \( n \).
Remark 3.14 Recall that the operation of symmetric difference is associative.

1. (for 3.15) Suppose \( B_1 \subseteq B_2 \) are Boolean algebras with \( a \in B_1 \), and \( b \neq c \) are in \( B_2 \), and \( \{b, c\} \) is independent over \( B_1 \) in \( B_2 \). Then

The element \( (b \triangle c) \triangle a \in B_2 \) is independent from \( B_1 \).

Starting from infinite independent sequences \( b_1, b_2 \in P_1^M \) and an infinite independent sequence of \( a_{d,n} \) we can prove by induction that the \( F_n^M(d) \) (as defined in Definition 3.13. 2a) are independent.

2. (for 3.18) Let \( D \) be an ultrafilter on a Boolean algebra \( B \). Note:

\( a \in D \) if and only if \( b \triangle c \notin D \).

If \( a_0, a_1, a_2 \in B \) are distinct and \( (a_0 \notin D \iff a_1 \in D) \) then at least one of \( a_i \triangle a_j \notin D \) (since the intersection over all pairs \( i, j \) of the \( a_i \triangle a_j \) is empty).

More importantly for our use later, \( (a_0 \in D \iff a_1 \in D) \iff (a_0 \triangle a_1 \triangle a_2) \in D \leftrightarrow a_2 \in D \).

3. (for 3.18) 2) implies that if \( D \) is an ultrafilter of \( B_2 \) and \( (b \in D \iff c \in D) \) and \( a \notin D \) then

\( b \triangle c \notin D \)

\( (b \triangle c) \triangle a \notin D \).

We will show in Theorem 3.15 that members of \( \mathbb{M}_2 \) are in \( K_1 \) and then in Theorem 3.18 that there are structures in \( \mathbb{M}_2 \) that are in \( K_2 \). Two main features distinguish \( K_1 \) from \( K_{\omega} \). The \( F_n^M(d) \) retain the intersection properties from \( K_{\omega} \) but also must be independent; membership of an \( M \) in \( K_1 \) from [BS17a] must be witnessed by the construction for a countable substructure \( M' \subset M \) of a family of finite Boolean algebras satisfying Definition 3.1.2 and 3.

Theorem 3.15 If \( M \in \mathbb{M}_2 \), then \( M \in K_1 \).

Proof. Let \( Y \subset P_2^M \) and \( X \subset P_1^M \) be finite ; we shall find \( N \in K_{\omega \cap \aleph_0}^1 \) such that \( Y \cup X \subseteq N \subseteq M \); this suffices. As, \( K_1 \) is defined to be the direct limits of of finitely generated structures in \( K_{\omega \cap \aleph_0}^1 \).

Our two main tasks are to find such an \( N \) in which i) the \( P_n^N \) satisfy property 6 of Definition 3.1 and ii) there is a sequence of finite Boolean algebras \( B_{k_n} \) witnessing 2 and 3 of Definition 3.1. First we attack i).

Let the sequence \( \langle (\alpha_d, \eta_d, a_d, k) : d \in P_2^M, k < \omega \rangle \) witness \( M \in \mathbb{M}_2 \) as in Definition 3.13. Let \( \langle d_i : i < n \rangle \) enumerate \( Y \) without repetition and denote, for \( i < n, \eta_{d_i} \) by \( \eta_i \) and \( \alpha_d \) by \( \alpha_i \). Without loss, the \( \langle \eta_i(0) : i < n \rangle \) are non-decreasing. Fix \( k_1 \) such that

- \( k_1 \geq n_M(d_i) \) (see Definition 3.13. B) for all \( i < n \),

- \( \langle \eta_i \mid k_1 : i < n \rangle \) are distinct for \( i < n \), and
\[X_0 = X \cup \{F^M_k(d_i) : i < |Y| = n, k \leq n_M(d_i)\} \text{ is contained in the subalgebra generated by } \{b_\nu : \eta_\nu \uparrow k_1 \not\equiv \nu \text{ for } i < |Y|\} \cup \{b_0\} \cup P^M_{4,1}.

(By clause C of Definition 3.13, none of the \(F^M_k(d_i) = b_\sigma\) where \(\sigma(0)\) is equal to an \(\alpha_\ell\) with \(i < n\). We can avoid \(X\) by choosing \(k_1\) big enough.)

To establish claim i) we need the following claim.

**Claim 3.16** \(A = (F^M_k(d_i) : k \geq k_1 \text{ and } i < n)\) is independent in \(P^M_1\) over \(X_0\) modulo the atoms.

**Proof.** We prove this claim by showing by induction on \(\ell \leq |Y| = n:\)

\[(\oplus_\ell) \ A_\ell = (F^M_k(d_i) : k \geq k_1 \text{ and } i < \ell)\]

is independent in \(P^M_1\) over \(X_\ell\) modulo the atoms, where we have defined \(X_0\) and for \(0 < \ell \leq |Y|,\)

\[X_{\ell+1} = X_\ell \cup \{F^M_k(d_i) : k < k_1, i < \ell\}.

Note that \(A = A_\ell.\) The independence of \(A_\ell\) over all the \(F^M_k(d_i)\) with \(d_i \in Y\) for \(k < n_M(d_i)\) is clear since they are in \(X_0.\) For \(i < \ell \leq |Y|,\) the induction on \(\ell\) shows incrementally, at stage \(\ell,\) the independence of the tail of the \(F^M_k(d_i)\) over the \(F^M_k(d_i)\) for \(n_M(d_i) \leq k < k_1\)

Now by Definition 3.13.C, \(X_\ell \subseteq W_\ell\) is contained in \(\tilde{B}_\ell,\) the subalgebra generated by

\[Z_\ell = \{b_\nu : \nu_{d_\ell} \uparrow k_{\ell+1} \not\equiv \nu\} \cup \{b_0\} \cup P^M_{4,1}.

We verify this assertion by careful comparison of the definition of \(Z_\ell\) with that of \(W_\ell\) in Definition 3.13.C. Think of the \(\tilde{B}_\ell \subseteq B_\ell\) as omitting certain cones among the \(b_\nu.\) The **crucial point** is that \(b_{\nu_1[d_\ell,n]}\) and \(b_{\nu_2[d_\ell,n]}\) are not in \(B_\ell.\) Thus, Claim 3.8.2 and Definition 3.13.B imply the infinite set \(\{b_{\nu[i,k]} : i \in \{0,1\}, k \geq k_1, i < \omega\}\) is independent over \(\tilde{B}_\ell.\) For convenience we write \(a_{k,i}\) for \(a_{d_\ell,i}\). Looking at the first term in the union in Definition 3.13.C, and comparing \(X_\ell\) with the generators of \(W_\ell\) in that definition, each \(a_{k,i}\) for \(k < \ell, i < \omega\) is in \(\tilde{B}_\ell,\) and for each \(k < \ell\) and \(i < n(d_\ell), F^M_k(d_i) \in \tilde{B}_\ell\) (3rd term of the definition of \(Z_\ell\)). Compare also the definition of \(\eta_{d_\ell}\) in A(b) of Definition 3.13.

Using the crucial point, we also claim \(F^M_k(d_i) \in \tilde{B}_\ell\) when \(\ell < k_3\) for \(i \geq n(d_\ell).\)

As, setting \(d = d_\ell\) for notational simplicity,

\[F^M_n(d_k) = F^M_n(d) = (b_{\nu_1[d,n]} \triangle b_{\nu_2[d,n]}) \triangle a_{d,n},

by Lemma 3.13.B.(b). For each \(i\) and \(k, \nu_i[d_i,k]\) is \(\leq\) above \(\alpha_{d_i}\), so \(V = \{\nu_i[d_i,k] : i < \ell, k < \omega\}\) is independent from \(B_\ell\) and \(X_\ell \subseteq B_\ell;\) thus, \(V\) is independent from \(X_\ell.\) Since we noticed the \(a_{k,i} \in \tilde{B}_\ell,\) Lemma 3.14 implies \(\{F^M_k(d_i) : k \geq k_2\}\) is independent over \(\tilde{B}_\ell.\) By the induction hypothesis we finish.

Applying Claim 3.8.3, \(a \leq (b_{\nu_1[d,n]} \triangle b_{\nu_2[d,n]}),\) for \(n \geq n(d)\), and by hypothesis, the \(a_{d,n}\) satisfy the same condition. Thus, for sufficiently large \(n, a \not\in F^M_n(d_k).\)
This completes task i). To accomplish task ii) and finish the proof of Theorem 3.15 by satisfying conditions 2-4 of Definition 3.1, we must find a sequence of finite Boolean algebras $B_n$ witnessing that $X \cup Y$ is contained in a member of $K_{<\aleph_0}$.

Recall that $M_\nu$ is generated by $\{b_\nu : \nu \in T\} \cup P_{4,1}^{M_\nu} \cup \{b_{<\nu}\}$. As $X$ is finite, there is a $k_2$ such that $X$ is contained in the finite subalgebra of $P_{1}^{M_\nu}$ generated by $\{b_\nu : \nu \in T, \text{lg}(\nu) < k_2\} \cup P_{4,1}^{M_\nu}$.

We now choose $N \subseteq M$ with $P_1^N = \bigcup_{m \geq k_2} B_m$, $P_0^N = \{G_1^{-1}(a) : a \in P_{4,1}^M \cap P_1^N\}$ and $P_2^N = Y$.

Define the $B_m$ for $m \geq k_2, N \in K_1$ as follows. Let $B_{k_2}$ be the subalgebra of $M_\nu$ generated by $X \cup \{F_k^M(d_\ell) : k \leq k_2, \ell < |Y|\}$. For $m \geq k_2$, let $B_m$ be generated by $X \cup \{F_k^M(d_\ell) : k < m, \ell < |Y|\}$. Without loss of generality (using the choice of $b$ from Claim 3.8), we can demand each $\mathbb{B}_{k_1}$ is a finite free Boolean algebra. This sequence witnesses that $M \in K_1$. \hfill \Box_{3.15}

Now we show $\mathbb{M}_2$ is non-empty and at least one member satisfies all the tasks. In case 4) of this argument we address the requirement that $\text{uf}(M_\alpha) = \emptyset$ and so $\text{uf}(M) = \emptyset$ as well.

**Notation 3.17** We can enumerate $T$ as $\langle t_\alpha : \alpha < \lambda \rangle$ such that each task appears $\lambda$ times, as we assumed in Hypothesis 3.6 that $\lambda = \lambda^{\aleph_0}$.

For Theorem 3.18, realizing all the tasks, $\lambda > 2^{\aleph_0}$ suffice; the requirement in Lemma 2.1.5 that $\lambda = 2^\nu$ is used to get maximal models. The object of case 3) is to ensure that the final model is in $K_2$; case 4) shows $\text{uf}(M) = \text{uf}(M_\alpha) = \emptyset$.

**Theorem 3.18** There is an $M \in \mathbb{M}_2$ that satisfies all the tasks. Thus $M \in K_2$ and is $P_0$-maximal.

Proof. We choose $M_\alpha$ by induction on $\alpha \leq \lambda$ such that:

1. $w_\alpha$ witnesses $M_\alpha \in \mathbb{M}_2$ (Definition 3.13). And for $\beta < \alpha$, $w_\alpha$ extends $w_\beta$.

That is, for $d \in P_2^{M_\alpha}, \alpha_d[w_\alpha] = \alpha_d[w_\beta], \eta_d[w_\alpha] = \eta_d[w_\beta]$, etc.

2. $P_2^{M_\alpha} \subseteq P_2^{M_\beta}$ has cardinality at most $|\alpha| + 2^{\aleph_0}$.

3. If $\alpha = \beta + 1$ and $t_\beta$ is relevant to $M_\beta$, $M_\alpha$ satisfies task $t_\beta$.

case 1 If $\alpha = 0$, set $M_0 = M_* \upharpoonright (P_0^{M_*} \cup P_1^{M_*})$.

case 2 Take unions at limits.

case 3 $\alpha = \beta + 1$ and $t_\beta \in T_1$; say, $t_\beta = (N_1, N_2)$.

If $N_1$ is not embedded in $M_\beta$ then the task is irrelevant and let $M_\alpha = M_\beta$. Let $\langle c_i : i < m \rangle$ enumerate $P_2^{N_2} - P_1^{N_1}$ and $\langle d_i : i < m \rangle$ enumerate the first $m$ elements of $P_2^{M_\beta} - P_2^{M_*}$.

By induction, since $M_\beta \in \mathbb{M}_2$ there are witnesses $w_\beta = (\alpha_d, \eta_d, \alpha_d)$ (formally $\langle \alpha_{d_\beta}, \eta_{d_\beta}, \alpha_{d_\beta} \rangle$) for $d \in P_2^{M_\beta}$. By Definition 3.13.C, we can fix $U_\alpha \subseteq \lambda$ of cardinality $\leq |\alpha| + 2^{\aleph_0}$ such that:

{Ualph}
\( \{ a_{d,k} : k < \omega, d \in P_{2}^{M_\beta} \} \cup \{ b_\nu : (\exists d \in P_{2}^{M_\beta})(\alpha_d) \leq \nu \in \mathcal{T}_\alpha \} \cup P_{4,1}^{M_r} \)

is included in the subalgebra of \( M_\alpha \) generated by the
\[
\{ b_\rho : \exists \beta \in U_\alpha, \langle \beta \rangle \leq \rho \in \mathcal{T}_\beta \} \cup \{ b_0 \} \cup P_{4,1}^{M_r}.
\]

Let \( M_\alpha \) extend the universe of \( M_\beta \) by adding \( \langle d_i : i < m \rangle \subset P_{2}^{M_\beta} \). Note that the domain of \( M_\alpha \) is a subset of \( M_\beta \), but \( M_\alpha \) is not a substructure of \( M_\beta \); we are about to define the \( F_k^{M_\alpha} \) at the \( d_i \) for \( i < m \). Let \( k_\ast \) be large enough and let \( B \) be a finite Boolean sub-algebra of \( P_{2}^{N_2} \) and \( b_\ast \in B \) be as in Definition 3.1.3 of \( K_1 \). In particular \( b_\ast \) is a finite union of atoms of \( P_1^{M_\ast} \), which are in \( P_1^{N_2} \), and \( P_1^{N_2} \) is generated freely over \( P_1^{N_1} \cup B \) by \( \{ F_k^{N_2}(c_i) : k_\ast \leq k < \omega, i < m \} \).

To extend the witnesses to \( M_\alpha \), let \( \langle (\beta_i, \eta_i) : i < m \rangle \) be such that the \( \beta_i \) are a strictly increasing list of the first \( m \) even members of \( \lambda - U_\alpha \) with \( \eta_i \in \mathcal{T}_\beta \). Let \( a_{d_i,k} \) be the 0 of \( P_1^{M_\ast} \) for \( i < m, k < \omega \).

We first map \( B \) into \( M_\beta \); map atoms \( a \in P_{1}^{N_2} - P_{4,1}^{N_1} \) into atoms \( a' \in P_{2}^{M_\beta} - P_{4,1}^{M_\beta} \). Then map the finitely many \( F_k^{N_2}(c_i) \) for \( k < n_\ast, i < m \) to \( b_{k,i} \) which are in \( P_{1}^{N_1} \) and independent over \( B_{n_\ast} \) for \( N_1 \). Now let \( F_k^{M_\ast}(d_i) \) be the join of \( b_{k,i} \) with all the \( a \in P_{1}^{N_1} \) that lie below \( F_k^{N_2}(c_i) \) and the \( a' \) such that \( a' \in P_{4,1}^{N_1} - P_{4,1}^{N_1} \) and \( a' \leq F_k^{N_2}(c_i) \).

Now, the \( \{ b_{(n,i,k)\triangleright<1} \cup b_{(n,i,k)\triangleright<1} : i < m \} \) are independent\(^4\) over \( P_1^{N_1} \).

So we can define \( h_\beta \) to embed the Boolean algebra \( P_{1}^{N_2} \) into \( P_{1}^{M_\ast} \) over \( P_1^{N_1} \) such that \( k \geq k_\ast \) implies
\[
h_\beta(F_k^{N_2}(c_j)) = b_{n_\ast,k} \| b_{n_\ast,k}.
\]

By Claim 3.8.3, since the \( \beta_i \) are even, for each \( a \in P_{4,1}^{M_\ast} \), for some \( n, \eta, \rho \in \mathcal{T}_\alpha \) with \( \lg(\eta) \geq n \) and \( \lg(\rho) \geq n \) then \( a \leq F_k^{M_\ast} \) if and only if \( a \leq F_k^{M_\ast} \). For each \( j \) and \( k, b_{n_\ast,k} \| b_{n_\ast,k} = (b_{n_\ast,k} \| b_{n_\ast,k}) \| 0_{P_1^{M_\ast}} \). So setting \( a_{d_i,k} = 0 \) for \( i < m \), we have:
\[
F_k^{M_\ast}(d_j) = (b_{n_\ast,k} \| b_{n_\ast,k}) \| 0_{P_1^{M_\ast}},
\]
and for each \( a \in P_{4,1}^{M_\ast} \), for some \( n, a \not\in P_1^{M_\ast}, F_k^{M_\ast}(d_j) \). Thus, \( M_\alpha \in M_1 \) and so in \( M_\alpha \in M_2 \) as required.

\(^4\)Suppose one takes any partition of an independent set and chooses for each block one element which is a finite Boolean combination of elements from that block. Then, that set of elements is independent.
case 4 \( \alpha = \beta + 1 \) and \( t_\beta \in T_2 \); say, \( t_\beta = c \).

We define \( M_\alpha \). Recalling Definition 3.7, we have a witness \( \langle d^\beta_{d,k}, \eta^\beta_d, \alpha^\beta_d \rangle \) that \( M_\beta \in \mathbb{M}_2 \); we extend it to a witness for \( M_\alpha \). Let \( \gamma \) be an even ordinal such that \( \gamma \neq \alpha_d \) for any \( d \in P_2^{M_\beta} \) and \( \nu(0) \neq \gamma \) if \( b_\nu = a_{d,k} \) for some \( k < \omega \) and \( d \in P_2^{M_\beta} \). Let \( \langle d_\gamma; \eta \in \lim \mathcal{T}_\gamma \rangle \) be a set of pairwise distinct elements of \( P_2^{M_\beta} - P_2^{M_\beta} \). And, let \( M_\alpha \) be generated by \( M_\beta \cup \{ d_\eta; \eta \in \lim \langle \mathcal{T}_\gamma \rangle \} \).

To define \( F_k^{M_\alpha}(d_\eta) \), for each \( \eta \in \lim \mathcal{T}_\gamma \) and \( k < \omega \), choose \( i_0 < i_1 \leq 2 \) that are different from \( \eta(k) \). Recalling \( c = t_\beta \), let

\[
F_k^{M_\alpha}(d_\eta) = (b_{\eta|k\backslash \alpha_0} \triangle b_{\eta|k\cap \gamma_1}) \triangle (F_k^{M_\tau}(c)).
\]

Thus, for the \( d \in P_2^{M_\alpha} - P_2^{M_\beta} \) chosen towards satisfying \( t_\beta = c \), we have set \( \langle \alpha^\beta_d, \eta^\beta_d, \alpha^\beta_d \rangle = (\gamma, d_\eta, F_k^{M_\alpha}(c)) \).

It is routine to show \( M_\alpha \in \mathbb{M}_2 \). We must show \( M_\alpha \) satisfies task \( t_\beta \). For this, suppose \( D \) is an ultrafilter on \( P_1^{M_\beta} \) such that the set \( S_{c}^{M_\beta}(D) = \{ n: F_n^{M_\alpha}(c) \in D \} \) is infinite (Definition 2.2.1). Define \( \eta^D \in \lim \mathcal{T}_\gamma \) by induction\(^5\); \( \eta^D(0) = \gamma \). By Remark 3.14.2 one of the three elements \( b_{(\gamma,i)\triangle b_{(\gamma,j)}} \), for \( i \neq j \) and \( i, j < 3 \), must not be in \( D \). Let \( \eta^D(1) \) be the other member of \( \{ 0, 1, 2 \} \). For \( k \geq 1 \), suppose \( \nu = \eta^D \cup k \) has been defined. Again, by Remark 3.14.2 one of the three elements \( b_{\nu|k}\triangle b_{\nu|j} \), for \( i \neq j \) and \( i, j < 3 \), must not be in \( D \). Let \( \eta^D(k) \) be the other member of \( \{ 0, 1, 2 \} \). For the infinitely many \( n \) with \( F_n^{M_\alpha}(c) \in D \), we have \( F_n^{M_\alpha}(d_{\eta^D}) \in D \).

To show that \( t_\beta \) is satisfied by \( d_{\eta^D} \), we now verify Definition 1.2.7: for every \( a \in P_1^{M_\beta} \), for all but finitely many \( n \), \( F_n^{M_\alpha}(d_{\eta^D}) \land a = 0_{P_1^{M_\alpha}} \). As \( M_\alpha \in K_{\mathcal{L}_1} \), by Definition 1.2.7 we have for every large enough \( n \), \( P_1^{M_\alpha} = P_1^{M_\alpha} \land a = 0_{P_1^{M_\alpha}} \). Now, recall from 3.8.3, that for every \( a \in P_1^{M_\alpha} \) and the even ordinals \( \alpha < \lambda \), there is an \( n \), such that for any \( \nu, \rho \in \mathcal{T}_\alpha \) with \( \lg(\eta) \geq n \) and \( \lg(\rho) \geq n, a \leq \rho_{P_1^{M_\alpha}} \), \( b_\nu \), and only if \( a \leq \rho_{P_1^{M_\alpha}} \), \( b_\rho \). As \( \gamma \) is even, it follows that for every \( a \in P_1^{M_\alpha} \), and large enough \( n, a \land F_k^{M_\alpha}(d_{\eta^D}) = 0_{P_1^{M_\alpha}} \). Since this argument holds for each \( D \) such that \( S_{c}^{M_\beta}(D) \in \mathcal{K}_{\mathcal{L}_1} \) is infinite, we have verified \( t_\beta = c \).

\[ \square \]

**Conclusion 3.19** We have found a \( P_0 \)-maximal \( M \in K_2 \) with all \( |P_1^{M}| = \lambda \). As in [BS17a], for every \( \lambda \) less than the first measurable, since \( M \in K_2 \) implies \( |M| \leq 2^{P_0^M} \), there is a maximal model \( M \in K_{\mathcal{L}_2} \) with \( 2^{\lambda} \leq |M| < 2^{2^\lambda} \).

**Remark 3.20** Note that the model \( M \) contains uncountably many elements \( d_\eta \in P_2^{M} \), which were constructed in case 4, such that for some \( \alpha_d \), each of the \( \eta(0) = \alpha_d \), but \( \eta \) and \( \eta' \) differ at \( k \geq k_\gamma \), and \( F_k^{M_\alpha}(d_\eta) = F_k^{M_\alpha}(d_\eta') \) as, \( F_k^{M_\alpha}(d_\eta) = (b_{\eta|k\cap \gamma_0} \triangle b_{\eta|k\cap \gamma_1}) \triangle (F_k^{M_\beta}(c)) \). This contradicts nice freeness. In contrast, the \( P_0 \)-maximal model

\(^5\)This argument is patterned on the simple black box in Lemma 1.5 of [She], but even simpler.
constructed in [BS17a] using diamond, was $K_1$-free for subalgebras of cardinality $< \lambda$.

**Question 3.21**

1. Is there a $\kappa < \mu$, where $\mu$ is the first measurable, such that if a complete sentence has a maximal model in cardinality $\kappa$, it has maximal models in cardinalities cofinal in $\mu$?

2. Is there a complete sentence that has maximal models cofinally in some $\kappa$ with $\beth_{\omega_1} < \kappa < \mu$ where $\mu$ is the first measurable, but no larger models are maximal. Could the first inaccessible be such a $\kappa$?

**References**


