Hanf Numbers and Presentation Theorems in AEC

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1 Introduction

Grossberg [Gro02, Conjecture 9.3] has raised the question of the existence of Hanf numbers for joint embedding and amalgamation in Abstract Elementary Classes (AEC). Various authors have given lower bounds (discussed in Section 5), usually for the disjoint version of these properties. Here we show a strongly compact cardinal κ is an upper bound for various such Hanf numbers. We define 4 kinds amalgamation properties(with various cardinal parameters) in the next section and a 5th at the end of Section 3.

Recall that the Hanf number of the property P (depending on one cardinal parameter) for AEC's with Löwenheim-Skolem number[1] μ is the least κ such that if $\boldsymbol{K}$ is such an AEC and $\boldsymbol{K}$ has a model of cardinality $\geq \kappa$ with property P it has such models in all larger cardinals. Our main result is the following: {mt}

Theorem 1.1. *Let κ be strongly compact and $\boldsymbol{K}$ be an AEC with Löwenheim-Skolem number less than κ. If $\boldsymbol{K}$ satisfies[2] AP/JEP/DAP/DJEP for models of size $[\mu, <\kappa)$,*

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[1] For simplicity we write Löwenheim-Skolem number for the sup of the Löwenheim-Skolem number and the cardinality of the vocabulary of $\boldsymbol{K}$. See Chapter 4 of [Bal09].

[2] This alphabet soup is decoded in Definition 1.2.

then $\boldsymbol{K}$ satisfies AP/JEP/DAP/DJEP/NDJEP for all models of size $\geq \mu$.

Our first attack on this problem combined Boney's recognition [Bon] that arbitrary AEC's can be interpreted into theories in $\mathbb{L}_{\kappa,\omega}$ for sufficiently large κ, and with the fact that, for κ strongly compact[3] , the usual first order syntactic characterization of amalgamation (e.g. 6.5.1 of [Hod93]), etc. holds in $L_{\kappa,\omega}$. This intuition gives a partial but unsatisfactory result (Section 3). The dissatisfaction stems from two sources. On the one hand, we found a shorter semantic proof of a stronger result (Section 2). On the other, the argument in Section 3 requires the restriction to disjoint amalgamations and indeed, a modification of the presentation theorem to achieve that for disjoint amalgamation. This weakness of the syntactic argument stems from the *ad hoc* nature of the Skolem expansion in Shelah's presentation theorem. This led us to the notion of a 'relational presentation' theorem (Theorem 4.3). We then can interpret an arbitrary AEC into an infinitary logic using natural $LS(\boldsymbol{K})$-ary predicates, the description of entire models. While this approach give at least *a priori* weaker bounds on Hanf numbers, it has several other advantages. It is sensitive to good behavior (e.g. stability) of the AEC and reduces the number of arbitrary choices in the standard proof of the presentation theorem.

In addition, we get some associated results for smaller large cardinals. There is a pattern of connections between specific kinds of transfers with weakly compact, measurable, and strongly compact cardinals. We illustrate it only in Section 4, but there are analogs in sections 2 and 3. Section 5 catalogs the known lower bounds on the Hanf numbers studied; for countable vocabularies they are all below $\beth_{\omega_1}$. We begin with a brief Section 1.1 establishing vocabulary and some basic results.

Of course, Theorem 1.1 implies the weaker version where μ is set to 0 in the statement. But we are interested in the behavior of these properties on the tail and the existence of examples which fail e.g the joint embedding property below a certain μ and satisfy it beyond are easy to come by. For example, take the disjunction of sentences ϕ_1, ϕ_2 in disjoint vocabularies such that ϕ_1 has no model above μ and ϕ_2 is categorical in all powers. So the result as stated covers many more cases than just assuming $JEP(< \kappa)$.

1.1 Preliminaries

{prelim}

We discuss the relevant background of AECs, especially for the case of disjoint amalgamation. {mdef}

Definition 1.2. *We consider several variations on the joint embedding property, written JEP or JEP$[\mu, \kappa)$.*

[3]Motivated by work of Brooke-Taylor and Rosický [BTR15] or Boney and Unger [BU], uses of κ strongly compact/measurable/weakly compact when $\kappa > LS(\boldsymbol{K})$ can be replaced by κ almost strongly compact/almost measurable/almost weakly compact to get the same result. We present the arguments here with the stronger assumptions for clarity.

1. *An AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the joint embedding property, JEP, (on the interval $[\mu, \kappa)$) if any two models (of size at least μ and less than κ) can be $\boldsymbol{K}$-embedded into a larger model.*

2. *If the embeddings witnessing the joint embedding property can be chosen to have disjoint ranges, then we call this the* disjoint embedding property *and write $DJEP$.*

3. *An AEC $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation property, AP, (on the interval $[\mu, \kappa)$) if, given any triple of models $M_0 \prec M_1, M_2$ (of size at least μ and less than κ), M_1 and M_2 can be $\boldsymbol{K}$-embedded into a larger model by embeddings that agree on M_0.*

4. *If the embeddings witnessing the amalgamation property can be chosen to have disjoint ranges except for M_0, then we call this the* disjoint amalgamation property *and write $DJAP$.* {ECG}

Definition 1.3. 1. *A* finite diagram *or* $EC(T, \Gamma)$-class *is the class of models of a first order theory T which omit all types from a specified collection Γ of complete types in finitely many variables over the empty set.*

2. *Let Γ be a collection of first order types in finitely many variables over the empty set for a first order theory T in a vocabulary τ_1. A $PC(T, \Gamma)$* class *is the class of reducts to $\tau \subset \tau_1$ of models of a first order theory τ_1-theory T which omit all members of the specified collection Γ of partial types.*

   We write $PC\Gamma$ to denote such a class without specifying either T or Γ.

Our basic theorem gives a Hanf number for the disjoint embedding property; we give some extensions below. We first require the following tool, which is proved in this form as [Bal09, Theorem 4.15] (less detailed versions appear in [Gro, She09] and was first stated in [She87].

2 Semantic arguments

{semanticarg}

It turns out that the Hanf number computation for the amalgamation properties is immediate from Boney's "Łoś' Theorem for AECs" [Bon, Theorem 4.3]. We will sketch the argument for completeness. For convenience here, we take the following of the many equivalent definitions of strongly compact; it is the most useful for ultraproduct constructions.

Definition 2.1 ([Jec06].20)**.** *The cardinal κ is strongly compact iff every κ-complete filter can be extended to a κ-complete ultrafilter. Equivalently, for every $\lambda \geq \kappa$, there is a fine[4], κ-complete ultrafilter on $P_\kappa\lambda = \{\sigma \subset \lambda : |\sigma| < \kappa\}$.*

[4] U is fine iff $G(\alpha) := \{z \in P_\kappa(\lambda) | \alpha \in z\}$ is an element of U for each $\alpha < \lambda$.

For this paper, the reader can take "essentially below κ" to mean "$LS(K) < \kappa$." {losaec}

Fact 2.2 (Łoś' Theorem for AECs)**.** *Suppose K is an AEC essentially below κ and U is a κ-complete ultrafilter on I. Then K and the class of K-embeddings are closed under κ-complete ultraproducts and the ultrapower embedding is a K-embedding.*

The argument for this theorem has two main steps. First, use Shelah's presentation theorem to interpret the AEC into $L_{\kappa,\omega}$ and then use the fact that $L_{\kappa,\omega}$ classes are closed under ultraproduct by *κ-complete* ultraproducts. {mt1}

Theorem 2.3. *Let κ be strongly compact and $\boldsymbol{K}$ be an AEC with Löwenheim-Skolem number less than κ.*

- *If $\boldsymbol{K}$ satisfies $AP(< \kappa)$ then $\boldsymbol{K}$ satisfies AP.*
- *If $\boldsymbol{K}$ satisfies $JEP(< \kappa)$ then $\boldsymbol{K}$ satisfies JEP.*
- *If $\boldsymbol{K}$ satisfies $DAP(< \kappa)$ then $\boldsymbol{K}$ satisfies DAP.*

Proof: We first sketch the proof for the first item, AP, and then note the modifications for the other two.

Suppose that $\boldsymbol{K}$ satisfies $AP(< \kappa)$ and consider a triple of models (M, M_1, M_2) with $M \prec_{\boldsymbol{K}} M_1, M_2$ and $|M| \leq |M_1| \leq |M_2| = \lambda \geq \kappa$. Now we will use our strongly compact cardinal. An *approximation* of (M, M_1, M_2) is a triple $\boldsymbol{N} = (N^{\boldsymbol{N}}, N_1^{\boldsymbol{N}}, N_2^{\boldsymbol{N}}) \in (K_{<\kappa})^3$ such that $N^{\boldsymbol{N}} \prec M, N_\ell^{\boldsymbol{N}} \prec M_\ell, N^{\boldsymbol{N}} \prec N_\ell^{\boldsymbol{N}}$ for $\ell = 1, 2$. We will take an ultraproduct indexed by the set X below of approximations to the triple (M, M_1, M_2). Set

$$X := \{\boldsymbol{N} \in (\boldsymbol{K}_{<\kappa})^3 : \boldsymbol{N} \text{ is an approximation of } (M, M_1, M_2)\}$$

For each $\boldsymbol{N} \in X$, $AP(< \kappa)$ implies there is an amalgam of this triple. Fix $f_\ell^{\boldsymbol{N}} : N_\ell^{\boldsymbol{N}} \to N_*^{\boldsymbol{N}}$ to witness this fact. For each $(A, B, C) \in [M]^{<\kappa} \times [M_1]^{<\kappa} \times [M_2]^{<\kappa}$, define

$$G(A, B, C)] := \{\boldsymbol{N} \in X : A \subset N^{\boldsymbol{N}}, B \subset N_1^{\boldsymbol{N}}, C \subset N_2^{\boldsymbol{N}}\}$$

These sets generate a κ-complete filter on X, so it can be extended to a κ-complete ultrafilter U on $P_\kappa X$; note that this ultrafilter will satisfy the appropriate generalization of fineness, namely that $G(A, B, C)$ is always a U-large set. Since this ultrafilter is fine, Łoś' Theorem for AECs implies that

$$h : M \to \Pi N^{\boldsymbol{N}}/U$$
$$h_\ell : M_\ell \to \Pi N_\ell^{\boldsymbol{N}}/U \qquad \text{for } \ell = 1, 2$$

Since these maps have a uniform definition, they agree on their common domain M. Furthermore, we can average the $f_\ell^{\boldsymbol{N}}$ maps to get ultraproduct maps

$$\Pi f_\ell^{\boldsymbol{N}} : \Pi N_\ell^N/U \to \Pi N_*^{\boldsymbol{N}}/U$$

and the maps $f_\ell^{\boldsymbol{N}}$ for $\ell = 1, 2$ agree on $\Pi N^{\boldsymbol{N}}/U$ since each of the individual functions do. As each N_ℓ embeds in $\Pi N_\ell^N/U$ the composition of the f and h maps gives the amalgam.

There is no difficulty if one of M_0 or M_1 has cardinality $< \kappa$; many of the approximating triples will have the same first or second coordinates but this causes no harm. Similary, we get the JEP transfer if $M_0 = \emptyset$. And we can transfer disjoint amalgamation since in that case each $N_1^{\boldsymbol{N}} \cap N_2^{\boldsymbol{N}} = N^{\boldsymbol{N}}$ and this is preserved by the ultraproduct. $\dagger_{2.3}$

3 $EC(T_1, \Gamma)$-syntactic Approach

{syn1}

In this section, we discuss results that can be obtained directly by (or with minor variations on) Shelah's Presentation Theorem. Our arguments use some of the specifics of the proof, so we have included an outline in Section 3. For full details, consult [Bal09, Theorem 4.15]. The assertion (and argument) for part 3 is new in this paper; the argument is embedded in the proof of Lemma 3.3. To prepare for it, we review the proof:

{pres}

Fact 3.1 (Shelah's Presentation Theorem)**.** *If $\boldsymbol{K}$ is an AEC (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$) with Löwenheim-Skolem-Skolem-number $\mathrm{LS}(\boldsymbol{K})$, there is a vocabulary $\tau_1 \supseteq \tau$ with cardinality $|\mathrm{LS}(\boldsymbol{K})|$, a first order τ_1-theory T_1 and a set Γ of at most $2^{\mathrm{LS}(\boldsymbol{K})}$ partial types such that:*

$$\boldsymbol{K} = \{M'|\tau : M' \models T_1 \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, the $\prec_{\boldsymbol{K}}$ relation satisfies the following conditions:

1. *if M' is a τ_1-substructure of N' where M', N' satisfy T_1 and omit Γ then $M'|\tau \prec_{\boldsymbol{K}} N'|\tau$;*
2. *if $M \prec_{\boldsymbol{K}} N$ there is an expansion of N to a τ_1-structure N' such that M is the universe of a τ_1-substructure of N';*
3. *More strongly, if $M \prec N \in \boldsymbol{K}$ and $M' \in EC(T_1, \Gamma)$ such that $M'|\tau = M$, then there is $N' \in EC(T_1, \Gamma)$ such that $M' \subset N'$ and $N'|\tau = N$.*

Proof Outline of Fact 3.1: Let τ' contain n-ary function symbols F_i^n for $n < \omega$ and $i < \mathrm{LS}(\boldsymbol{K})$. We take as T' the theory which asserts that for each $i < \lg(\boldsymbol{a})$, $F_i^n(\boldsymbol{a}) = a_i$. The types in Γ are precisely the quantifier-free types of tuples $a_1, \ldots, a_n$ such that either

- $\{F_i^n(a_1, \ldots, a_n) : i < LS(K)\}$ does not enumerate a model in $\boldsymbol{K}$; or
- there is a subtuple $\mathbf{b} \subset \mathbf{a}$ that enumerates something that is not a strong substructure of the above model.

Now one must show that each model in $\boldsymbol{K}$ can be expanded to meet these conditions. It follows from the closure of AECs under unions of chains that they are also closed under unions of directed systems. Thus, for each $M \in \boldsymbol{K}$, we can write it as the union of a directed system, called a *cover*, $\{M_{\mathbf{a}} \in K_{LS(K)} : \mathbf{a} \in {}^{<\omega}M\}$ with the property that $\mathbf{a} \in M_{\mathbf{a}}$. After we pick a cover, we interpret the functions $\{F_i^n : i < LS(K), n < \omega\}$ such that $\{F_i^n(a_1, \ldots, a_n) : i < LS(K)\}$ is some enumeration of the universe of $M_{a_1,\ldots,a_n}$.

Now one must verify each of the moreover's. The first two are in [Bal09] in exactly this form. We verify the third in the proof of in the proof of Lemma 3.3.

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The outline for proving that κ is a Hanf number is to prove a triple equivalence of the following form:

1. A property (DAP, $DJEP$, etc.) holds in $\boldsymbol{K}_{<\lambda}$.
2. The property holds in $(EC(T_1, \Gamma), \subset)_{<\lambda}$.
3. A corresponding syntactic condition holds on some $(L_1)_{LS(\boldsymbol{K})^+,\omega}$-theory of size $< \lambda$.

Then the compactness of the logic transfers the *syntactic property* 3) from $\boldsymbol{K}_{<\lambda}$ to all of $\boldsymbol{K}$ via (3). The equivalence of (2) and (3) is typically easy, as is the implication (2) $\to$ (1). However, when working to show (1) $\to$ (2) with Shelah's Presentation Theorem, disjointness becomes a necessary condition. This is because of the choices made in the presentation; if different choices are made for elements that the amalgamation wants to identify, this is problematic. However, some results are still possible for DAP, $DJEP$, and a new property, $NDJEP$, which we discuss at the end of this section.

Each property has a corresponding syntactic characterization, as mentioned in (3) above.

Definition 3.2. • Ψ *has* $<\lambda$-DAP satisfiability *iff for any expansion by constants* $\mathbf{c}$ *and all sets of atomic and negated atomic formulas (in* $\tau(\Psi)\cup\{\mathbf{c}\}$*)* $\delta_1(\mathbf{x},\mathbf{c})$ *and* $\delta_2(\mathbf{y},\mathbf{c})$ *of size* $<\lambda$*, if* $\Psi\wedge\exists\mathbf{x}\,(\bigwedge\delta_1(\mathbf{x},\mathbf{c})\wedge\bigwedge x_i\neq c_j)$ *and* $\Psi\wedge\exists\mathbf{y}\,(\bigwedge\delta_2(\mathbf{y},\mathbf{c})\wedge\bigwedge y_i\neq c_j)$ *are separately satisfiable, then so is*

$$\Psi\wedge\exists\mathbf{x},\mathbf{y}\left(\bigwedge\delta_1(\mathbf{x},\mathbf{c})\wedge\bigwedge\delta_2(\mathbf{y},\mathbf{c})\wedge\bigwedge_{i,j}x_i\neq y_j\right)$$

• Ψ *has* $<\lambda$-DJEP satisfiability *iff for all sets of atomic and negated atomic formulas (in* $\tau(\Psi)$*)* $\delta_1(\mathbf{x})$ *and* $\delta_2(\mathbf{y})$ *of size* $<\lambda$*, if* $\Psi\wedge\exists\mathbf{x}\bigwedge\delta_1(\mathbf{x})$ *and* $\Psi\wedge\exists\mathbf{y}\bigwedge\delta_2(\mathbf{y})$ *are separately satisfiable, then so is*

$$\Psi\wedge\exists\mathbf{x},\mathbf{y}\left(\bigwedge\delta_1(\mathbf{x})\wedge\bigwedge\delta_2(\mathbf{y})\wedge\bigwedge_{i,j}x_i\neq y_j\right)$$

We now outline the argument for $DJEP$; the others are similar. Note that (2) $\rightarrow$ (1) for DAP has been shown by Hyttinen and Kesälä [HK06, 2.16] and $NDJEP$ requires a slight modification of Shelah's Presentation Theorem; see Lemma 3.6 below. We provide more details in the next section. {syntrans}

Lemma 3.3. *Suppose that* $\boldsymbol{K}$ *is an AEC,* $\lambda>LS(\boldsymbol{K})$*, and* T_1 *and* Γ *are from Shelah's Presentation Theorem. Let* Φ *be the* $L_{LS(\boldsymbol{K})^+,\omega}$ *theory that asserts the satisfaction of* T_1 *and omission of each type in* Γ*. Then the following are equivalent:*

1. $\boldsymbol{K}_{<\lambda}$ *has* $DJEP$*.*
2. $(EC(T_1,\Gamma),\subset)_{<\lambda}$ *has DJEP.*
3. Φ *has* $<\lambda$*-*$DJEP$*-satisfiability.*

Proof:

(1) $\leftrightarrow$ (2): First suppose that $\boldsymbol{K}_{<\lambda}$ has DJEP. Let $M_0^*,M_1^*\in EC(T_1,\Gamma)_{<\lambda}$ and set $M_\ell:=M_\ell^*\mid\tau$. By disjoint embedding for $\ell=0,1$, there is $N\in\boldsymbol{K}$ such that each $M_\ell\prec N$. Our goal is to expand N to be a member of $EC(T_1,\Gamma)$ in a way that respects the already existing expansions. The expansions found in the standard proof give us, for each $\mathbf{a}\in{}^nM_1^*$ and for each $\mathbf{a}\in{}^nM_2^*$, some $M_{\mathbf{a}}^*$ whose universe is enumerated by $\{F_i^{\ell(\mathbf{a})}(\mathbf{a}):i<LS(K)\}$ with those functions being interpreted in whichever structure $\mathbf{a}$ lives; if it's in the intersection, since M_0^* is a τ_1 substructure of M_1^* and M_2^* these functions have the same values. If not, by induction on length of $\mathbf{b}\in N$, using the disjointness[5], one can choose $N_{\mathbf{b}}\in\boldsymbol{K}_{LS(K)}$ such that

[5] It is perfectly possible for $a_1\in M_1\setminus M_0$ and $a_2\in M_2\setminus M_0$ to be identified by an ordinary $\boldsymbol{K}$-amalgamation even though they have different τ_1-types over the empty set. {noap}

(a) $N_{\mathbf{b}} \prec N$;

(b) if $\mathbf{a} \subset \mathbf{b}$, then $N_{\mathbf{a}} \prec N_{\mathbf{b}}$; and

(c) if $\mathbf{b} \in {}^n M_1^*$ or $\mathbf{b} \in {}^n M_2^*$, then $N_{\mathbf{b}} = M_{\mathbf{b}}^*$.

Now, to expand N, we want to define $\{F_i^n : i < LS(K), n < \omega\}$ on it such that $|N_{\mathbf{b}}| = \{F_i^{\ell(\mathbf{b})}(\mathbf{b}) : i < LS(K)\}$ and that $F_i^{\ell(\mathbf{b})}(\mathbf{b}) = b_i$ for $i < \ell(\mathbf{b})$. If $\mathbf{b} \in {}^n M_1^*$ or $\mathbf{b} \in \cup^n M_2^*$, then this is already done. Otherwise, there is no restriction on $F_i^n(\mathbf{b})$, so define it arbitrarily to enumerate the right set. This gives an expansion N^* such that

- $M_1^*, M_2^* \subset N^*$; and
- for every $\mathbf{b} \in {}^n N$, $\{F_i^n(\mathbf{b}) : i < LS(K)\} \restriction L = N_{\mathbf{b}} \in \boldsymbol{K}$ and for $\mathbf{a} \subset \mathbf{b}$, $N_{\mathbf{a}} \prec N_{\mathbf{b}}$.

Thus, $N^* \in EC(T_1, \Gamma)$ contains M_1^* and M_2^* as desired.

Second, suppose that $EC(T_1, \Gamma)$ has λ-DJEP. Let $M_0, M_1 \in \boldsymbol{K}$; WLOG, $M_0 \cap M_1 = \emptyset$. Using Shelah's Presentation Theorem, we can expand to $M_0^*, M_1^* \in EC(T_1, \Gamma)$. Then we can use disjoint embedding to find $N^* \in EC(T_1, \Gamma)$ such that $M_1^*, M_2^* \subset N^*$. By Shelah's Presentation Theorem 3.1.(1), $N := N^* \restriction L$ is the desired model.

(2) $\leftrightarrow$ (3): First, suppose that Φ has $< \lambda$-DJEP satisfiability. Let $M_0^*, M_1^* \in EC(T_1, \Gamma)$ be of size $< \lambda$. Let $\delta_0(\mathbf{x})$ be the quantifier-free diagram of $M_0^*$and $\delta_1(\mathbf{y})$be the quantifier-free diagram of M_1^*. Then $M_0^* \vDash \Phi \wedge \exists \mathbf{x} \bigwedge \delta_0(\mathbf{x})$; similarly, $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_1(\mathbf{y})$ is satisfiable. By the satisfiability property, there is N^* such that

$$N^* \vDash \Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_0(\mathbf{x}) \wedge \bigwedge \delta_1(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Then $N^* \in EC(T_1, \Gamma)$ and contains disjoint copies of M_0^* and M_1^*, represented by the witnesses of $\mathbf{x}$ and $\mathbf{y}$, respectively.

Second, suppose that $(EC(T_1, \Gamma), \subset)_{<\lambda}$ has DJEP. Let $\Phi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ be as in the hypothesis of $< \lambda$-DJEP satisfiability. Let M_0^* witness the satisfiability of the first and M_1^* witness the satisfiability of the second; note both of these are in $EC(T_1, \Gamma)$. By DJEP, there is $N \in EC(T_1, \Gamma)$ that contains both as substructures. This witnesses

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Note that the formulas in δ_1 and δ_2 transfer up because they are atomic or negated atomic.

†

{sctrans}

Lemma 3.4. *Assume κ is strongly compact and let $\Psi \in L_{\kappa,\omega}(\tau_1)$ and $\lambda > \kappa$. If Ψ has $< \kappa$-DJEP-satisfiability, then Ψ has $< \lambda$-DJEP-satisfiability.*

Obviously the converse (for $\Psi \in L_{\infty,\omega}$)) holds without any large cardinals.

Proof: Let $\delta_1(\mathbf{x}), \delta_2(\mathbf{y})$ be sets of atomic and negated atomic formulas in τ_1 of size $< \lambda$ and $\mathbf{a}, \mathbf{b}$ be sequences of constants not in τ_1 of length $< \lambda$ such that $\Psi \wedge \bigwedge \delta_1(\mathbf{a})$ and $\Psi \wedge \bigwedge \delta_2(\mathbf{b})$ are satisfiable. We want to show that $\Psi \cup \{\delta_1(\mathbf{a})\} \cup \{\delta_2(\mathbf{b})\} \cup \{a_i \neq b_j\}$ is satisfiable. Since Ψ has $< \kappa$-DJEP satisfiability, each $< \kappa$ sized subtheory is satisfiable. Note that this is an $L_{\kappa,\omega}$ theory so, since κ is strongly compact, the whole theory is satisfiable. †

Proof of Theorem 1.1 for DAP and $DJEP$: We first complete the proof for DJEP. By Lemma 3.3, $< \kappa$-DJEP implies that Φ has $< \kappa$-DJEP satisfiability. By Lemma 3.4, Φ has $< \lambda$-DJEP satisfiability for every $\lambda \geq \kappa$. Thus, by Lemma 3.3 again, $\boldsymbol{K}$ has DJEP.

The proof for DAP is exactly analogous. †

The definition of *nearly disjoint joint embedding property* ($NDJEP$ for short) might seem artificial, but it is grounded in algebraic examples[6]. Groups, fields, modules, etc. come with identities that must be mapped to the same thing in any joint embedding, so full $DJEP$ is impossible. However, it is possible that this is the *only* obstacle to $DJEP$. Thus, $NDJEP$ allows for these constants (and the substructures they generate) to overlap as long as this is the only overlap. Note that $DJEP$ is an instance of $NDJEP$ that expands by no constants. The theory of $(\mathbb{Z}, +, 0)$ is an example showing sometimes extra constants are needed; it has no prime model while $(\mathbb{Z}, +, 0, 1)$ does (see [BBGK73]).
After giving the definition, we vary the proof above to prove Theorem 1.1 for this case. {ndjwpdef}

Definition 3.5.

1. *For A a set of constants $\{c_i : i < \mu\}$ not in the vocabulary , we say that $\boldsymbol{K}$ is an A-AEC iff every $M \in \boldsymbol{K}$ is associated with a unique expansion M_A such that, in the resulting AEC, the substructures generated by the constants are canonically isomorphic, i. e., via the map $c_i^M \mapsto c_i^N$.*
2. *We say that an A-AEC $\boldsymbol{K}$ has the* nearly disjoint joint embedding property *[on the interval $[\mu, \kappa)$] iff any $M_1, M_2 \in \boldsymbol{K}$ [of size at least μ and less than κ] can be jointly embedded such that their overlap is exactly the substructure generated by the constants in A.*

We modify Shelah presentation theorem. The proof is as usual, except that no choices are made on the canonical substructure represented by the constants.

[6]The condition is built into the fundamental assumption in [BKL14].

{ptprime}

Lemma 3.6 (Modified Shelah's Presentation Theorem)**.** *If K is an A-AEC, then the T_1 and Γ from the Presentation Theorem 3.1 can be chosen such that the Skolem functions do not have the interpretation of A in their domain or range.*

Proof Sketch: The proof goes as normal. The differences are

- for each M, we choose a cover $\{M_{\mathbf{a}} \in \boldsymbol{K}_{LS(\boldsymbol{K})} : \mathbf{a} \in {}^{<\omega}(M - M_0)\}$ where M_0 is the substructure generated by $\{c^M : c \in A\}$; since M_0 appears in every submodel of M, this set is still a cover.
- rather than choosing $\{F_i^n(\mathbf{a}) : i < LS(K)\}$ to enumerate all of $|M_{\mathbf{a}}|$, we only enumerate $|M_{\mathbf{a}}| - |M_0|$.
- we add the constants from A to the vocabulary τ_1.

Since every model has an isomorphic copy of M_0, the proof can proceed as before. †

Here is the syntactic characterization of NDJEP

Definition 3.7.

*Ψ has $< \lambda$-*NDJEP satisfiability *iff for all sets of atomic and negated atomic formulas (in $\tau(\Psi)$) $\delta_1(\mathbf{x})$ and $\delta_2(\mathbf{y})$ of size $< \lambda$, if $\Psi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Psi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ are separately satisfiable, then so is*

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} \left(x_i \neq y_j \vee \bigvee_k x_i = y_j = c_k \right) \right)$$

Now the proof of the Hanf number for NDJEP proceeds as in Section 3 using Lemma 3.6.

Theorem 3.8. *Let $\boldsymbol{K}$ be strongly compact and $\boldsymbol{K}$ be an AEC with $LS(\boldsymbol{K}) < \kappa$. If $\boldsymbol{K}$ satisfies $[\mu, < \kappa) - NDJEP$, then it satisfies $\geq \mu - NDJEP$.*

4 The relational presentation theorem

{relpres}

We modify Shelah's Presentation Theorem by eliminating the two instances where an arbitrary choice must be made: the choice of models in the cover and the choice of an enumeration of each covering model. This elimination leads to a canonical expansion of each model which is functorial[7] between the two categories of models. (This concept is introduced in Vasey [Vasa, Definition 3.1] as an *abstract Morleyization*.)

However, there is a price to pay for this canonicity. In order to remove the choices, we must add predicates of arity $LS(K)$ and the relevant theory must allow $LS(K)$-ary quantification, potentially putting it in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, where $\kappa = LS(K)$; contrast this with a theory of size $\leq 2^\kappa$ in $\mathbb{L}_{\kappa^+,\omega}$ for Shelah's version. As a possible silver lining, these arities can actually be brought down to $\mathbb{L}_{(I(\boldsymbol{K},\kappa)+\kappa)^+,\kappa^+}$. Thus, properties of the AEC, such as the number of models in the Löwenheim-Skolem cardinal are reflected in the presentation, while this has no effect on the Shelah version.

We fix some notation. Let $\boldsymbol{K}$ be an AEC in a vocabulary τ and let $\kappa = LS(\boldsymbol{K})$. We assume that $\boldsymbol{K}$ contains no models of size $< \kappa$. The same arguments could be done with $\kappa > LS(\boldsymbol{K})$, but this case reduces to applying our result to $\boldsymbol{K}_{\geq\kappa}$.

We fix a collection of compatible enumerations for models $M \in K_\kappa$. ***Compatible enumerations*** means that each M has an enumeration of its universe, denoted $\mathbf{m}^M = \langle m_i^M : i < \kappa \rangle$, and, if $M \cong M'$, there is some fixed isomorphism $f_{M,M'} : M \cong M'$ such that $f_{M,M'}(m_i^M) = m_i^{M'}$ and if $M \cong M' \cong M''$, then $f_{M,M''} = f_{M',M''} \circ f_{M,M'}$.

For each isomorphism type $[M]_\cong$ and $[M \prec N]_\cong$ with $M, N \in K_\kappa$, we add to τ

$$R_{[M]}(\mathbf{x}) \text{ and } R_{[M\prec N]}(\mathbf{x};\mathbf{y})$$

as κ-ary and $\kappa 2$-ary predicates to form τ_*.

A skeptical reader might protest that we have made many arbitrary choices so soon after singing the praises of our choiceless method. The difference is that all choices are made prior to defining the presentation theory, T^*.

Once T^* is defined, no other choices are made.

The goal of the theory T^* is to recognize every strong submodel of size κ and every strong submodel relation between them via our predicates. This is done by expressing in the axioms below concerning sequences $\mathbf{x}$ of length at most κ the following properties connecting the canonical enumerations with structures in $\boldsymbol{K}$.

$$R_{[M]}(\mathbf{x}) \text{ holds iff } x_i \mapsto m_i^M \text{ is an isomorphism}$$

$$R_{[M\prec N]}(\mathbf{x},\mathbf{y}) \text{ holds iff } x_i \mapsto m_i^M \text{ and } y_i \mapsto m_i^N \text{ are isomorphisms and } x_i = y_j \text{ iff } m_i^M = m_j^N$$

[7] A functorial expansion of an AEC $\boldsymbol{K}$ in a vocabulary τ is an AEC $\hat{\boldsymbol{K}}$ in a vocabulary $\hat{\tau}$ extending τ such that i) each $M \in \boldsymbol{K}$ has a unique expansion to a $\hat{M} \in \hat{\boldsymbol{K}}$, ii) if $f : M \approx M'$ then $f : \hat{M} \approx \hat{M}'$, and iii) if M is a strong substructure of M' for $\boldsymbol{K}$, then $\hat{M}$ is strong substructure of $\hat{M}'$ for $\hat{\boldsymbol{K}}$.

Note that, by the coherence of the isomorphisms, the choice of representative from $[M]_{\cong}$ doesn't matter. Also, we might have $M \cong M'$; $N \cong N'$; $M \prec N$ and $M' \prec N'$; but not $(M, N) \cong (M', N')$. In this case $R_{[M \prec N]}$ and $R_{[M' \prec N']}$ are different predicates.

We now write the axioms for T^*. *A priori* they are in the logic $\mathbb{L}_{(2^\kappa)^+,\kappa^+}(\tau_*)$ but the theorem states a slightly finer result. To aid in understanding, we include a description prior to the formal statement of each property. {tspah}def}

Definition 4.1. *The theory T^* in $\mathbb{L}_{(I(\boldsymbol{K},\kappa)+\kappa)^+,\kappa^+}(\tau_*)$ is the collection of the following schema:*

{1}

1. If $R_{[M]}(\mathbf{x})$ holds, then $x_i \mapsto m_i^M$ should be an isomorphism.
If $\phi(z_1, \ldots, z_n)$ is an atomic or negated atomic τ-formula that holds of $m_{i_1}^M, \ldots, m_{i_n}^M$, then include

$$\forall \mathbf{x} \left(R_{[M]}(\mathbf{x}) \to \phi(x_{i_1}, \ldots, x_{i_n}) \right)$$

{2}

2. If $R_{[M \prec N]}(\mathbf{x}, \mathbf{y})$ holds, then $x_i \mapsto m_i^M$ and $y_i \mapsto m_i^N$ should be isomorphisms and the correct overlap should occur.
If $M \prec N$ and $i \mapsto j_i$ is the function such that $m_i^M = m_{j_i}^N$, then include

$$\forall \mathbf{x}, \mathbf{y} \left(R_{[M \prec N]}(\mathbf{x}, \mathbf{y}) \to \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i} \right) \right)$$

{7}

3. Every $< \kappa$-tuple is covered by a model.
If $\alpha < \kappa$, include the following where $\lg(\mathbf{x}) = \alpha$ and $\lg(\mathbf{y}) = \kappa$

$$\forall \mathbf{x} \exists \mathbf{y} \left(\bigvee_{[M]_{\cong} \in K_\kappa/\cong} R_{[M]}(\mathbf{y}) \wedge \bigwedge_{i<\alpha} \bigvee_{j<\kappa} x_i = y_{j_i} \right)$$

{8}

4. If $R_{[N]}(\mathbf{x})$ holds and $M \prec N$, then $R_{[M \prec N]}(\mathbf{x}^\circ, \mathbf{x})$ should hold for the appropriate subtuple $\mathbf{x}^\circ$ of $\mathbf{x}$.
If $M \prec N$ and $\pi : \kappa \to \kappa$ is the map so $m_i^M = m_{\pi(i)}^N$, then denote $\mathbf{x}^\pi$ to be the subtuple of $\mathbf{x}$ such that $x_i^\pi = x_{\pi(i)}$ and include

$$\forall \mathbf{x} \left(R_{[N]}(\mathbf{x}) \to R_{[M \prec N]}(\mathbf{x}^\pi, \mathbf{x}) \right)$$

{9}

5. Coherence: If $M \subset N$ are both strong substructures of the whole model, then $M \prec N$.
If $M \prec N$ and $m_i^M = m_{j_i}^N$, then include

$$\forall \mathbf{x}, \mathbf{y} \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i} \to R_{[M \prec N]}(\mathbf{x}, \mathbf{y}) \right)$$

Remark 4.2. *We have intentionally omitted the converse to Definition 4.1.(1), namely*

$$\forall \mathbf{x} \left(\bigwedge_{\phi(z_{i_1},\ldots,z_{i_n}) \in tp_{qf}(M/\emptyset)} \phi(x_{i_1},\ldots,x_{i_n}) \to R_{[M]}(\mathbf{x}) \right)$$

because it is not true. The "toy example" of a nonfinitary AEC–the $L(Q)$-theory of an equivalence relation where each equivalence class is countable–gives a counter-example.

For any $M^* \vDash T^*$, denote $M^* \restriction \tau \in K$ by M. {preb}{b}

Theorem 4.3 (Relational Presentation Theorem)**.** *1. If $M^* \vDash T^*$ then $M^* \restriction \tau \in K$. Further, for all $M_0 \in K_\kappa$, we have $M^* \vDash R_{[M_0]}(\mathbf{m})$ implies that $\mathbf{m}$ enumerates a strong substructure of M.* {a}

2. *Every $M \in K$ has a unique expansion M^* that models T^*.* {c}
3. *If $M \prec N$, then $M^* \subset N^*$.* {d}
4. *If $M^* \subset N^*$ both model T^*, then $M \prec N$.* {e}
5. *If $M \prec N$ and $M^* \vDash T$ such that $M^* \mid \tau = M$, then there is $N^* \vDash T$ such that $M^* \subset N^*$ and $N^* \mid \tau = N$.*

Moreover, this is a functorial expansion in the sense of Vasey [Vasa, Definition 3.1] and $(\mathrm{Mod}\, T^*, \subset)$ is an AEC except that it allows κ-ary relations.

Note that although the vocabulary τ^* is κ-ary, the structure of objects and embeddings from $(\mathrm{Mod}\, T^*, \subset)$ still satisfies all of the category theoretic conditions on AECs, as developed by Lieberman and Rosicky [LR]. This is because $(\mathrm{Mod}\, T^*, \subset)$ is equivalent to an AEC, namely $\boldsymbol{K}$, via the forgetful functor.

Proof: (1): We will build a $\prec$-directed system $\{M_{\mathbf{a}} \subset M : \mathbf{a} \in {}^{<\omega}M\}$ that are members of K_κ. We don't (and can't) require in advance that $M_{\mathbf{a}} \prec M$, but this will follow from our argument.
For singletons $a \in M$, taking $\mathbf{x}$ to be $\langle a : i < \kappa \rangle$ in (4.1.3), implies that there is $M'_a \in K_\kappa$ and $\mathbf{m}^a \in {}^\kappa M$ with $a \in \mathbf{m}^a$ such that $M \vDash R_{[M'_a]}(\mathbf{m}^a)$. By (1), this means that $m_i^a \mapsto m_i^{M'_a}$ is an isomorphism. Set $M_a := \mathbf{m}^a$.[8]
Suppose $\mathbf{a}$ is a finite sequence in M and $M_{\mathbf{a}'}$ is defined for every $\mathbf{a}' \subsetneq \mathbf{a}$. Using the union of the universes as the $\mathbf{x}$ in (4.1.3), there is some $N \in K_\kappa$ and $\mathbf{m}^{\mathbf{a}} \in {}^\kappa M$ such that

- $|M_{\mathbf{a}'}| \subset \mathbf{m}^{\mathbf{a}}$ for each $\mathbf{a}' \subsetneq \mathbf{a}$.

[8]We mean that we set M_a to be τ-structure with universe the range of $\mathbf{m}^a$ and functions and relations inherited from M'_a via the map above.

- $M \vDash R_{[N]}(\mathbf{m}^{\mathbf{a}})$.

By (4.1.4), this means that $M \vDash R_{\overline{M}_{\mathbf{a}'} \prec \overline{N}}(\mathbf{m}^{\mathbf{a}'}, \mathbf{m}^{\mathbf{a}})$, after some permutation of the parameters. By (2) and (1), this means that $M_{\mathbf{a}'} \prec N$; set $M_{\mathbf{a}} := \mathbf{m}^{\mathbf{a}}$.

Now that we have finished the construction, we are done. AECs are closed under directed unions, so $\cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$. But this model has the same universe as M and is a substructure of M; thus $M = \cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$.

For the further claim, suppose $M^* \vDash R_{[M_0]}(\mathbf{m})$. We can redo the same proof as above, but using $\mathbf{m}$ as the cover of any finite subtuple of it[9]. Thus, by the AEC axioms, we have

$$\mathbf{m} = M_{\mathbf{a}} \prec \bigcup_{\mathbf{a}' \in {}^{<\omega} M} M_{\mathbf{a}'} = M$$

(2): First, it's clear that $M \in K$ has an expansion; for each $M_0 \prec M$ of size κ, make $R_{[M_0]}(\langle m_i^{M_0} : i < \kappa \rangle)$ hold and, for each $M_0 \prec N_0 \prec M$ of size κ, make $R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$ hold. Now we want to show this expansion is the unique one.
Suppose $M^+ \vDash T^*$ is an expansion of M. We want to show this is in fact the expansion described in the above paragraph. Let $M_0 \prec M$. By (4.1.3) and (1) of this theorem, there is $N_0 \prec M$ and $\mathbf{n} \in {}^{\kappa} M$ such that

- $M^+ \vDash R_{[N_0]}(\mathbf{n})$
- $|M_0| \subset \mathbf{n}$

By coherence, $M_0 \prec \mathbf{n}$. Since $n_i \mapsto m_i^{N_0}$ is an isomorphism, there is $M_0^* \cong M_0$ such that $M_0^* \prec N_0$. Note that $T^* \models \forall \mathbf{x} R_{[M_0^*]}(\mathbf{x}) \leftrightarrow R_{[M_0]}(\mathbf{x})$. By (4.1.4),

$$M^+ \vDash R_{[M_0^* \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \mathbf{n})$$

By (4.1.2), $M^+ \vDash R_{[M_0^*]}(\langle m_i^{M_0} : i < \kappa \rangle)$, which gives us the conclusion by the further part of (1) of this theorem.

Similarly, if $M_0 \prec N_0 \prec M$, it follows that

$$M^+ \vDash R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$$

Thus, this arbitrary expansion is actually the intended one.

(3): Apply the uniqueness of the expansion and the transitivity of $\prec$.

[9]Note that it would suffice to use it once.

(4): As in the proof of (1), we can build $\prec$-directed systems $\{M_{\mathbf{a}} : \mathbf{a} \in {}^{<\omega}M\}$ and $\{N_{\mathbf{b}} : \mathbf{b} \in {}^{<\omega}N\}$ of submodels of M and N, so that $M_{\mathbf{a}} = N_{\mathbf{a}}$ when $\mathbf{a} \in {}^{<\omega}M$. From the union axioms of AECs, we see that $M \prec N$.

(5): This follows from (3), (4) of this theorem and the uniqueness of the expansion.

Recall that the map $M^* \in \operatorname{Mod} T^*$ to $M^* \restriction \tau \in \boldsymbol{K}$ is a an abstract Morleyization if it is a bijection such that every isomorphism $f : M \cong N$ in $\boldsymbol{K}$ lifts to $f : M^* \cong N^*$ and $M \prec N$ implies $M^* \subset N^*$. We have shown that this is true of our expansion. †

Remark 4.4. *The use of infinitary quantification might remind the reader of the work on the interaction between AECs and $\mathbb{L}_{\infty,\kappa^+}$ by Shelah [She09, Chapter IV] and Kueker [Kue08] (see also Boney and Vasey [BV] for more in this area). The main difference is that, in working with $\mathbb{L}_{\infty,\kappa^+}$, those authors make use of the semantic properties of equivalence (back and forth systems and games). In contrast, particularly in the following transfer result we look at the syntax of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$.*

Proposition 4.5. $(K, \prec)$ *has λ-amalgamation [joint embedding, etc.] iff* $(\operatorname{Mod} T^*, \subset)$ *has λ-amalgamation [joint embedding, etc.].*

Proof: First, suppose K has λ-amalgamation and let $M_0^* \subset M_1^*, M_2^*$. Then there is $N \in K$ and K-embeddings $f_\ell : M_\ell \to_{M_0} N$ for $\ell = 1, 2$. By Theorem 4.3.2, there is a canonical expansion N^* of N. Similarly, f_ℓ lifts to a τ_*-embedding, using Theorem 4.3.3. Thus we have λ-amalgamation in $(\operatorname{Mod} T^*, \subset)$.
Second, suppose $(\operatorname{Mod} T^*, \subset)$ has λ-amalgamation and let $M_0 \prec M_1, M_2$. By Theorem 4.3.2, 3, and 5, we have $M_0^* \subset M_1^*, M_2^*$. Then we can amalgamate these as τ^*-structures and, by 4.3.1 and 4, this amalgam reducts to an amalgam of the original models in K. †

Now we show the transfer of amalgamation between different cardinalities using the technology of this section.

Notation 4.6. *Let $M_0^* \subset M_1^*, M_2^*$ be τ^*-structures. We define the* amalgamation diagram $AD(M_1^*, M_2^*/M_0^*)$ *to be*

$$\{\phi(\boldsymbol{c}_{\mathbf{m}_0}, \boldsymbol{c}_{\mathbf{m}_1})) : \phi \text{ is quantifier-free from } \tau^* \text{ and for } \ell = 0 \text{ or } 1, M_\ell^* \vDash \phi_{\ell(\boldsymbol{c}_{\mathbf{m}_0}, \boldsymbol{c}_{\mathbf{m}_1})}\}$$

in the vocabulary $\tau^* \cup \{c_m : m \in M_1 \cup M_2\}$ where each constant is distinct except for the common submodel M_0 and $\mathbf{c_m}$ denotes the finite sequence of constants $c_{m_1}, \ldots, c_{m_n}$. {synap}

Claim 4.7. *Amalgamating $M_0 \prec M_1, M_2$ is equivalent to finding a model of*

$$T^* \cup AD(M_1^*, M_2^*/M_0^*)$$

Proof: An amalgam of $M_0 \prec M_1, M_2$ is canonically expandable to an amalgam of $M_0^* \subset M_1^*, M_2^*$, which is precisely a model of $T^* \cup AD(M_1^*, M_2^*/M_0^*)$.†

Definition 4.8. *We say $M \in K_{<\kappa}$ is a $<\kappa$-*amalgamation base *($<\kappa$-a.b) if any pair of models of cardinality $<\kappa$ extending M can be amalgamated over M.*

This gives us the following results syntactically.

Proposition 4.9. *Suppose $LS(K) < \kappa$.*

- *If κ is weakly compact and $M \in K_\kappa$ is the increasing union of a chain of $<\kappa$-a.b.'s of length κ, then M is a κ-a.b.*
- *If κ is measurable and $M \in K_\lambda$ is the increasing union of a chain of $<\lambda$-a.b.'s of length κ or a union of such a chain of λ-a.b.'s, then M is a λ-a.b.*
- *If κ is λ-strongly compact and every strong submodel of $M \in K_{\leq\lambda}$ is a $<\kappa$-a.b., then M is a $\leq\lambda$-a.b.*

Proof: The proof of the different parts are essentially the same: take a valid amalgamation problem over M and formulate it syntactically via Claim 4.7 in $\mathbb{L}_{\kappa,\kappa}(L^*)$. Then use the appropriate syntactic compactness for the large cardinal (see [CK73, Exercise 4.2.6] for the lesser known measurable version) to conclude the satisfiability of the appropriate theory.
For instance, suppose κ is weakly compact and $M = \cup_{i<\kappa} M_i$ where $M_i \in K_{<\kappa}$ is a $<\kappa$-a.b. Suppose $M \prec M^1, M^2$ is an amalgamation problem from K_κ. Find resolutions $\langle M_i^\ell \in K_{<\kappa} : i < \kappa\rangle$ with $M_i \prec M_i^\ell$ for $\ell = 1, 2$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{i<\kappa} \left(T^* \cup AD(M_i^{1*}, M_i^{2*}/M_i^*)\right)$$

and is of size κ. Each member of the union is satisfiable (by Claim 4.7 because M_i is a $<\kappa$-a.b.) and of size $<\kappa$, so $T^* \cup AD(M^{1*}, M^{2*}/M^*)$ is satisfiable. Since $M^1, M^2 \in K_\kappa$ were arbitrary, M is a κ-a.b. †

From this, we get the following corollaries computing upper bounds on the Hanf number for the $\leq\lambda$-AP.

Corollary 4.10. *Suppose $LS(K) < \kappa$.*

- *If κ is weakly compact and K has $<\kappa$-AP, then K has $\leq\kappa$-AP.*
- *If κ is measurable,* cf $\lambda = \kappa$, *and K has $<\lambda$-AP, then K has $\leq\lambda AP$.*
- *If κ is λ-strongly compact and K has $<\kappa$-AP, then K has $\leq\lambda$-AP.*

Moreover, when κ is strongly compact, we can imitate the proof of [MS90, Corollary 1.6] to show that being an amalgamation base follows from being a $<\kappa$-existentially closed model of T^*.

5 The Big Gap

{biggap-sec}

Intuitively, Hanf's principle is that if a certain property can hold for only set-many objects then it is eventually false. He refines this twice. First, if $\mathcal{K}$ a *set* of collections of structures $\boldsymbol{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\boldsymbol{K}, \lambda)$ means some member of $\boldsymbol{K}$ with cardinality λ satisfies P then there is a cardinal κ_P such that for any $\boldsymbol{K} \in \mathcal{K}$, if $\phi(\boldsymbol{K}, \kappa')$ holds for some $\kappa' \geq \kappa_P$, then $\phi(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. Secondly, he observed that if the property P is closed down for sufficiently large members of each $\boldsymbol{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Morley (plus the Shelah presentation theorem) gives a decisive concrete example of this principle to AEC's. Any AEC in a countable vocabulary with countable Löwenheim-Skolem number with models up to $\beth_{\omega_1}$ has arbitrarily large models. And Morley [Mor65] gave easy examples showing this bound was tight for arbitrary sentences of $L_{\omega_1,\omega}$. But it was almost 40 years later that Hjorth [Hjo02, Hjo07] showed this bound is also tight for *complete*-sentences of $L_{\omega_1,\omega}$. And a fine point in his result is interesting.

We say a ϕ *characterizes* κ, if there is a model of ϕ with cardinality κ but no larger. Further, ϕ *homogeneously* [Bau74] characterizes κ if ϕ is a complete sentence of $L_{\omega_1,\omega}$ that characterizes κ, contains a unary predicate U such that if M is the countable model of ϕ, every permutation of $U(M)$ extends to an automorphism of M (i.e. $U(M)$ is a set of absolute indiscernibles.) and there is a model N of ϕ with $|U(N)| = \kappa$.

In [Hjo02], Hjorth found, by an inductive procedure, for each $\alpha < \omega_1$, a countable (finite for finite α) set S_α of complete $L_{\omega_1,\omega}$-sentences such that some $\phi_\alpha \in S_\alpha$ characterizes $\aleph_\alpha$[10]. This procedure was nondeterministic in the sense that he showed one of (countably many if α is infinite) sentences worked at each $\aleph_\alpha$; it is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. In [BKL14], we show a modification of the Laskowski-Shelah example (see [LS93, BFKL13]) gives a family of $L_{\omega_1,\omega}$-sentences ϕ_r, such that ϕ_r homogeneously characterizes $\aleph_r$ for $r < \omega$. Thus for the first time [BKL14] establishes in ZFC, the existence of specific sentences ϕ_r characterizing $\aleph_r$.

Another significant instance of Hanf's observation is Shelah's proof in [She99a] that if $\mathcal{K}$ is taken as all AEC's $\boldsymbol{K}$ with $LS_{\boldsymbol{K}}$ bounded by a cardinal κ, then there is such an eventual Hanf number for categoricity in a successor. Boney [Bon] places an upper bound on this Hanf number as the first strongly compact above κ. This followed from establishing a similar upper bound for the Hanf number for tameness.

Vasey [Vasb] establishes the Hanf number for both categoricity in for universal classes (in the AEC sense) satisfying amalgamation in countable vocabularies to be at most $\beth_{\beth_{(2^\omega)^+}}$.

[10]Malitz [Mal68] (under GCH) and Baumgartner [Bau74] had earlier characterized the $\beth_\alpha$ for countable α.

Note that the definition of a Hanf number for tameness is more complicated as tameness is fundamentally a property of two variables: $\boldsymbol{K}$ is $(<\chi,\mu)$-tame if for any $N \in \boldsymbol{K}_\mu$, if the Galois types p and q over N are distinct, there is an $M \prec N$ with $|M| < \chi$ and $p \upharpoonright M \neq q \upharpoonright M$.

Thus, we define the *Hanf number for $< \kappa$-tameness* to be the minimal λ such that the following holds:

if $\boldsymbol{K}$ is an AEC with $LS(\boldsymbol{K}) < \kappa$ that is $(<\kappa,\mu)$-tame for *some* $\mu \geq \lambda$, then it is $(<\kappa,\mu)$-tame for arbitrarily large[11] μ.

The results of [Bon] show that Hanf number for $<\kappa$-tameness is κ when κ is strongly compact[12]. However, this is done by showing a much stronger "global tameness" result that ignores the hypothesis: *every* AEC $\boldsymbol{K}$ with $LS(\boldsymbol{K}) < \kappa$ is $(<\kappa,\mu)$-tame for all $\mu \geq \kappa$. Boney and Unger [BU], building on earlier work of Shelah [She], have shown that this global tameness result is actually an equivalence (in the almost strongly compact form). Also, due to monotonicity results for tameness, the Boney results show that the Hanf number for $<\lambda$-tameness is at most the first almost strongly compact above λ (if such a thing exists). However, establishing a ZFC upper bound or any kind of lower bound seems out of reach at this moment.

In this paper, we have established a similar upper bound for a number of amalgamation-like properties. Moreover, although it is not known beforehand that the classes are eventually downward closed, that fact falls out of the proof. In all these cases, the known lower bounds (i. e., examples where AP holds initially and eventually fails) are far smaller.

We state the results for countable Löwenheim-Skolem numbers, although the [BKS09, KLH14] results generalize to larger cardinalities.

The best lower bounds for the disjoint amalgamation property is $\beth_{\omega_1}$ as shown in [KLH14] and [BKS09]. In [BKS09], Baldwin, Kolesnikov, and Shelah gave examples of $L_{\omega_1,\omega}$-definable classes that had disjoint embedding up to $\aleph_\alpha$ for every countable α (but did not have arbitrarily large models). Kolesnikov and Lambie-Hanson [KLH14] show that for the collection of all coloring classes (again $L_{\omega_1,\omega}$-definable when α is countable) in a vocabulary of a fixed size κ, the Hanf number for amalgamation (equivalently in this example disjoint amalgamation) is precisely $\beth_{\kappa^+}$ (and many of the classes have arbitrarily large models). In [BKL14], Baldwin, Koerwein, and Laskowski construct, for each $r < \omega$, a *complete* $L_{\omega_1,\omega}$-sentence ϕ^r that has disjoint 2-amalgamation up to and including $\aleph_{r-2}$; disjoint amalgamation and even amalgamation fail in $\aleph_{r-1}$ but hold (trivially) in $\aleph_r$; there is no model in $\aleph_{r+1}$.

[11]One might want $<\kappa$-tameness on a tail, rather than on an unbounded class of cardinals. Under some natural type extension properties (following, for instance, from no maximal models and amalgamation), the two notions are equivalent.

[12]This can be weakened to almost strongly compact; see Brooke-Taylor and Rosický [BTR15] or Boney and Unger [BU].

The joint embedding property and the existence of maximal models are closely connected[13]. The main theorem of [BKS] asserts: If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Thus, a lower bound on the Hanf number for either maximal models of the joint embedding property is again $\beth_{\omega_1}$. Again, the result is considerably more complicated for complete sentences. But [BS15] show that there is a sentence ϕ in a vocabulary with a predicate X such that if $M \models \phi$, $|M| \leq |X(M)|^+$ and for every κ there is a model with $|M| = \kappa^+$ and $|X(M)| = \kappa$. Further they note that if there is a sentence ϕ that homogenously characterizes κ, then there is a sentence ϕ' with a new predicate B such that ϕ' also characterizes κ, B defines a set of absolute indiscernibles in the countable model, and there are models M_λ for $\lambda \leq \kappa$ such that $(|M|, |B(M_\lambda)|) = (\kappa, \lambda)$. Combining these two with earlier results of Souldatos [Sou13] one obtains several different ways to show the lower bound on the Hanf number for a complete $L_{\omega_1,\omega}$-sentence having maximal models is $\beth_{\omega_1}$. In contrast to [BKS], all of these examples have no models beyond $\beth_{\omega_1}$.

Here are some of the most striking problems in this area.

Question 5.1. 1. *Can one calculate an upper bound on these Hanf numbers for 'amalgamation' in ZFC? Can[14] the gaps in the upper and lower bounds of the Hanf numbers reported here be closed in ZFC? Will smaller large cardinal axioms suffice for some of the upper bounds? Does categoricity help?*

2. *Can[15] one define in ZFC a sequence of sentences ϕ_α for $\alpha < \omega_1$, such that ϕ_α characterizes $\aleph_\alpha$?*
3. *(Shelah) If $\aleph_{\omega_1} < 2^{\aleph_0}$ $L_{\omega_1,\omega}$-sentence has models up to $\aleph_{\omega_1}$, must it have a model in $2^{\aleph_0}$. (He proves this statement is consistent in [She99b]).*
4. *(Souldatos) Is any cardinal except $\aleph_0$ characterized by a complete sentence of $L_{\omega_1,\omega}$ but not homogeneously.*

[13]Note that, under joint embedding, the existence of a maximal model is equivalent to the non-existence of arbitrarily large models

[14]Grossberg initiated this general line of research.

[15]This question seems to have originated from discussions of Baldwin, Souldatos, Laskowski, and Koerwien.

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