Beyond First Order Logic: From number of structures to structure of numbers

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1 Introduction

Model theory studies classes of structures. These classes are usually a collection of structures that satisfy an (often complete) set of sentences of first order logic. Such sentences are created by closing a family of basic relations under finite conjunction, negation and quantification over individuals. *Non-elementary logic* enlarges the collection of sentences by allowing longer conjunctions and some additional kinds of quantification. In this paper we first describe for the general mathematician the history, key questions, and motivation for the study of non-elementary logics and distinguish it from first order model theory. We give more detailed examples accessible to model theorists of all sorts. We conclude with questions about countable models which require only a basic background in logic.

For the last 50 years most research in model theory has focused on first order logic. Motivated both by intrinsic interest and the ability to better describe certain key mathematical structures (e.g. the complex numbers with exponentiation), there has recently been a revival of 'non-elementary model theory'. We develop contrasts between first order and non-elementary logic in a more detailed way than just noting 'failure of compactness'. We explain the sense in which we use the words syntax and semantics in Section 2. Many of the results and concepts in this paper will reflect a tension between these two viewpoints. In particular, as we move from the study of classes that are defined syntactically to those that are defined semantically, we will be searching for a replacement for the fundamental notion of first order model theory, i.e. the notion of a complete theory. Section 2 also defines the basic notions of non-elementary model theory. Section 3 describes some of the research streams in more detail and illuminates some of the distinctions between elementary and non-elementary model theory. Subsection 3.1 describes the founding result of modern first order model theory, Morley's categoricity theorem, and sketches Shelah's generalization of it to $L_{\omega_1,\omega}$. The remainder of Section 3 discusses several lines of work in AEC and provides specific mathematical examples that illustrate some key model theoretic notions. We describe concrete examples explaining the concepts and problems in non-elementary model theory and a few showing connections with other parts of mathematics. Two of these illustrate the phrase 'to structure of numbers' in the title. Example 3.3.4, initiated by Zilber, uses infinitary methods to study complex exponentiation and covers of Abelian varieties. Example 4.3.7 studies models of Peano Arithmetic and the notion of elementary end-extension. This is the first study of models of Peano arithmetic as an AEC. Section 4 contains new results and explores the proper analogy to complete theory for AECs; it answers a question asked

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by David Kueker and includes Kossak's example of a class of models of PA interesting from the standpoint of AEC.

Neither of the standard approaches, $L_{\kappa,\omega}$-definable class or abstract elementary class (AECs), has been successful in studying the countable models of an infinitary sentence. The first approach is too specific. It rapidly reduces to a complete infinitary sentence which has only one countable model. Results so far in studying general AECs give little information about countable models. We seek to find additional conditions on an AEC that lead to a fruitful study of the class of countable models. In particular we would like to find tools for dealing with one famous and one not so famous problem of model theory. The famous problem is Vaught's conjecture. Can a sentence of $L_{\omega_1,\omega}$ have strictly between $\aleph_0$ and $2^{\aleph_0}$ countable models? The second problem is more specific. What if we add the condition that the class is $\aleph_1$-categorical; can we provide sufficient conditions for having less than $2^{\aleph_0}$ countable models; for actually counting the number of countable models? We describe two sets of concepts for addressing this issue; unfortunately so far not very successfully. The first is the notion of a simple finitary AEC and the second is an attempt to define a notion of a 'complete AEC', which like a complete first order theory imposes enough uniformity to allow analysis of the models but without trivializing the problem to one model.

One thesis of this paper is that the importance of non-elementary model theory lies not only in widening the scope of applications of model theory but also in shedding light on the essence of the tools, concepts, methods and conventions developed and found useful in elementary model theory.

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2 Non-elementary model theory

In this section we study the history of non-elementary model theory during the second half of the twentieth century and compare that to the development of more 'mainstream' first order model theory. We identify two different trends in the development. In both the 'elementary' and non-elementary cases the focus of research has moved from 'syntactic' consideration towards 'semantic' ones - we will explain what we mean by this. We see some of the cyclic nature of science. Non-elementary classes bloom in the 60's and 70's; the bloom fades for some decades, overshadowed by the success and applications arising from the 'elementary' field. But around the turn of the 21st century, innovative examples and further internal developments lead to a rebirth.

We will focus on some 'motivating questions' that have driven both the elementary and non-elementary approaches, such as the categoricity transfer problem. While counting models seems a rather mundane problem, new innovations and machinery developed for the solution have led to the recognition of systems of invariants that are new to mathematics and in the first order case to significant mathematical advances in e.g. number theory [Bou99]. It is hoped that the deeper developments of infinitary logic will have similar interactions with core mathematics. Boris Zilber's webpage contains many beginnings.

2.1 Syntax and semantics

The distinction between *syntax* and *semantics* has been present throughout the history of modern logic starting from the late 19th century: *completeness theorems* build a bridge between the two by asserting that a sentence is provable if and only if it is true in all models. By syntax we refer to the formalism of logic, objects of language as strings of symbols and deductions as manipulations of these strings according to certain rules. Semantics, however, has to do with interpretations, 'meaning' and 'sense' of the language. By the *semantics* for a language we mean a 'truth definition' for the sentences of the language, a description of the conditions when a structure is considered to be a model for that sentence. 'Semantic properties' have to do with properties of such models.

In fact these two notions can also be seen as methodologies or attitudes toward logic. The extreme (formalist) view of the syntactic method avoids reference to any 'actual' mathematical objects or meaning for the statements of the language, considering these to be 'metaphysical objects'. The semantic attitude is that logic arises from the tradition of mathematics. The method invokes a trace of Platonism, a search for the 'truth' of statements with less regard for formal language. The semantic method would endorse 'proof in metamathematics or set theory' while the syntactic method seeks a 'proof in some formal system'. Traditionally model theory is seen as the junction of these two approaches. Chang and Keisler[CK73] write: *universal algebra + logic = model theory.* Juliette Kennedy[Ken] discusses ideas of 'formalism freeness', found in the work of Kurt Gödel. Motivated by incompleteness and hence the 'failure' of first order logic to capture truth and reasoning, Gödel asked if there is some (absolute) concept of proof (or definability) 'by all means imaginable'. One interpretation of this absolute notion (almost certainly not Gödel's) is as the kind of semantic argument described above. We will spell out this contrast in many places below.

Model theory by definition works with the semantic aspect of logic, but the dialectics between the syntactic and semantic attitudes is central. This becomes even clearer when discussing questions arising from *non-elementary model theory.* Non-elementary model theory studies formal languages other than 'elementary' or first order logic; most of them extend first order. We began by declaring that model theory studies classes of models. Traditionally, each class is the collection of models that satisfy some (set of) sentence(s) in a particular logic. Abstract Elementary Classes provide new ways of determining classes: a class of structures in a fixed vocabulary is characterized by semantic properties. The notion of AEC does not designate the models of a collection of sentences in some formal language, although many examples arise from such syntactic descriptions. In first order logic, the most fruitful topic is classes of models of complete theories. A *theory* T is a set of sentences in a given language. We say that T is *complete*, if for every sentence ϕ in the language, either T implies ϕ, or T implies $\neg\phi$. In Section 4 we seek an analogue to completeness for AEC.

Model-Theoretic Logics, edited by Barwise and Feferman [BF85], summarizes the early study of non-elementary model theory. In this book, 'abstract model theory' is a study comparing different logics with regard to such properties as interpolation, expansions, relativizations and projections, notions of compactness, Hanf and Löwenheim-Skolem numbers.

A vocabulary[1] L consists of constant symbols, relation symbols and function symbols, which have a prescribed number of arguments (arity). An L-structure consists of a universe, which is a set, and interpretations for the symbols in L. When L' is a subset of a vocabulary L, and M is an L-structure, we can talk about the reduct of M to L', written $M \restriction L'$. Then M is the expansion of $M \restriction L'$ to L. If M and N are two L-structures, we say that M is an L*-substructure* of N if the domain of M is contained in the domain of N and the interpretations of all the symbols in L agree in M and the restriction of N to M.

A *formal language* or logic in the vocabulary L is a collection of formulas that are built by certain rules from the symbols of the vocabulary and from some 'logical symbols'. In this paper we focus on countable vocabularies but don't needlessly restrict definitions.

L*-terms* are formed recursively from variables and the constant and function symbols of the vocabulary by composing in the natural manner. With a given interpretation for the constants and assignment of values for the variables in a structure, each term designates an element in the structure..

[1] Another convention specifies the vocabulary by a small Greek letter and the L with decorations describes the particular logic. What we call a vocabulary is sometimes called a language. We have written language or logic for the collections of sentences; more precisely this might be called the language and the logic would include proof rules and even semantics.

An *atomic formula* is an expression $R(t_1, \dots t_n)$ where R is an n-ary relation symbol (including equality) of the vocabulary and each t_i is a term.

Definition 2.1.1 (The language $L_{\lambda\kappa}$) *Assume that L is a vocabulary. The language $L_{\lambda\kappa}$ consists of formulas $\phi(\bar{x})$, where the free variables of the formula are contained in the sequence $\bar{x}$ and where the formulas are built with the following operations:*

- *$L_{\lambda\kappa}$ contains all atomic formulas in the vocabulary L.*
- *If $\phi(\bar{x})$, $\psi(\bar{x})$ are in $L_{\lambda\kappa}$, then the negation $\neg\phi(\bar{x})$ and implication $(\phi(\bar{x}) \to \psi(\bar{x}))$ are in $L_{\lambda\kappa}$.*
- *If $\phi_i(\bar{x})$ is in $L_{\lambda\kappa}$ for every i in the index set I, and $|I| < \lambda$ the conjunction $\bigwedge_{i\in I} \phi_i(\bar{x})$ and disjunction $\bigvee_{i\in I} \phi_i(\bar{x})$ are in $L_{\lambda\kappa}$.*
- *If $\phi(y_i, \bar{x})$ is in $L_{\lambda\kappa}$ for each i in the well-ordered index set I, and $|I| < \kappa$, then the quantified formula $(Q_i y_i)_{i\in I}\phi(\bar{x})$ is in $L_{\lambda\kappa}$, where each quantifier Q_i is either $\forall$ ('for all y_i') or $\exists$ ('there exists y_i').*

First order logic is the language $L_{\omega\omega}$, i.e. only finite operations are allowed. We define that $L_{\infty}\kappa$ is the union of all $L_{\lambda\kappa}$ for all cardinal numbers λ.

The languages $L_{\lambda\omega}$ allowing only finite strings of quantifiers are much better behaved. We will later introduce *abstract elementary classes* generalizing, among other things, classes of structures definable with a sentence in $L_{\lambda\omega}$. The definition of the *truth* of a formula in a structure is crucial. For a formula $\phi(\bar{x})$, with the sequence $\bar{x}$ containing all the free variables of ϕ, we define what it means that the formula $\phi(\bar{x})$ is true in an L-structure M with the variables $\bar{x}$ interpreted in a particular way as elements $\bar{a}$, written $M \models \phi(\bar{a})$. The definition is done by induction on the complexity of the formula, following the inductive definition of the formula in Definition 2.1.1.

Definition 2.1.2 (The language $L(Q)$) *The language $L(Q)$ is formed as the first order logic $L_{\omega\omega}$, but we allow also formulas of the form $Qy\phi(y, \bar{x})$ with the following truth definition: $M \models Qy\phi(y, \bar{a})$ if there are uncountable many $b \in M$ such that $M \models \phi(b, \bar{a})$.*

Definition 2.1.3 (Elementary substructure with respect to a fragment) *A subset $\mathscr{F} \subseteq L$ is a* fragment *of some formal language L if it contains all atomic formulas and is closed under subformulas, substitution of variables with L-terms, finite conjunction and disjunction, negation and the quantifiers $\forall$ and $\exists$, applied finitely many times. For two L-structures M and N, we say that M is an $\mathscr{F}$-elementary substructure of N, written*

$$M \preccurlyeq_{\mathscr{F}} N,$$

if M is an L-substructure of N and for all formulas $\phi(\bar{x})$ of $\mathscr{F}$ and sequences $\bar{a}$ of elements in M, $M \models \phi(\bar{a})$ if and only if $N \models \phi(\bar{a})$.

Definition 2.1.4 (Elementary class and PC-class) *An* elementary class *$\mathbb{K}$ of L-structures is the class of all models of a given theory in first order logic. A* pseudoelementary *(PC) class $\mathbb{K}$ is the class of reducts $M \restriction L$ of some elementary class in a larger vocabulary $L' \supseteq L$.*

We say that a formal language (logic) L is compact if whenever a set of sentences is *inconsistent*, that is, has no model, then there is some finite subset which already is inconsistent. This is a crucial property that, along with the upwards Löwenheim-Skolem property, fails in most non-elementary logics.

The Löwenheim-Skolem number and the Hanf number is defined for a formal logic L (i.e. 'the Löwenheim-Skolem or Hanf number of L'). In the following definitions $\mathbb{K}$ is a class definable with a sentence of L, $\preccurlyeq_{\mathbb{K}}$ is given as the $\mathscr{F}$-elementary substructure relation in some given fragment $\mathscr{F}$ of L, usually the smallest fragment containing the sentence defining $\mathbb{K}$, and the collection $\mathscr{C}$ is the collection of all classes definable with a sentence L.

Definition 2.1.5 (Löwenheim-Skolem number) *The Löwenheim-Skolem number* $\mathrm{LS}(\mathbb{K})$ *for a class of structures* $\mathbb{K}$ *and a relation* $\preccurlyeq_{\mathbb{K}}$ *between the structures is the smallest cardinal number* λ *with the following property: For any* $M \in \mathbb{K}$ *and a subset* $A \subseteq M$ *there is a structure* $N \in \mathbb{K}$ *containing* A *such that* $N \preccurlyeq_{\mathbb{K}} M$ *and* $|N| \leq \max\{\lambda, |A|\}$.

Definition 2.1.6 (Hanf Number) *The Hanf number* H *for a collection* $\mathscr{C}$ *of classes of structures is the smallest cardinal number with the property: for any* $\mathbb{K} \in \mathscr{C}$*, if there is* $M \in \mathbb{K}$ *of size at least* H*, then* $\mathbb{K}$ *contains arbitrarily large structures.*

Modern model theory begins in the 1950's. Major achievements in the mid 60's and early 70's included Morley's categoricity transfer theorem in 1965 [Mor65] and Shelah's development of stability theory [She78]. These works give results on counting the number of isomorphism types of structures in a given cardinality and establishing invariants in order to classify the isomorphism types. Such invariants arise naturally in many concrete classes: the dimension of a vector space or the transcendence degree of an algebraically closed field are prototypical examples. A crucial innovation of model theory is to see how to describe structures by *families* of dimensions. The general theory of dimension appears in e.g. ([She78, Pil96]); it is further developed and applied to valued fields in [HHM08].

Non-elementary model theory thrived in the mid 60's and early 70's. Results such as Lindström theorem in 1969, Barwise's compactness theorem for *admissible fragments* of $L_{\omega_1\omega}$ published in 1969, Mostowski's work on generalized quantifiers in 1957 [Mos57] and Keisler's beautiful axiomatization of $L(Q)$ in [Kei70] gave the impression of a treasury of new formal languages with amenable properties, a possibility to extend the scope of definability and maybe get closer to the study of provability with 'all means imaginable'. However, the general study turned out to be very difficult. For example, the study of the languages $L_{\lambda\kappa}$ got entangled with the set-theoretical properties of the cardinals λ and κ. Since the real numbers are definable as the unique model of a sentence in $L_{2^\omega,\omega}$, the continuum hypothesis would play a major role. But perhaps the study was focused too much on the syntax and trying to study the model theory of *languages*? Why not study the properties of classes of structures, defined semantically. One might replace compactness with, say, closure under unions of chains?

One can argue that a major achievement of non-elementary model theory has been to *isolate* properties that are crucial for classifying structures, properties that might not be visible to a mathematician working with only a specific application or even restricted to the first order case. Excellence (see below) is a crucial example. Some examples of applications of non-elementary model theory to 'general mathematics' are presented in the chapter 'Applications to algebra' by Eklof in [BF85]. In many of these applications we can see that some class of structures is definable in $L_{\omega_1\omega}$ or in $L_{\infty\omega}$ and then the use the model theory of these languages to, for example, count the number of certain kind of structures or classify them in some other way. Barwise writes in *Model-theoretic logics* [BF85]:

> Most important in the long run, it seems, is where logic contributes to mathematics by leading to the formation of concepts that allow the right questions to be asked and answered. A simple example of this sort stems from 'back-and-forth arguments' and leads to the concept of partially isomorphic structures, which plays such an important role in extended model theory. For example, there is a classical theorem by Erdos, Gillman and Henriksen; two real-closed fields of order type η_1 and cardinality $\aleph_1$ are isomorphic. However, this way of stating the

> theorem makes it vacuous unless the continuum hypothesis is true, since without this hypothesis there are no fields which satisfy both hypotheses. But if one looks at the proof, there is obviously something going on that is quite independent of the size of the continuum, something that needs a new concept to express. This concept has emerged in the study of logic, first in the work of Ehrenfeucht and Fraïssé in first-order logic, and then coming into its own with the study of infinitary logic. And so in his chapter (in the book [BF85]), Dickmann shows that the theorem can be reformulated using partial isomorphisms as: Any two real-closed fields of order type η_1, of any cardinality whatsoever, are strongly partially isomorphic. There are similar results on the theory of abelian torsion groups which place Ulm's theorem in its natural setting. ... Extended model theory provides a framework within which to understand existing mathematics and push it forward with new concepts and tools.

One of the foundational discoveries of abstract model theory was Per Lindström's theorem that first order logic is the strongest logic which has both the compactness property and a countable Löwenheim-Skolem number. In order to study such concepts as 'the strongest logic', one has to define the notion of an 'abstract logic'. The book [BF85] presents the syntax as a crucial part: an abstract logic is a class of sentences with a satisfaction relation between the sentences and the structures, where this relation satisfies certain properties. However, Barwise comments on Lindström's formulation of his theorem [Lin69]:

> To get around the difficulties of saying just what a logic is, they dealt entirely with classes of structures and closure conditions on these classes, thinking of the classes definable in some logic. That is, they avoided the problem of formulating a notion of a logic in terms of syntax, semantics, and satisfaction, and dealt purely with their semantic side.

Lindström defined a logic to be a non-empty set of objects called sentences, but the role of these is only to name a class of structures as 'structures modeling one sentence'. Then it is possible to define for example compactness as the property that if a countable intersection of such classes is empty, then already some finite intersection must be empty.

Saharon Shelah built on these insights and introduced Abstract Elementary Classes in [She87]. Semantic properties of a class of structures $\mathbb{K}$ and a relation $\preccurlyeq_{\mathbb{K}}$ are prescribed, which are sufficient to isolate interesting classes of structures. But more than just the class is described; the relation $\preccurlyeq$ between the structures in $\mathbb{K}$ provides additional information that, as examples in Subsection 3.3 illustrate, may be crucial.

Definition 2.1.7 *For any vocabulary τ, a class of τ-structures $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an* abstract elementary class *(AEC) if*

1. *Both $\mathbb{K}$ and the binary relation $\preccurlyeq_{\mathbb{K}}$ are closed under isomorphism.*
2. *If $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$, then $\mathscr{A}$ is a substructure of $\mathscr{B}$.*
3. *$\preccurlyeq_{\mathbb{K}}$ is a partial order on $\mathbb{K}$.*
4. *If $\langle \mathscr{A}_i : i < \delta \rangle$ is an $\preccurlyeq_{\mathbb{K}}$-increasing chain:*
   (a) $\bigcup_{i<\delta} \mathscr{A}_i \in \mathbb{K}$;
   (b) for each $j < \delta$, $\mathscr{A}_j \preccurlyeq_{\mathbb{K}} \bigcup_{i<\delta} \mathscr{A}_i$
   (c) if each $\mathscr{A}_i \preccurlyeq_{\mathbb{K}} \mathscr{M} \in \mathbb{K}$, then $\bigcup_{i<\delta} \mathscr{A}_i \preccurlyeq_{\mathbb{K}} \mathscr{M}$.
5. *If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$, $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and $\mathscr{A} \subseteq \mathscr{B}$ then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.*

6. *There is a Löwenheim-Skolem number* LS($\mathbb{K}$) *such that if* $\mathscr{A} \in \mathbb{K}$ *and* $B \subset \mathscr{A}$ *a subset, there is* $\mathscr{A}' \in \mathbb{K}$ *such that* $B \subset \mathscr{A}' \preccurlyeq_{\mathbb{K}} \mathscr{A}$ *and* $|\mathscr{A}'| = |B| + \mathrm{LS}(\mathbb{K})$.

When $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$, we say that $\mathscr{B}$ is an $\mathbb{K}$-extension of $\mathscr{A}$ and $\mathscr{A}$ is an $\mathbb{K}$-submodel of $\mathscr{B}$. If $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ and $f : \mathscr{A} \to \mathscr{B}$ an embedding such that $f(\mathscr{A}) \preccurlyeq_{\mathbb{K}} \mathscr{B}$, we say that f is a $\mathbb{K}$-*embedding*. Category-theoretic versions of the axioms are studied by Kirby [Kir08], Liebermann [Lie09] and Beke and Rosick [BR10].

A basic example of an AEC is the class of models defined by some sentence $\phi \in L_{\infty\omega}$, where $\preccurlyeq_{\mathbb{K}}$ is taken as the elementary substructure relation in the smallest fragment of $L_{\infty\omega}$ containing ϕ. Then the Löwenheim-Skolem number is the size of the fragment. An even simpler example is that of an elementary class, where ϕ is a complete theory in first order logic.

A class defined with a sentence in $L_{\omega_1\omega}(Q)$ with the quantifier $Qx\phi(x)$ standing for 'there exists uncountably many x such that $\phi(x)$ holds' can be thought as an AEC. The natural syntactic notion of elementary submodel is inadequate but substitutes are available (see chapter 5 of [Bal09]). Arbitrary psuedoelementary classes are often not AEC. (E.g. If $\mathbb{K}$ is the class of all structures A in a language L with a single unary predicate such that $|A| \leq 2^{|U(A)|}$ then $\mathbb{K}$ fails to be an AEC with respect to L-elementary submodel because it is not closed under unions of chains. (See 4.29 of [Bal09].)

In contemporary first order model theory, the most fundamental concept is the class of models of a complete theory in first order logic. This can be seen as a form of *focusing*; instead of studying different vocabularies, expansions and projections, one fixes one class: the class of differentially closed fields of fixed characteristic (see [MMP96]) or the class of models of 'true' arithmetic. This focus on classes and of properties determining 'similar' classes has become a crucial tool in applications to algebra. The difference from the 'Lindström-style' study of classes of structures is significant: we do not study many classes of structures each corresponding to the 'models of one sentence', but focus on a fixed class, 'models of a theory'. *Abstract elementary classes*, which will be one of the main notions studied in this paper, takes the 'semantic view' to the extreme by eliminating the syntactic definition.

3 Several research lines in non-elementary logic

3.1 Categoricity transfer in $L_{\omega,\omega}$ and $L_{\omega_1\omega}$

Definition 3.1.1 (Categoricity) *Let κ be a cardinal. We say that a class of structures $\mathbb{K}$ is κ-categorical if there is exactly one model of size κ in $\mathbb{K}$, up to isomorphism.*

A theory T is κ-categorical, if $Mod(T)$, the class of models of T, is κ-categorical.

The transition to the focus on classes of models begins with Morley's theorem:

Theorem 3.1.2 (Morley's categoricity transfer theorem) *Assume that T is a complete theory in $L_{\omega\omega}$, where L is countable. If there exists an uncountable cardinal κ such that T is κ-categorical, then T is λ-categorical for all uncountable cardinals λ.*

Categoricity transfer will be our first example of a motivating question in the history of model theory. Its proof gave many new tools and concepts that are nowadays contained in every basic course in model theory. Furthermore, both the tools and the theorem itself have been generalized to different frameworks. A categoricity transfer theorem for elementary classes in an uncountable vocabulary was proved by Shelah in [She74] (announced in 1970): if the language has cardinality κ and a theory is categorical in some uncountable cardinal greater than κ then it is categorical in all cardinalities greater than κ. This widening of scope led to many tools, such as weakly minimal sets and a greater focus on the properties of individual formulas, that proved fruitful for countable vocabularies. We will look more closely at some of the many extensions of categoricity results to non-elementary classes.

We consider a syntactical *type* in some logic $\mathscr{L}$ as a collection of $\mathscr{L}$-formulas in some finite sequence of variables $\bar{x}$ with *parameters* from a given subset A of a structure M such that an element $\bar{b}$ in an $\mathscr{L}$-elementary extension N of M realizes (simultaneously satisfies) p. If no such sequence exists in a model N, we say that the type is *omitted* in N. In elementary classes, the compactness theorem implies all *finitely consistent* such collections p of formulas really are *realized*. If there is a structure N and a finite sequence $\bar{b} \in N$ such that $M \preccurlyeq N$ and

$$p = \{\phi(\bar{x}, \bar{a}) : N \models \phi(\bar{b}, \bar{a})\}.$$

then p is called a *complete type* over A for two reasons. Semantically: it gives a complete description of the relation of b and A. Syntactically; every formula $\phi(\bar{x}, \bar{a})$ over A or its negation is in p.

An essential concept in Morley's argument is a *saturated structure* M: M is *saturated* if all *consistent* types over parameter sets of size strictly less than $|M|$ are realized in M. Two saturated models of T of size κ are always isomorphic. Morley shows that if T is categorical in some uncountable power, saturated models exist in each infinite cardinality. Then he concludes that if T is *not* categorical in some uncountable power λ, there is a model of power λ which is *not* saturated or even $\aleph_1$-saturated; some type over a countable subset is omitted. But then he shows that if some model of uncountable power λ omits a type over a countable set, then in any other uncountable power κ some model omits the type. Hence, T cannot be categorical in κ either. This method, *saturation transfer*, generalizes to many other frameworks. While proving saturation transfer for elementary classes he introduced many new concepts such as a *totally transcendental theory* ($\aleph_0$*-stable* theory), *prime models over sets* and *Morley sequences*.

Keisler generalized many of the ideas from Morley's proof to the logic $L_{\omega_1\omega}$; see [Kei71]. He studies a class of structures $(\mathbb{K}, \preccurlyeq_{\mathscr{F}})$, where $\mathbb{K}$ is definable with a sentence in $L_{\omega_1\omega}$ and $\mathscr{F}$ is some countable fragment of $L_{\omega_1\omega}$ containing the sentence. He uses a concept of *homogeneity*, which is closely related to saturation.

Definition 3.1.3 *For L-structures M and N and a fragment $\mathscr{F}$ of $L_{\omega_1\omega}$, $A \subset M$ a subset and $f : A \to N$ a function, write*

$$(M, A) \equiv_{\mathscr{F}} (N, f(A))$$

if for every formula $\phi(\bar{x}) \in \mathscr{F}$ and every $\bar{a} \in A$,

$$M \models \phi(\bar{a}) \text{ if and only if } N \models \phi(f(\bar{a})).$$

A model is $(\kappa, \mathscr{F})$-homogeneous, if for every set $A \subseteq M$ of cardinality strictly less than κ and every function $f : A \to M$, if

$$(M, A) \equiv_{\mathscr{F}} (M, f(A)),$$

then for all $b \in M$ there exists $c \in M$ such that

$$(M, A \cup \{b\}) \equiv_{\mathscr{F}} (M, f(A) \cup \{c\}).$$

Keisler proved the following theorem (Theorem 35 of [Kei71]):

Theorem 3.1.4 *(Keisler 1971, announced in 1969) Let $\mathscr{F}$ be a countable fragment of $L_{\omega_1\omega}$, $T \subseteq \mathscr{F}$ a set of sentences and $\kappa, \lambda > \omega$. Assume that:*

1. *T is κ-categorical.*

2. *For every countable model M of T, there are models N of T of arbitrarily large power such that $M \preccurlyeq_{\mathscr{F}} N$.*

3. *Every model M of power κ is $(\omega_1, \mathscr{F})$-homogeneous.*

Then T is λ-categorical. Moreover, every model of T of power λ is $(\lambda, \mathscr{F})$-homogeneous.

One stage in the transition from strictly syntactic to semantic means of defining classes is Shelah's version of Theorem 3.1.4. To understand it, we need the following fact, which stems from Chang, Scott and Lopez-Escobar (see for example [C.C68] from 1968); the current formulation is Theorem 6.1.8 in the book [Bal09].

Theorem 3.1.5 *(Chang-Scott-Lopez-Escobar) Let ϕ be a sentence in $L_{\omega_1\omega}$ in a countable vocabulary L. Then there is a countable vocabulary L' extending L, a first-order L' theory T and a countable collection Σ of L'-types such that reduct is a 1-1 map from the models of T which omit Σ onto the models of ϕ.*

A crucial point is that the infinitary aspects are translated to a first order context, at the cost of expanding the vocabulary. If ϕ is a complete sentence, the pair (T, Σ) can be chosen so that the associated class of models is the class of atomic models of T (every tuple realizes a principal type). Saharon Shelah generalized this idea to develop a more general context, *finite diagrams* [She70]. A *finite diagram* D is a set of types over the empty set and the class of structures consists of the models which only realize types from D. Shelah defined a structure M to be (D, λ)-homogeneous if it realizes only types from D and is $(|M|, L_{\omega,\omega})$-homogenous (in the sense of Definition 3.1.3). He (independently) generalized Theorem 3.1.4 to finite diagrams. His argument, like Keisler's, required the assumption of homogeneity. Thus, [She70] is the founding paper of *homogeneous model theory*, which was further developed in for example [GL02], [BL03], [HS01], [HS00]. The compact case ('Kind II' in [She75]) was transformed into the study of continuous logics and abstract metric spaces [BYACU08] and finally generalized to metric abstract elementary classes [HH09]. These last developments have deep connections with Banach space theory.

Baldwin and Lachlan in 1971[BL71] give another method for first order categoricity transfer. They develop some *geometric* tools to study structures of a theory categorical in some uncountable cardinal: any model of such a theory is prime over a *strongly minimal* set and the isomorphism type is determined by a certain *dimension* of the strongly minimal set. This gives a new proof for the Morley theorem for elementary classes but also the *Baldwin-Lachlan Theorem*: if an elementary class is categorical in some uncountable cardinal, it has either just one or $\aleph_0$-many countable models. The geometric analysis of uncountably categorical elementary classes was developed even further by Zilber (see [Zil93], earlier Russian version [Zil84]), giving rise to *geometric stability theory.* We discuss the number of countable models of an $\aleph_1$-categorical *non-elementary* class in Section 4.

A further semantic notion closely tied to categoricity is Shelah's 'excellence'. Excellence is a kind of generalized amalgamation; (details in [Bal09]). The rough idea is to posit a type of unique *prime models* over certain *independent diagrams* of models. 'Excellence' was discovered independently by Boris Zilber while studying the model theory of an algebraically closed field with *pseudoexponentiation*, (a homomorphism from $(F, +)$ to $(F^*, \cdot)$. He defines the notion of a quasiminimal excellent (qme) class by 'semantic conditions'; Kirby [Kir10] proved they can be axiomatized in $L_{\omega_1\omega}(Q)$. Zilber showed any qme class is categorical in all uncountable powers and finds such a class of pseudo-exponential fields. Natural algebraic characterizations of excellence have been found in context of algebraic groups by Zilber and Bays [Zil06], [BZ00], [Bay]. Excellence implies that the class of structures has models in all cardinalities, has the *amalgamation property*, (Definition 3.2.2) and admits full categoricity transfer. Zilber's notion of 'excellence' specializes Shelah's notion of excellence for sentences in $L_{\omega_1\omega}$, invented while proving the following general theorem for transferring categoricity for sentences in $L_{\omega_1\omega}$ [She83][2]. The theorem uses a minor assumption on cardinal arithmetic.

Theorem 3.1.6 *(Shelah 1983) Assume that* $2^{\aleph_n} < 2^{\aleph_{n+1}}$ *for all* $n < \omega$. *Let* $\phi \in L_{\omega_1\omega}$ *be a sentence which has an uncountable model, but strictly less than the maximal number of models in each cardinality* $\aleph_n$ *for* $0 < n < \omega$. *Then the sentence is* excellent.

(The following doesn't need the cardinal arithmetic assumption:) Assume that a sentence ϕ *in* $L_{\omega_1\omega}$ *is excellent and categorical in some uncountable cardinality. Then* ϕ *is categorical in every uncountable cardinality.*

The excellence property is defined only for complete sentences in $L_{\omega_1\omega}$, more precisely for the associated classes of *atomic models* (each model omits all non-isolated types) of a first order theory T in an extended vocabulary. Excellent classes have been further studied by several authors [Les05a], [HL06] [GH89]. Theorem 3.1.6, expounded in [Bal09], extends easily to incomplete sentences:

Corollary 3.1.7 *Assume that* $2^{\aleph_n} < 2^{\aleph_{n+1}}$ *for all* $n < \omega$. *Let* $\phi \in L_{\omega_1\omega}$ *be a sentence which is categorical in* $\aleph_n$ *for each* $n < \omega$. *Then* ϕ *is categorical in every cardinality.*

Shelah and Hart [HS90], made more precise in [BK09], show the necessity of considering categoricity up to $\aleph_\omega$; there are examples of $L_{\omega_1\omega}$-sentences ϕ_n which are categorical in each $\aleph_k$ for $k \leq n$ but have the maximal number of models in $\aleph_{n+1}$. However, it is not known whether the assumption on cardinal arithmetic can be removed from the theorem.

In the discussion above we isolated properties such as *homogeneity* and *excellence*, which enable one to prove categoricity transfer theorems. More important, they support the required tools for classifying and analyzing structures with model-theoretic methods; both generated subfields: *homogeneous model theory* and *model theory for excellent classes.* These properties have applications to 'general mathematics': $L_{\infty\omega}$-free algebras [MS89] for homogeneous model theory or Zilber's pseudoexponentiation and the work on covers of Abelian varieties [Zil05] for excellence. We argue that finding such *fundamental properties* for organizing mathematics is one of the crucial tasks of model theory.

The investigation of $L_{\omega_1,\omega}$ surveyed in this section makes no assumption that the class studied has large models; the existence of large models is deduced from sufficient categoricity in small cardinals. Shelah pursues a quite different line in [She09]. He abandons the syntactic hypothesis of definability in a specific logic. In attempting to prove eventual categoricity, he chooses smaller AEC's in successive cardinalities. Thus he attempts to construct a smaller class which is categorical in all powers. Crucially, there is no assumption that there are arbitrarily large models in this work.

3.2 Abstract elementary classes and Jónsson classes

In this section, we discuss a line which very distinct from those summarized in the last paragraph of Section 3.1; the classes are assumed to have models in arbitrarily large cardinalities.

Abstract elementary classes arise from very different notions $\preccurlyeq_{\mathbb{K}}$, which do not necessarily have a background in some logic traditionally studied in model theory. If a class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an AEC, many tools of model theory can be applied to study that class. The first essential observation is that an analog of the Chang-Scott-Lopez-Escobar Theorem 3.1.5 holds for any AEC. Here, purely semantic conditions on a class imply it has a syntactic definition.

[2]The important first order notion of the OTOP discussed in Subsection 3.3 was derived from the earlier concept of excellence for $L_{\omega_1,\omega}$.

Theorem 3.2.1 *(Shelah) Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an abstract elementary class of* L*-structures, where* $|L| \leq \mathrm{LS}(\mathbb{K})$*. There is a vocabulary* $L' \supseteq L$ *with cardinality* $|\mathrm{LS}(\mathbb{K})|$*, a first order* L'*-theory* T *and a set* Σ *of at most* $2^{\mathrm{LS}(\mathbb{K})}$ *partial types such that* $\mathbb{K}$ *is the class of reducts of models of* T *omitting* Σ *and* $\preccurlyeq_{\mathbb{K}}$ *corresponds to the* L'*-substructure relation between the expansions of structures to* L'*.*

To extend the notion of Hanf Number to AEC, take $\mathscr{C}$ in Definition 2.1.6 as the collection of all abstract elementary classes for a fixed vocabulary and a fixed Löwenheim-Skolem number. (For a more general account of Hanf Numbers see page 32 of [Bal09].) There is an interesting interplay between syntax and semantics: we can compute the *Hanf number* for AECs with a given LS($\mathbb{K}$) (See Definition 2.1.6), a semantically defined class. But the proof relies on the methods available only for an associated syntactically defined class of structures in an extended vocabulary.

The following properties of an AEC play a crucial role in advanced work:

Definition 3.2.2 (Amalgamation and Joint embedding) 1. *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has the* amalgamation property *(AP), if it satisfies the following:*

If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$*,* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$*,* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ *and* $\mathscr{B} \cap \mathscr{C} = \mathscr{A}$*, there is* $\mathscr{D} \in \mathbb{K}$ *and a map* $f : \mathscr{B} \cup \mathscr{C} \to \mathscr{D}$ *such that* $f \restriction \mathscr{B}$ *and* $f \restriction \mathscr{C}$ *are* $\mathbb{K}$*-embeddings.*

2. *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has* the joint embedding property *(JEP) if for every* $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ *there is* $\mathscr{C} \in \mathbb{K}$ *and* $\mathbb{K}$*-embeddings* $f : \mathscr{A} \to \mathscr{C}$ *and* $g : \mathscr{B} \to \mathscr{C}$*.*

The notion of AEC is naturally seen as a generalization of Jónsson's work in the 50's on universal and homogeneous-universal relational systems; we introduce new terminology for those AEC's close to his original notion.

Definition 3.2.3 (A Jónsson class) *An abstract elementary class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a* Jónsson class *if* $\mathbb{K}$ *has arbitrarily large models, and* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has the joint embedding and amalgamation properties.*

The models of a first order theory under elementary embedding form a Jónsson class in which complete first order type (over a model) coincides exactly with the Galois types described below and the usual notion of a monster model is the one we now explain.

A standard setting, stemming from Jónsson's [J6́0] version of Fraïssé limits of classes of structures, builds a 'large enough' *monster model* $\mathfrak{M}$ (or universal domain) for an elementary class of structures via amalgamation and unions of chains. A monster model is *universal* and *homogeneous* in the sense that

- All 'small enough' structures can be elementarily embedded in $\mathfrak{M}$ and
- all *partial elementary maps* from $\mathfrak{M}$ to $\mathfrak{M}$ with 'small enough' domain extend to automorphisms of $\mathfrak{M}$.

Here 'small enough' refers to the possibility to find all structures 'of interest' inside the monster model; further cardinal calculation can be done to determine the actual size of the monster model.

The situation is more complicated for AEC. We consider here *Jónsson classes*, where we are able to construct a *monster model*. However, even then the outcome differs crucially from the monster in elementary classes, since we get only *model-homogeneity*, that is, the monster model for a Jónsson class is a model $\mathfrak{M}$ such that

- For any 'small enough' model $M \in \mathbb{K}$ there is a $\mathbb{K}$-embedding $f : M \to \mathfrak{M}$.
- Any isomorphism $f : M \to N$ between 'small enough' $\mathbb{K}$-elementary substructures $M, N \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ extends to an automorphism of $\mathfrak{M}$.

The first order case has homogeneity over sets; AEC's have homogeneity only over models.

The first problem in stability theory for abstract elementary classes is to define 'type', since now it cannot be just a collection of formulas. We note two definitions of *Galois type.*

Definition 3.2.4 (Galois type) *1. For an arbitrary AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *and models* $M \preccurlyeq_{\mathbb{K}} N \in \mathbb{K}$ *consider the following relation for triples* $(\bar{a}, M, N)$*, where* $\bar{a}$ *is a finite tuple in* N*:*

$$(\bar{a}, M, N) \equiv (\bar{b}, M, N')$$

if there are a model $N'' \in \mathbb{K}$ *and* $\mathbb{K}$*-embeddings* $f : N \to N''$, $g : N' \to N''$ *such that* $f \restriction M = g \restriction M$ *and* $f(\bar{a}) = \bar{b}$*. Take the transitive closure of this relation. The equivalence class of a tuple* $\bar{a}$ *in this relation, written* $\mathrm{tp^g}(\bar{a}, M, N)$ *is called the Galois type of* $\bar{a}$ *in* N over M*.*

2. Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a Jónsson class and* $\mathfrak{M}$ *is a fixed monster model for the class. We say that the tuples* $\bar{a}$ *and* $\bar{b}$ *in* $\mathfrak{M}$ *have the same Galois type over a subset* $A \subseteq \mathfrak{M}$*,*

$$\mathrm{tp^g}(\bar{a}/A) = \mathrm{tp^g}(\bar{b}/A),$$

if there is an automorphism f *of* $\mathfrak{M}$ *fixing* A *pointwise such that* $f(\bar{a}) = \bar{b}$*.*

Fruitful use of Definition 3.2.4.2 depends on the class having the amalgamation property over the 'parameter sets'. Thus, even with amalgamation, there is a good notion of Galois types only over models and not over arbitrary subsets.

The monster model is λ-saturated for a 'big enough' λ. That is, all Galois-types over subsets of size $\leq \lambda$ are realized in $\mathfrak{M}$. When M is a $\mathbb{K}$-elementary substructure of the monster model $\mathfrak{M}$, the two notions of a Galois type $\mathrm{tp^g}(\bar{a}, M, \mathfrak{M})$ agree. As in the first order case, the set of realization of a Galois-type of $\bar{a}$ (over a model) is exactly *the orbits of the tuple* $\bar{a}$ under automorphisms of $\mathfrak{M}$ fixing the model M pointwise. That is,

$$\mathrm{tp^g}(\bar{a}, M, \mathfrak{M}) = \mathrm{tp^g}(\bar{b}, M, \mathfrak{M})$$

if and only if there is an automorphism f of $\mathfrak{M}$ fixing M pointwise such that $f(\bar{a}) = \bar{b}$. Furthermore, if $N \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ is any $\mathbb{K}$-extension of M containing $\bar{a}$, $\mathrm{tp^g}(\bar{a}, M, N)$ equals $\mathrm{tp^g}(\bar{a}, M, \mathfrak{M}) \cap N$. Hence in Jónsson classes we fix a monster model $\mathfrak{M}$ and use a simpler notation for a Galois type, $\mathrm{tp^g}(\bar{a}/M)$, which abbreviates $\mathrm{tp^g}(\bar{a}, M, \mathfrak{M})$. Since we can also study automorphisms of $\mathfrak{M}$ fixing some *subset* A of $\mathfrak{M}$, also the notion of a Galois type over a set A becomes amenable. But over sets, the two forms are not equivalent.

The notion of Galois type lacks many properties that the compactness of first order logic guarantees for first order types. In the first order case, we can always realize a *union* of an increasing chain of types in the monster model and types have *finite character*: the types of $\bar{a}$ and $\bar{b}$ agree over a subset A if and only if they agree over every finite subset of A. Many such nice properties disappear for arbitrary Galois types. But we restrict to better-behaved Jónsson classes. Grossberg and VanDieren [GV06] isolated the concept of *tameness* that is crucial in the study of categoricity transfer for Jónsson classes.

Definition 3.2.5 (Tameness) *We say that a Jónsson class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* (κ, λ)*-tame for* $\kappa \leq \lambda$ *if the following are equivalent for a model* M *of size at most* λ*:*

- $\mathrm{tp^g}(\bar{a}/M) = \mathrm{tp^g}(\bar{b}/M)$*,*
- $\mathrm{tp^g}(\bar{a}/M') = \mathrm{tp^g}(\bar{b}/M')$ *for each* $M \preccurlyeq_{\mathbb{K}} M$ *with* $|M'| \leq \kappa$*.*

Furthermore, we say that the class is κ-tame *if it is* (κ, λ)*-tame for all cardinals* λ *and* tame *if it is* $\mathrm{LS}(\mathbb{K})$*-tame.*

Giving up compactness also has benefits: 'non-standard structures' that realize unwanted types, which are forced by compactness, can now be avoided. For example, we might study real vector spaces in a two sorted language and demand that the reals be standard.

The first 'test question' for AECs was to ask if one can prove a categoricity transfer theorem. Shelah stated the following conjecture:

Conjecture 3.2.6 *There exists a cardinal number* κ *(depending only on* $\mathrm{LS}(\mathbb{K})$*) such that if an AEC with a given number* $\mathrm{LS}(\mathbb{K})$ *is categorical in some cardinality* $\lambda > \kappa$*, then it is categorical in every cardinality* $\lambda > \kappa$*.*

Shelah introduced the notion of a Jónsson class (not the name) in 1999 [She99] and proved the following categoricity transfer result. (Part II [Bal09]).

Theorem 3.2.7 *(Shelah) Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be a Jónsson class. Then there is a calculable cardinal* H_2*, depending only on* $\mathrm{LS}(\mathbb{K})$*, such that if* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in some cardinal* $\lambda^+ > \mathrm{H}_2$*, then* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in all cardinals in the interval* $[\mathrm{H}_2, \lambda^+]$*.*

We remark that this almost settles the Categoricity Conjecture for Jónsson classes: for each such AEC with a fixed Löwenheim-Skolem number LS, let $\mu_{\mathbb{K}}$ be the sup (if it exists) of the successor cardinals in which $\mathbb{K}$ is categorical. Since there does not exists a proper class of such AECs, there is a supremum for such $\mu_{\mathbb{K}}$, denote this number $\Lambda(\mathrm{LS})$. Now if a Jónsson class with Löwenheim-Skolem number LS is categorical in some successor cardinal $\lambda > \mu = \sup(\Lambda(\mathrm{LS}), \mathrm{H}_2)$, it is categorical in all cardinals in $[\mathrm{H}_2, \lambda^+]$, and in arbitrarily large successor cardinals, and hence in all cardinals above H_2. Two problems remain in this area. Remove the restriction to successor cardinals in Theorem 3.2.7; this would avoid the completely non-effective appeal to $\Lambda(\mathrm{LS})$. Make a more precise calculation of the cardinal H_2 in the successor case (Problem D.1.5 of [Bal09].)

Shelah proves a downward categoricity transfer theorem and also shows categoricity for $\lambda^+ > H_2$ implies certain kind of 'tameness' for Galois types over models of size $\leq H_2$, which enables the transfer of categoricity up to all cardinals in the interval $[\mathrm{H}_2, \lambda^+]$. Grossberg and VanDieren separated out the upward categoricity transfer argument, and realized that tameness was the only additional condition needed to transfer categoricity arbitrarily high. The downward step uses the *saturation transfer* method, where saturation is with respect to *Galois types*; the upwards induction uses the *dimension method.*

Theorem 3.2.8 *(Grossberg and VanDieren) Assume that a* χ*-tame Jónsson class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in* λ^+*, where* $\lambda > \mathrm{LS}(\mathbb{K})$ *and* $\lambda \geq \chi$*. Then* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is categorical in each cardinal* $\geq \lambda^+$*.*

Lessmann [Les05b] extends the result to $\mathrm{LS}(\mathbb{K})^+$-categoricity when $\mathrm{LS}(\mathbb{K}) = \aleph_0$. The restriction to countable Löwenheim cardinal number reflects a significant combinatorial obstacle. In these two results the categoricity transfer is only from successor cardinals and the proof is essentially an induction on dimension. In Subsection 3.4 we discuss further use of the saturation transfer method for simple, finitary AECs by Hyttinen and Kesälä in [Kes06].

3.3 The stability classification: First order vrs non-elementary

One of the major themes of contemporary model theory is the notion of classification theory. Classification is used in two senses. On the one hand models in a particular class can be classified by some assignment of structural invariants. On the other hand, the classes of models[3] are split into different groups according to common properties, which may be

[3]The word class is vastly overloaded in this context. In first order logic, a complete theory is a natural unit. In studying infinitary logic, the natural unit often becomes an AEC (in the first order case this would be the class of models of the theory.)

semantic or syntactic; many examples are given below. Shelah (e.g. [She09]) has stressed the importance of certain properties of theories, those which are dividing lines: both the property and its negation has strong consequences. In the following we discuss various important classes of theories and emphasize those properties which are dividing lines.

Saharon Shelah originated *stability theory* for elementary classes [She78] and produced much of the early work. Now however, the field embraces much of model theory and the tools are pervasive in modern applications of model theory. Among the many texts are: [Bue91],[Bal88],[Pil96].

We can define *stability* in λ, as the property that there are no more than λ many distinct complete types over any subset of size λ. However, stability has many equivalent definitions in elementary classes. A remarkable consequence of the analysis is that counting the number of types is related to the geometry of the structures in the class. For example, if the class of structures is stable in any cardinal at all, one can define a notion of *independence* between arbitrary subsets of any model, which is a useful tool to analyze the properties of the structures in the class. The importance of such a notion of independence is well established and such *independence calculus* has been generalized to some unstable elementary classes such as classes given by *simple* [Wag00] or *NIP* theories [Adl]. Stability theory has evolved to such fields as *geometric stability theory*[Pil96], which is the major source for applications of model theory to 'general mathematics'.

Stability theory divides classes into four basic categories. This division is called the *stability hierarchy*:

1. $\aleph_0$-stable classes;
2. superstable classes, that is, classes stable from some cardinal onwards;
3. stable classes, that is, stable in at least one cardinal;
4. unstable classes.

In elementary classes $\aleph_0$-stable classes are stable in all cardinalities and hence we get a hierarchy of implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. Uncountably categorical theories are always $\aleph_0$-stable whereas non-superstable classes have the maximal number of models in each uncountable cardinal. An $\aleph_0$-stable or superstable class can also have the maximal number of models, e.g., if it has one of the properties DOP or OTOP, discussed in Examples 3.3.2 and 3.3.3.

Developing stability theory for non-elementary classes is important not only because it widens the scope of applications but also because it forces further analysis of the tools and concepts developed for elementary classes. Which of the tools are there only because first order logic 'happens' to be compact and which could be cultivated to extend to non-elementary classes? Especially, can we *distinguish* some core properties enabling the process? What are the problems met in, say, categoricity transfer or developing independence calculus? Why does the number of *types* realized in the structure seem to affect the geometric properties of structures and can we analyze the possible geometries arising from different frameworks? For example, Hrushovski [Hru89] proved a famous theorem in geometric stability theory: under assumptions of a logical nature the geometry given by the notion of independence on the realizations of a regular type, must fall into one of three natural categories involving group actions. In the available non-elementary versions of the same theorem ([HLS05],[HK10]), we cannot rule out a fourth possibility: existence of a so-called *non-classical group*, a non-abelian group admitting an ω-homogeneous pre-geometry. We can identify some quite peculiar properties of such groups. It remains open whether such groups can exist at all.

We present here some examples of AECs where the choice of the relation $\preccurlyeq_{\mathbb{K}}$ affects the placement of the class in the stability hierarchy. How 'coincidental' is the division of elementary classes according to the stability hierarchy? The placement of a class of structures in the hierarchy has been shown to affect a huge number of properties that at first sight do not seem to have much to do with the number of types. Which of these connections are 'deep' or 'semantic', or especially, which extend to non-elementary frameworks? Can an appropriate hierarchy be found?

The moral of these examples is that properties of the 'same' class of structures might look different if definitions in logics with more expressive power are allowed or a different notion $\preccurlyeq_{\mathbb{K}}$ for an abstract elementary class is chosen.

Example 3.3.1 (Abelian groups) Let $\mathbb{K}$ be the class of all abelian groups. $(\mathbb{K}, \prec_{\mathbb{K}})$ is an $\aleph_0$-stable AEC with the notion $\prec_{\mathbb{K}}$ as the substructure relation.

However, the same class of structures is strictly stable (stable but not superstable) if we take as $\preccurlyeq_{\mathbb{K}}$ the following notion: $M \preccurlyeq_{\mathbb{K}} N$ if and only if M is a subgroup and

$$\text{for each } a \in M \text{ and } n \in \mathbb{N} \setminus \{0\},\ n \text{ divides } a \text{ in } M \text{if and only if } n \text{ divides } a \text{ in } N.$$

The model theory of abelian groups is studied in Eklof [EF72].The study of AECs induced by tilting and co-tilting modules in Baldwin, Eklof, and Trlifaj [BET07],[Trl09] provides a more semantic notion of $\preccurlyeq_{\mathbb{K}}$ and the classes of Abelian groups are strictly superstable except in one degenerate case.

A number of properties in first order classification theory induce 'bad behavior' for an elementary class of structures, signaled by the existence of the maximal number of models in a given cardinality. The most basic of these are instability and unsuperstability. Others include OTOP, 'the omitting types order property' and DOP 'the dimensional order property', with a version ENI-DOP, which gives many countable models. Especially, these play a role in classifying countable complete first order theories; their negations NOTOP, NDOP and ENI-NDOP have 'good' implications, from the viewpoint of classification theory; they aid in the assigning of invariants.

One equivalent definition for unstability is that there is a formula which in the models of a first order theory defines an infinite ordering. Then by compactness, the elementary class must contain models interpreting various different orderings, which (nontrivially) forces the number of models to the maximum. Similarly the properties DOP and OTOP cause certain kind of orderings to appear in the structures; but, the orderings are not defined by a single first order formula. Just as in Example 3.3.1, the unsuperstability of the class of abelian groups is not visible to quantifier-free formulas, the only ones 'seen' by the substructure-relation, OTOP and DOP are a form of instability not visible to first order formulas.

The following two examples illustrate the properties OTOP and DOP. In each case we 'define' an arbitrary graph (e.g. an ordering) on $P \times P$ by describing a column above each point of the plane. The two methods of description, by a type or a single formula, distinguish OTOP and DOP.

Example 3.3.2 (An example with OTOP) Let the vocabulary L consist of two predicates P and Q and ternary relations R_n for each $n < \omega$.

By ternary predicates $R_n(x, y, z)$ we define a decreasing chain of sets $R_n(a, b, z)$ of subsets of Q over each pair (a, b) in $P \times P$. The sets $R_0(a, b, z)$ are disjoint as the pairs (a, b) vary. And there is exactly one element $c_n^{a,b}$ in $R_n(a, b, z)$ but not in $R_{n+1}(a, b, z)$. Thus the types $p_{ab}(x) = \{R_n(a, b, x), x \neq c_n^{a,b} : n < \omega\}$ can be independently omitted or realized.

The resulting elementary class is $\aleph_0$-stable but it has the maximal number of models in each infinite cardinality. Any directed graph (especially any ordering) can be coded by a structure the following way:

$$\text{there exists an edge from } x \text{ to } y \ \Leftrightarrow\ \exists z \bigwedge_{n<\omega} R_n(x, y, z).$$

We can study the same class $\mathbb{K}$ of structures but replace first order elementary substructure by $\preccurlyeq_{\mathbb{K}}$, elementary submodel in a fragment of $L_{\omega_1\omega}$ containing all first order formulas and the formula

$$\phi(x,y) = \exists z \bigwedge_{n<\omega} R_n(x,y,z).$$

Now the relation $\preccurlyeq_{\mathbb{K}}$ 'sees' the complexity caused by the formula, and the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is unstable in the sense of the fragment. But this means it is also unstable as an abstract elementary class. As, the Galois types (See Definition 3.2.4 for Galois type) always refine syntactic types if the submodel notion has a syntactic definition.

This example also has ENI-DOP and thus DOP. From ENI-DOP, we can define another notion $\preccurlyeq_{\mathbb{K}}$ for that class so that the new AEC is unstable but still has Löwenheim-Skolem number $\aleph_0$. Namely, let $M \preccurlyeq_{\mathbb{K}} N$ if M is an elementary substructure of N and whenever there are only finitely many z such that $M \models \bigwedge_{n<\omega} R_n(x,y,z)$, then the number of such elements z is not increased in N.

Example 3.3.3 (An example with DOP) Let the vocabulary L consist of predicates X_1, X_2 and P and two binary relation symbols π_1 and π_2. We define a theory in first order logic, with definable projections from P to each X_i and study the dimensions of pre-images of pairs in $X_1 \times X_2$. We require that

- The universe of a structure consists of three disjoint infinite predicates X_1, X_2 and P,
- the relations π_i determine surjective functions $\pi_i : P \to X_i$ for $i = 1, 2$ and
- for each $x \in X_1$ and $y \in X_2$ there are infinitely many $z \in P$ such that $\pi_1(z) = x$ and $\pi_2(z) = y$.

Again we get an $\aleph_0$-stable elementary class, which is $\aleph_0$-categorical but has the maximal number of models in each uncountable cardinality. Now we can code an ordering $(I, <)$ on the pairs $(x_i, y_i)_{i\in I}$ in an uncountable model so that

$$(x_i, y_i) < (x_j, y_j) \text{ if and only if } \{z \in P : \pi_1(z) = x_i \text{ and } \pi_2(z) = y_j\} \text{ is uncountable.}$$

Furthermore, we get an unstable abstract elementary class for the same class of structures $\mathbb{K}$ as follows: we strengthen the notion $\preccurlyeq_{\mathbb{K}}$ so that $M \preccurlyeq_{\mathbb{K}} N$ implies that for all pairs (x, y) in the set $X_1 \times X_2$ of the structure M, if there are only countably many z in the set P of M such that $\pi_1(z) = x$ and $\pi_2(z) = y$, then no such z is added to the set P of the structure N. Since automorphisms of the monster model must preserve the cardinalities of sets described on the right hand side of the above displayed equivalence, the class is unstable for Galois types. This notion $\preccurlyeq_{\mathbb{K}}$ does not have finite character (Definition 3.4.1) and the new class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has Löwenheim-Skolem number $\aleph_1$.

Similar phenomena appear in *differentially closed fields* of characteristic zero, whose elementary theory is $\aleph_0$-stable with ENI-DOP, and thus DOP. They have the maximal number of models in each infinite cardinality. See the survey articles by Marker [Mar00],[Mar07].

The following examples exhibit the difference between a traditional first order approach and a non-elementary approach.

Example 3.3.4 (Exponential maps of abelian varietes) Martin Bays, Misha Gavrilovich, Anand Pillay, and Boris Zilber [BZ00, Bay, Gav06] study 'exponential maps' or 'universal group covers'

$$\pi : (\mathbb{C}^g, +) \to A(\mathbb{C}),$$

where $(\mathbb{C}^g, +)$ is the additive group of the complex numbers to power g and $A(\mathbb{C})$ is an abelian variety. The kernel Λ of π is a free abelian subgroup of $(\mathbb{C}^g, +)$. Two approaches appear in the work: the structures modeling the first order theory of such a map and the structures modeling the $L_{\omega_1\omega}$-theory. The $L_{\omega_1\omega}$-sentence describing the map is quasi-minimal excellent and so categorical in each uncountable cardinality. All the models of the sentence share the same Λ and are determined up to the transcendence degree of the field interpreted in $A(\mathbb{C})$. However, the first order theory is also 'classifiable', it is superstable with NDOP and NOTOP and is 'shallow', although not categorical. Each model of the first-order theory is described by choosing a lattice Λ and a transcendence degree for the field in $A(\mathbb{C})$.

In this case, the non-elementary framework was understood first; the elementary class gives a little more information. Both depend on rather deep algebraic number theory. This topic is an offshoot of trying to understand the model theory of the complex exponentiation $\exp : (\mathbb{C}, +, \times) \to (\mathbb{C}, +, \times)$, which has a very ill-behaved theory in first order logic; see [Bal06, Zil04] for more discussion on the subject.

Example 3.3.5 (Valued fields) The recent book by Haskell, Hrushovski and Macpherson [HHM08] greatly develops the first order model theory of *algebraically closed valued fields*. The elementary class is unstable and not even simple, and hence the structure theory has involved developing new extensions of the stability-theoretic machinery investigating the class of theories without the independence property.

A valued field consists of a field K together with a homomorphism from its multiplicative group to an ordered abelian group Γ, which satisfies the ultrametric inequality. The problems in the elementary theory of valued fields reduce to that of the value group Γ and the so called *residue field*.

However, we can study valued fields as an AEC fixing the value group as $(\mathbb{R}, +, <)$ and taking all substructures as elementary substructures, requiring also that the value group stays fixed. This class is stable and contains those valued fields that are of most interest. The cases where $(\Gamma, +, <)$ is not embeddable to $(\mathbb{R}, +, <)$ are often called *Krull valuations*. They are forced to be in the scope of study in the first order approach since first order logic cannot separate them from the usual ones. The non-elementary class fixing the value group can be seen as 'almost compact'; see the work of Itaï Ben Yaacov [Yaa09].

3.4 Simple finitary AECs

Simple finitary AECs were defined particularly to study independence and stability theory in a framework without compactness. The idea was both to find a common extension for homogeneous model theory and the study of excellent sentences in $L_{\omega_1\omega}$ and also clarify the 'core' properties which support a successful dimension theory. The property *finite character* is essential for this analysis.

Definition 3.4.1 (Finite character) *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has* finite character *if for any two models* $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ *such that* $\mathscr{A} \subseteq \mathscr{B}$ *the following are equivalent:*

1. $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$
2. *For every finite sequence* $\bar{a} \in \mathscr{A}$ *there is a* $\mathbb{K}$*-embedding* $f : \mathscr{A} \to \mathscr{B}$ *such that* $f(\bar{a}) = \bar{a}$.

Definition 3.4.2 (Finitary AEC) *An abstract elementary class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* finitary *if* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a Jónsson class with countable Löwenheim-Skolem number that has finite character.*

Definition 3.4.2 slightly modifies Hyttinen, Kesälä [HK06]; in particular the formulation of finite character is from Kueker [Kue08b]. Elementary classes are finitary AECs. However, a class defined by an arbitrary sentence in $L_{\omega_1\omega}$, the relation $\preccurlyeq_{\mathbb{K}}$ being the one given by the corresponding fragment, may not have AP, JEP or even arbitrarily large models. A relation $\preccurlyeq_{\mathbb{K}}$ given by any fragment of $L_{\infty\omega}$ will have finite character. Most abstract elementary classes definable in $L_{\omega_1\omega}(Q)$ do not have finite character. An easy example of a class without finite character, due to Kueker [Kue08b], is a class of structures with a countable predicate P, where $M \preccurlyeq_{\mathbb{K}} N$ if and only if $M \subseteq N$ and $P(M) = P(N)$.

The notion of weak type is just Galois type with built-in finite character: two tuples $\bar{a}$ and $\bar{b}$ have the same *weak type* over a set A, written

$$\mathrm{tp}^{\mathrm{w}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{w}}(\bar{b}/A),$$

if they have the same Galois type over each finite subset $A' \subseteq A$. Furthermore, we say that a model M is *weakly saturated* if it realizes all weak types over subsets of size $< M$.

Basic stability theory with a categoricity transfer result for simple finitary AEC's is carried out in the papers [HK06], [HK07] and [HK]. However, some parts of the theory hold also for arbitrary Jónsson classes; this is expounded in [HK11]. Recently David Kueker [Kue08a] has clarified when AEC admit syntactic definitions and particularly the connection of finite character to definability in $L_{\infty\omega}$ definablity of AEC's; unlike in Theorem 3.2.1, no extra vocabulary is needed for these results.

Theorem 3.4.3 *(Kueker) Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an abstract elementary class with* $\mathrm{LS}(\mathbb{K}) = \kappa$. *Then,*

1. *The class* $\mathbb{K}$ *is closed under* L_{∞,κ^+}*-elementary equivalence.*
2. *If* $\mathrm{LS}(\mathbb{K}) = \aleph_0$ *and* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *contains at most* λ *models of cardinality* $\leq \lambda$ *for some cardinal* λ *such that* $\lambda^\omega = \lambda$*, then* $\mathbb{K}$ *is definable with a sentence in* L_{λ^+,ω_1}.
3. *If* $\kappa = \aleph_0$ *and* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has* finite character*, the class is closed under* $L_{\infty,\omega}$*-elementary equivalence.*
4. *Furthermore, if* $\kappa = \aleph_0$*,* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *has finite character and at most* λ *many models of size* $\leq \lambda$ *for some infinite* λ*,* $\mathbb{K}$ *is definable with sentence in* $L_{\lambda^+,\omega}$.

The notion of an *indiscernible sequence* of tuples further illustrates the distinction between the syntactic and semantic viewpoint. Classically a sequence is indiscernible if each increasing n-tupe of elements realize the same (syntactic) type. In AEC, a sequence $(\bar{a}_i)_{i<\kappa}$ is *indiscernible* over a set A (or A-indiscernible) if the sequence can be extended to any 'small enough' length $\kappa' > \kappa$ so that any order-preserving partial permutation of the larger sequence extends to an automorphism of the monster model fixing the set A.

Note that two tuples lying on the same A-indiscernible sequence is a much stronger condition than two tuples having the same Galois type over A. However, 'lying on the same sequence' is not a transitive relation and hence not an equivalence relation; the notion of *Lascar strong type* is obtained by taking the transitive closure of this relation.

Using indiscernible sequences we can define a notion of *independence* based on *Lascar splitting*[4]. Furthermore, we say that the class is *simple* if this notion satisfies that each

[4]The notions are defined 'for weak types' since they are preserved under the equivalence of weak types.

Definition 3.4.4 (Independence) *A type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ *Lascar-splits over a finite set* $E \subseteq A$ *if there is a strongly indiscernible sequence* $(\bar{a}_i)_{i<\omega}$ *such that* $\bar{a}_0, \bar{a}_1$ *are in the set* A *but*

$$\mathrm{tp}^{\mathrm{w}}(\bar{a}_0/E \cup \bar{a}) \neq \mathrm{tp}^{\mathrm{w}}(\bar{a}_1/E \cup \bar{a}).$$

We write that a set B *is* independent *of a set* C *over a set* A*, written*

$$B \downarrow_A C,$$

if for any finite tuple $\bar{a} \in B$ *there is a finite set* $E \subseteq A$ *such that for all sets* D *containing* $A \cup C$ *there is* $\bar{b}$ *realizing the type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A \cup C)$ *such that* $\mathrm{tp}^{\mathrm{w}}(\bar{b}/D)$ *does NOT Lascar-split over* E.

type is independent over its domain. Under further stability hypotheses (Both $\aleph_0$-stability [HK06],[HK] and superstability [HK07],[HK11] have been developed.) we get an *independence calculus* for subsets of the monster model. Unlike in elementary stability theory, stability or even categoricity does not imply simplicity; it is a further assumption. However, we show that if *any* reasonable independence calculus exists for arbitrary sets and not just over models, the class must be simple and the notion of independence must agree with the one defined by Lascar splitting, see [HK].

The saturation transfer method was further analyzed for simple, finitary AECs by Hyttinen and Kesälä in [Kes06]. It was noted there, that even without tameness, *weak saturation* transfers between different uncountable cardinalities. Assuming simplicity, they developed much of the stability theoretic machinery for these classes and hence were able to remove the assumption in Theorems 3.2.7 and 3.2.8 that the categoricity cardinal is a successor.

Theorem 3.4.5 *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple finitary AEC,* $\kappa > \omega$*, and each model of size* κ *is weakly saturated. Then*

1. *for any* $\lambda > \min\{(2^{\aleph_0})^+), \kappa\}$*, each model of size* λ *is weakly saturated.*

2. *Furthermore, each uncountable* $\aleph_0$*-saturated model is weakly saturated.*

If in addition $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* $\aleph_0$*-tame, all weakly saturated models with a common cardinality are isomorphic.*

What then is the role of finite character of $\preccurlyeq_{\mathbb{K}}$? If it happens that there are only countably many Galois types over any finite set (this holds for example if the class is $\aleph_0$-stable), the finite character property provides a 'finitary' sufficient condition for a substructure M of $\mathfrak{M}$ to be in $\mathbb{K}$: If all Galois types over finite subsets are realized in M, M is back-and forth-equivalent to an $\aleph_0$-saturated $\mathbb{K}$-elementary substructure N of $\mathfrak{M}$ with $|N| = |M|$; a chain argument and finite character give that $N \approx M$. Even without the condition on the number of Galois types, finite character enables many constructions involving building models from finite sequences. It implies, for example, that under simplicity and superstability, two tuples with the same *Lascar type* over a countable set can be mapped to each other by an automorphism fixing the set (i.e. they have the same Galois type over the set), see [HK11]. These Lascar types (also called *weak Lascar strong types*) are a major tool in geometric stability theory for finitary classes [HK10], since they have finite character.

4 Countable models and completeness

A famous open conjecture for elementary classes was stated by Vaught in [Vau61]:

Conjecture 4.0.6 (Vaught conjecture) *The number of countable models of a countable and complete first order theory must be either countable or* $2^{\aleph_0}$.

The conjecture can be resolved by the continuum hypothesis, which is independent of the axioms of set theory: If there is no cardinality between $\aleph_0$ and $2^{\aleph_0}$, the conjecture is trivially true. The problem is to determine the value in ZFC. Morley [Mor70] proved the most significant result: not just for first order theories but for any sentence of $L_{\omega_1\omega}$ the number of countable models is either $\leq \aleph_1$ or $2^{\aleph_0}$. He used a combination of descriptive set theoretic and model theoretic techniques. There has been much progress using descriptive set theory. The study of this conjecture has also lead to many new innovations in model theory: a positive solution for $\aleph_0$-stable theories was shown by Harrington, Makkai and Shelah in [SHM84] and a more general positive solution for superstable theories of finite rank by Buechler in [Bue08]. However, the full conjecture is still open. [Bal07] provides connections with the methods of this paper.

An easier question for elementary classes is the number of countable models of a theory, which has only one model, up to isomorphism, in some *uncountable* power. Morley [Mor67] showed that the number of countable models of an uncountably categorical elementary class must be countable. We consider as a useful 'motivating question'.

Question 4.0.7 *Must an AEC categorical in $\aleph_1$ or in some uncountable cardinal have only countably many countable models?*

As asked, the answer is opposite to the first order case. For example, we can define a sentence ψ in $L_{\omega_1\omega}$ as a disjunct of two sentences, one totally categorical and one having uncountably many countable models but no uncountable models. This problem does not occur in the first order case because categoricity implies completeness. $L_{\omega_1,\omega}$ poses two difficulties to this approach. First, deducing completeness from categoricity is problematic; there are several completions. Secondly, $L_{\omega_1,\omega}$-completeness is too strong; it implies $\aleph_0$ categoricity and there are interesting $\aleph_1$-categorical sentences that are not $\aleph_0$-categorical. But sentences like ψ lack 'good' semantic properties such as joint embedding. We might ask a further question: are there some semantic properties that allow the dimensional analysis of the Baldwin-Lachlan proof for an abstract elementary class? For example, does the question have a negative answer for, say, finitary AECs? (See Subsection 4.1.) What can we say on the number of countable models in different frameworks? Some results and conjectures were stated for *admissible* infinitary logics already by Kierstead in 1980 [Kie80].

For a non-elementary class with a better toolbox for dimension-theoretic considerations it might be possible to say more on such questions. For example, *excellent* sentences of $L_{\omega_1\omega}$ have a well-behaved model theory; but such sentences are *complete*, so their countable model is unique up to isomorphism. An essential benefit of the approach of *finitary abstract elementary classes* is that the framework also enables the study of incomplete sentences of $L_{\omega_1\omega}$. The Vaught conjecture is false for finitary abstract elementary classes: Kueker [Kue08a] gives an example, well-orders of length $\leq \omega_1$, where $\preccurlyeq_{\mathbb{K}}$ is taken as end-extension. This example has exactly $\aleph_1$ many countable models. The example is categorical in $\aleph_1$, but is not a finitary AEC since it does not have arbitrarily large models. However, we can transform the example to a finitary AEC, by adding a sort with a totally categorical theory; but we lose categoricity.

Contrast the semantic and syntactic approach. If we require definability in some specific language, $L_{\omega\omega}$ or $L_{\omega_1\omega}$, the Vaught conjecture is a hard problem, but it has an 'easy' solution under the 'semantic' requirements we have suggested, such as, a finitary AEC. Is there a similar difference for Question 4.0.7, maybe in the opposite direction? David Kueker had a special reason for asking question 4.0.7 for *finitary AECs.* Recall that by Theorem 3.4.3 d) that if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an AEC with finite character, $\mathrm{LS}(\mathbb{K}) = \aleph_0$, and $\mathbb{K}$ contains at most λ models of cardinality $\leq \lambda$, then it is definable in $L_{\lambda^+\omega}$. Hence if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is $\aleph_1$ categorical and has only countably many countable models, it is definable in $L_{\omega_1\omega}$. But under what circumstances can we gain this? Clearly if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is $\aleph_0$-categorical, this holds. Kueker asks the following, refining Question 4.0.7:

Question 4.0.8 *(Kueker) Does categoricity in some uncountable cardinal imply that a finitary AEC $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is definable with a sentence in $L_{\omega_1\omega}$?*

Answering the following question positively would suffice:

Question 4.0.9 *(Kueker) Does categoricity in some uncountable cardinal imply that a finitary AEC $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has only countably many countable models?*

Unfortunately, Example 4.1.1 gives a negative answer Question 4.0.9, leaving the first question open.

Kueker's results illuminate the distinction between semantic and syntactic properties. Abstract elementary classes were defined with only semantic properties in mind, Kueker provides additional semantic conditions which imply definability in a specific syntax. Thus, the ability to choose a notion of $\preccurlyeq_{\mathbb{K}}$ for an AEC to make it finitary has definability consequences. The concept of *finite character* concerns the relation $\preccurlyeq_{\mathbb{K}}$ between the models in an AEC; Kueker's results conclude definability for the class $\mathbb{K}$ of structures. He does prove some, but remarkably weaker, definability results without assuming finite character.

4.1 An example answering Kueker's second question

The following example is a simple, finitary AEC, which is categorical in each uncountable power but has uncountably many countable models. Hence the example gives a negative answer to Kueker's second question.

Example 4.1.1 *We define a language $L = \{Q, (P_n)_{n<\omega}, E, f)$, where Q and P_n are unary predicates, E is a ternary relation and f is a unary function. We consider the following axiomatization in $L_{\omega_1\omega}$:*

1. *The predicates Q and $\langle P_n : n < \omega\rangle$ partition the universe.*
2. *Q has at most one element.*
3. *If $E(x, y, z)$ then $x \in Q$ and z, y are not in Q.*
4. *If Q is empty, we have that for each $n < \omega$, $|P_{n+1}| \leq |P_n| + 1$.*
5. *If P_0 is nonempty, then Q is nonempty.*
6. *For all $x \in Q$, the relation $E(x, -, -)$ is a equivalence relation where each class intersects each P_n exactly once.*
7. *$f(x) = x$ for all $x \in Q$ and $y \in P_n$ implies $f(y) \in P_{n+1}$.*
8. *f is one-to-one.*
8. *For $x \in Q$, $y \in P_n$ and $z \in P_{n+1}$, $E(x, y, z)$ if and only if $f(y) = z$.*

Now we define the class $\mathbb{K}$ be the L-structures satisfying the axioms above and the relation $\preccurlyeq_{\mathbb{K}}$ to be the substructure relation.

The example has two kinds of countable models. When there is no element in Q, the predicate P_n may have at most n elements, and either $|P_{n+1}| = |P_n|$ or P_{n+1} is one element larger. If any P_n has more than n elements, the predicate Q gets an element. When there is an element x in Q, all predicates P_n have equal cardinality, since the relation $E(x, -, -)$ gives a bijection between the predicates.

Thus we can characterize the countable models of $\mathbb{K}$: There are countably many models with nonempty Q: one where each P_n is countably infinite and one where each P_n has size k for $1 \leq k < \omega$. If Q is empty, the model is characterized by a function $f : \omega \to \{0, 1\}$ so that $f(n) = 1$ if and only if $|P_{n+1}| > |P_n|$. Hence there are $2^{\aleph_0}$ countable models.

This example is an AEC with $\mathrm{LS}(\mathbb{K}) = \aleph_0$. The key to establish closure under unions of chains is to note that if the union of a chain has a nonempty Q, some model in the chain must already have one. This example clearly has finite character, joint embedding and arbitrarily large models. Furthermore, the class is categorical in all uncountable cardinals.

We prove that the class has amalgamation. For this, let M, M' and M'' be in $\mathbb{K}$ such that M' and M'' extend M. We need to amalgamate M' and M'' over M. The case where $Q(M)$ is nonempty is easier and we leave it as an exercise. Hence we assume that $Q(M)$ is empty. By taking isomorphic copies if necessary we may assume that the intersection $P_n(M'') \cap P_m(M')$ is $P_n(M)$ for $n = m$ and empty otherwise. Furthermore, we extend both M' and M'' if necessary so that $Q(M')$ and $Q(M'')$ become nonempty and each $P_n(M')$ and $P_n(M'')$ become infinite. We amalgamate as follows: For two elements $x \in P_n(M')$ and $y \in P_n(M'')$, if there is $k < \omega$ such that $f^k(x) = f^k(y)$ in $P_{n+k}(M)$, then we identify x and y. Otherwise, we take a disjoint union.

We prove that the class is simple. For this, define the following notion of independence for A, B, C subsets of the monster model:

$$A \downarrow_C B \Leftrightarrow \begin{array}{l} \text{For any } a \in A, b \in B \text{ if we have that } E(x,a,b), \\ \text{then there is } c \in C \text{ with } E(x,a,c). \end{array}$$

This notion satisfies invariance, monotonicity, finite character, local character, extension, transitivity, symmetry and uniqueness of free extensions. furthermore, $\bar{a} \not\downarrow_C B$ if and only if for some $D \supseteq B$ and every $\bar{b} \models \mathrm{tp}^{\mathrm{w}}(\bar{a}/C \cup B)$, the type $\mathrm{tp}^{\mathrm{w}}(\bar{b}/D \cup C)$ (Lascar-)splits over C. Hence the notion is the same as the independence notion defined for general finitary AECs. This ends the proof.

We can divide this AEC into two disjoint subclasses, both of which are AECs with the same Löwenheim-Skolem number. The class of models where there is no element in Q has uncountably many countable models and is otherwise 'badly-behaved'; all models are countable and the amalgamation property fails. However, the class of models where Q is nonempty, is an uncountably categorical finitary AEC with only countably many countable models. This resembles the example of the sentence in $L_{\omega_1\omega}$, mentioned in the beginning of this section, which was a disjunction of two sentences, a totally categorical one and one with uncountably many countable models and no uncountable ones. Is this 'incompleteness' the reason for categoricity not implying countably many countable models? Can we obtain the conjecture if we require the AEC to be somehow 'complete'? These concepts and questions are explored in the next section.

Jonathan Kirby recently suggested another example with similar properties. This example might feel more natural to some readers, since it consists of 'familiar' structures.

Example 4.1.2 Let $\mathbb{K}$ be the class of all fields of characteristic 0 which are either algebraically closed or (isomorphic to) subfields of the complex algebraic numbers $\mathbb{Q}^{alg}$. Let $\preccurlyeq_{\mathbb{K}}$ be the substructure relation. Then $\mathbb{K}$ is categorical in all uncountable cardinalities and has $2^{\aleph_0}$ countable models which all embed in the uncountable models. Also $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple finitary AEC. Further, this class can be divided into smaller AEC's. For example, we can take all algebraically closed fields of characteristic 0, *except* those isomorphic to subfields of $\mathbb{Q}^{alg}$ as one class and all fields isomorphic to a subfield of $\mathbb{Q}^{alg}$ as the other.

4.2 Complete, Irreducible and minimal AECs

We define several concepts to describe the 'completeness' or 'incompleteness' of an abstract elementary class. A nonempty collection $\mathbb{C}$ of structures of an AEC $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a *sub-AEC* of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, if

- $\mathbb{C}$ is an abstract elementary class with $\preccurlyeq_{\mathbb{C}} = \preccurlyeq_{\mathbb{K}} \cap \mathbb{C}^2$
- $\mathrm{LS}(\mathbb{K}) = \mathrm{LS}(\mathbb{C})$, that is, the Löwenheim-Skolem numbers are the same.

This allows both 'extreme cases' that $\mathbb{C}$ is $\mathbb{K}$ or that $\mathbb{C}$ consists of only one structure, up to isomorphism. The latter can happen if the only structure in $\mathbb{C}$ is of size $\mathrm{LS}(\mathbb{K})$ and is not isomorphic to a proper $\preccurlyeq_{\mathbb{K}}$-substructure of itself.

Definition 4.2.1 (Minimal AEC) *We say that an AEC is* minimal*, if it does not contain a proper sub-AEC.*

Definition 4.2.2 (Irreducible AEC) *We say that an AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* irreducible *if there are no two proper sub-AECs* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *of* $\mathbb{K}$ *such that* $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K}$*.*

Definition 4.2.3 (Complete AEC) *We say that an AEC* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* complete *if there are no two sub-AECs* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *of* $\mathbb{K}$ *such that* $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K}$ *and* $\mathbb{C}_1 \cap \mathbb{C}_2 = \emptyset$*.*

Example 4.1.1 is not complete, not irreducible and not minimal. The sub-AEC of Example 4.1.1, which contains the models where Q is nonempty, is also not complete: One abstract elementary class can be formed by taking all such models where each P_n is of equal size $\leq M$ for some finite M, and the rest of the models of the class form another AEC.

We make a few remarks that follow from the definitions.

Remark 4.2.4

1. *Minimality implies Irreducibility, which implies Completeness.*
2. *Minimality implies the joint embedding property for models of size* $\mathrm{LS}(\mathbb{K})$*.*
3. *Completeness and the amalgamation property imply joint embedding.*
4. *If* T *is a complete first order theory, then the elementary class of models of* T *is not necessarily complete in the sense above.*

Item 1 is obvious. Item 2 holds, since if there are a pair M_0, M_1 of models in $\mathbb{K}$ with size $\mathrm{LS}(\mathbb{K})$, which do not have a common extension, those structures of $\mathbb{K}$ which $\mathbb{K}$-embed M_0 form a proper sub-AEC. For item 3, note that if the class has the amalgamation property, the following classes are disjoint sub-AECs: $\{M \in \mathbb{K} : M$ can be jointly embedded with $M_0\}$ and $\{M \in \mathbb{K} : M$ cannot be jointly embedded with $M_0\}$. Furthermore, the amalgamation property gives that joint embedding for models of size $\mathrm{LS}(\mathbb{K})$ implies joint embedding for all models. Note that an $\aleph_1$ but not $\aleph_0$-categorical countable first order theory is not complete as an AEC.

Example 4.1.1 has joint embedding and amalgamation but is not complete or minimal, hence the implications of items 2 and 3 are not reversible. Is one or both of the implications of item 1 of Remark 4.2.4 reversible? If $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an $\aleph_0$-stable elementary class which is not $\aleph_0$-categorical, the class of $\aleph_0$-saturated models of T is a proper sub-AEC, so the class is not minimal. Example 4.3.7 below gives a class which is complete but not irreducible, minimal or $\aleph_0$-categorical. However, this example is not finitary: it does not have finite character or even arbitrarily large models.

To discuss the relationship between minimality and $\mathrm{LS}(\mathbb{K})$-categoricity, it is important to specify the meaning of $\mathrm{LS}(\mathbb{K})$-categoricity. We define an AEC to be $\mathrm{LS}(\mathbb{K})$-categorical, if it has only one model up to isomorphism, of size *at most* $\mathrm{LS}(\mathbb{K})$. We have forbidden smaller models because models of an AEC which are strictly smaller than the number $\mathrm{LS}(\mathbb{K})$ can cause quite irrational and one could say insignificant changes to the class. We could add, say, one finite model which is not embeddable in any member of the class; this would give non-minimality, since the one model constitutes an AEC. However, an AEC with one model of size $\mathrm{LS}(\mathbb{K})$ and no smaller models, is automatically minimal: For any sub-AEC $\mathbb{K}'$ we can show by induction on the size of the models in $\mathbb{K}$, using the union and Löwenheim-Skolem axioms, that all models of $\mathbb{K}$ are actually contained in $\mathbb{K}'$.

Here are some further questions:

Question 4.2.5

1. *If an AEC is uncountably categorical and* complete*, can it have uncountably many countable models?*
2. *Is there a minimal AEC which is not* $\mathrm{LS}(\mathbb{K})$*-categorical?*
3. *Is there an irreducible AEC which is not minimal?*

4.3 An example of models of Peano Arithmetic - Completeness does not imply Irreducibility

In this section we present an example of a class of models of Peano Arithmetic suggested by Roman Kossak. The example shows that completeness does not imply irreducibility. The properties of the class are from Chapters 1.10 and 10 of the book *The Structure of Models of Peano Arithmetic* [KS06].

A model M of Peano Arithmetic (PA) is *recursively saturated* if for all finite tuples $\bar{b} \in M$ and recursive types $p(v, \bar{w})$, if $p(v, \bar{b})$ is finitely realizable then $p(v, \bar{b})$ is realized in M. Clearly an elementary union of recursively saturated models is recursively saturated. For M, a nonstandard model of PA, define $SSy(M)$, the *standard system* of M, as follows:

$$SSy(M) = \{X \subseteq \mathbb{N} : \exists Y \text{ definable in } M \text{ such that } X = Y \cap \mathbb{N}\}$$

Lemma 4.3.1 *(Proposition 1.8.1 of [KS06]) Let N, M be two recursively saturated models of Peano Arithmetic. Then $M \equiv_{\infty\omega} N$ if and only if $M \equiv N$ and $SSy(M) = SSy(N)$.*

It follows that any countable recursively saturated elementary end-extension of a recursively saturated M is isomorphic to M.

We say $N \models PA$ is ω_1*-like*, if it has cardinality $\aleph_1$ and every proper initial segment of N is countable. We say that $N \models PA$ is an *elementary cut* in M if M is an elementary end-extension of N.

Theorem 4.3.2 *(Corollary 10.3.3 of [KS06]) Every countable recursively saturated model $M \models PA$ has $2^{\aleph_1}$ pairwise non-isomorphic recursively saturated ω_1-like elementary end-extensions.*

The following abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, has one countable model, $2^{\aleph_1}$ models of size $\aleph_1$ and no bigger models. We will use it to generate the counterexample.

Example 4.3.3 *Let M be a countable recursively saturated model of Peano Arithmetic. Let $\mathbb{K}$ be the smallest class, closed under isomorphism, containing M and all ω_1-like recursively saturated elementary end-extensions of M. Let $\preccurlyeq_{\mathbb{K}}$ be elementary end-extension.*

Lemma 4.3.4 *The AEC 4.3.3 does not have finite character.*

Proof: Let M be a recursively saturated countable model of PA. Let M' be a recursively saturated elementary substructure of M (not necessarily a cut) and let $\bar{a}$ be a finite tuple in M'. We construct a $\preccurlyeq_{\mathbb{K}}$-map $f : M' \to M$ fixing $\bar{a}$. When M' is not a cut we contradict finite character. For this, we will find an elementary cut M'' of M and an isomorphism $f : M' \to M''$ such that $f(\bar{a}) = \bar{a}$. Since M and M' are recursively saturated, both $(M, \bar{a})$ and $(M', \bar{a})$ are recursively saturated. Furthermore, $(M, \bar{a})$ is elementarily equivalent to $(M', \bar{a})$. Now let M'' be an elementary cut in M such that $(M, \bar{a})$ is an elementary end-extension of $(M'', \bar{a})$ and $(M'', \bar{a})$ is recursively saturated. Then $(M', \bar{a}) \cong (M'', \bar{a})$. □

From now on, let M be a fixed countable recursively saturated model of PA.

Now we construct a complete but not irreducible AEC. Let $\prec_{end}$ denote elementary end-extension. We define

$$M(a) = \bigcap \{K \prec_{end} M : a \in K\},$$

$$M[a] = \bigcup \{K \prec_{end} M : a \notin K\},$$

where $M[a]$ can be empty. Then let $gap(a)$ denote $M(a) \setminus M[a]$.

It is easy to see that an equivalent definition is the following: Let $\mathscr{F}$ be the set of definable functions $f : M \to M$ for which $x < y$ implies $x \leq f(x) \leq f(y)$. Let a be an element in M. The *gap*(a) in M is the smallest subset C of M containing a such that whenever $b \in C$, $f \in \mathscr{F}$ and $b \leq x \leq f(b)$ or $x \leq b \leq f(x)$, then $x \in C$.

We say that $N \models PA$ is *short* if it is of the form $N(a)$ for some $a \in N$. Equivalently, N has a *last gap.* A short model $N(a)$ is not recursively saturated, since it omits the type

$$p(v, a) = \{t(a) < v : t \text{ a Skolem term}\}.$$

If N is not short, it is called *tall.* The following three properties are exercises in [KS06].

1. The union of any ω-chain of end-extensions of short elementary cuts in M is tall.
2. Any tall elementary cut in M is recursively saturated and hence isomorphic to M.
3. If K is an elementary cut in M and is NOT recursively saturated, then $K = M(a)$ for some $a \in M$.

It follows also that the union of any ω-chain of elementary end-extensions of models isomorphic to short elementary cuts in M is isomorphic to M. For the following theorem, see [Smo81].

Theorem 4.3.5 *Two short elementary cuts $M(a)$ and $M(b)$ are not isomorphic if and only if the sets of recursive types realised in gap(a) and gap(b) respectively are* disjoint. *There are countably many pairwise non-isomorphic short elementary cuts in M.*

Lemma 4.3.6 *If $a \notin M_0$, the model $M(a)$ is isomorphic to some proper initial segment $M(a')$ of $M(a)$, which is an elementary cut of $M(a)$.*

Proof: Define the recursive type

$$p(x, a) = \{\phi(x) \leftrightarrow \phi(a) : \phi(x) \in L\} \cup \{t(x) < a : t \text{ is a Skolem term}\}.$$

Any finite subset of $\text{tp}(a/\emptyset)$ is realized in M_0 since $M_0 \prec M$. Thence $p(x, a)$ is consistent as M_0 is closed under the Skolem terms. Let $a' \in M$ realize $p(x, a)$. Then $\text{tp}(a') = \text{tp}(a)$ and $M(a') < a$. Hence $M(a)$ is isomorphic to $M(a')$ by Theorem 4.3.5. Furthermore, $M(a')$ is an elementary cut in $M(a)$. □

Lemma 4.3.6 implies elementary $\prec_{end}$-chains can be formed from isomorphic copies of one $M(a)$, when $a \notin M_0$. Hence, each of the following classes $\mathbb{K}_\alpha$ is an abstract elementary class extending the $\aleph_0$-categorical class $\mathbb{K}$ from the Example 4.3.3 and $\mathbb{K}_\alpha$ has α many countable models, where $\alpha \in \omega \cup \{\omega\}$.

Example 4.3.7 *Let α be a finite number or ω. Choose $(M(a_i))_{i<\alpha}$ to be pairwise non-isomorphic short elementary cuts in M, where each a_i is non-standard. Let $\mathbb{K}_\alpha$ be the smallest class, closed under isomorphism, containing $\mathbb{K}$ and $M(a_i)$ for all $1 \leq i < \alpha$. Let $\preccurlyeq_{\mathbb{K}}$ be elementary end-extension.*

The countable models of $\mathbb{K}_\alpha$ are exactly M and $M(a_i)$ for $1 \leq i < \alpha$. This class is closed under $\preccurlyeq_{\mathbb{K}}$-unions: if $\langle M_j,\ j < \beta \rangle$, is a $\preccurlyeq_{\mathbb{K}}$-chain of models in $\mathbb{K}_\alpha$, we have that for every countable limit ordinal β, $\bigcup_{j<\beta} M_\beta$ is tall and hence isomorphic to M, and if β is uncountable, the union is isomorphic to some ω_1-like recursively saturated model in $\mathbb{K}$. (Note that the union is also an end-extension of M.)

Any abstract elementary class containing a short elementary cut $M(a)$ for some $a \in M$ must contain M, as M is a union of models isomorphic to $M(a)$ elementarily end-extending each other. Hence any abstract elementary class containing $M(a)$ contains M.

It follows that $\mathbb{K}_\alpha$ is complete since it has no disjoint sub-AECs. Furthermore, the class $\mathbb{K}_\alpha$ is not irreducible for $\alpha > 2$, since we can divide it into two classes, one containing $M(a_i)$ but not $M(a_j)$ and one vice versa, for any $i \neq j < \alpha$.

However, Example 4.3.7 is neither a Jónsson class (all models have cardinality below the continuum) nor a finitary AEC. We ask:

Question 4.3.8 *Is there a Jónsson class which is complete but not irreducible or minimal? Furthermore, is there such a finitary AEC?*

References

[Adl] Hans Adler, *An introduction to theories without the independence property*, To appear in Archive for Mathematical Logic.

[Bal88] John T. Baldwin, *Fundamentals of stability theory*, Perspectives in Mathematical Logic, Springer-Verlag, 1988.

[Bal06] ______, *The complex numbers and complex exponentiation: Why infinitary logic is necessary!*, Available at http://www.math.uic.edu/ jbaldwin/pub/Bog05art.pdf, 2006.

[Bal07] John T. Baldwin, *The Vaught conjecture: do uncountable models count?*, Notre Dame Journal of Formal Logic **48** (2007), no. 1, 79–92 (electronic). MR MR2289898

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, AMS, 2009.

[Bay] Martin Bays, *Model theory of exponential maps of abelian varieties*, Talk at the meeting Geometric Model Theory, 25th-28th March 2010, Oxford, Slides available at `http://people.maths.ox.ac.uk/~bays/misc/borisfest/notes/`.

[BET07] John T. Baldwin, Paul C. Eklof, and Jan Trlifaj, $^{\perp}N$ *as an abstract elementary class*, Annals of Pure and Applied Logic **149** (2007), no. 1-3, 25–39. MR MR2364195 (2008i:03039)

[BF85] J. Barwise and S. Feferman, *Model-theoretic logics*, Springer-Verlag,New York, 1985.

[BK09] John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*, Israel Journal of Mathematics **170** (2009), 411–443. MR MR2506333

[BL71] J. T. Baldwin and A. H. Lachlan, *On strongly minimal sets*, The Journal of Symbolic Logic **36** (1971), 79–96. MR MR0286642 (44 #3851)

[BL03] Steven Buechler and Olivier Lessmann, *Simple homogeneous models*, Journal of the American Mathematical Society **16** (2003), no. 1, 91–121 (electronic). MR MR1937201 (2003i:03036)

[Bou99] E. Bouscaren (ed.), *Model theory and algebraic geometry : An introduction to E. Hrushovski's proof of the geometric mordell-lang conjecture*, Springer-Verlag, 1999.

[BR10] T. Beke and J. Rosick, *Abstract elementary classes and accessible categories*, Available at http://www.math.muni.cz/ rosicky/papers/elem3.pdf, 2010.

[Bue91] Steven Buechler, *Essential stability theory*, Springer-Verlag, 1991.

[Bue08] Steven Buechler, *Vaught's conjecture for superstable theories of finite rank*, Annals of Pure and Applied Logic **155** (2008), no. 3, 135–172. MR 2467429 (2009h:03041)

[BYACU08] I. Ben-Yaacov, A.J.Berenstein, C.W.Henson, and A. Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis Vol. 2 (Zoe Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), London Math Soc. Lecture Note Series Nr 350, Cambridge Univ Press, Almaty, 2008.

[BZ00] M. Bays and B.I. Zilber, *Covers of multiplicative groups of an algebraically closed field of arbitrary characteristic*, preprint: arXive math.AC/0401301, 200?

[C.C68] C.C.Chang, *Some remarks on the model theory of infinitary languages*, The syntax and semantics of infinitary languages (J. Barwise, ed.), Springer-Verlag, 1968, pp. 36–64.

[CK73] C.C. Chang and H.J Keisler, *Model theory*, North-Holland, 1973, 3rd edition 1990.

[EF72] Paul C. Eklof and Edward R. Fischer, *The elementary theory of abelian groups*, Annals of Pure and Applied Logic **4** (1972), 115–171. MR 0540003 (58 #27457)

[Gav06] Misha Gavrilovich, *Model theory of universal covering spaces of complex analytic varieties*, Ph.D. thesis, Balliol college Oxford, 2006, URL: `://people.maths.ox.ac.uk/bays/misha-thesis.pdf`.

[GH89] Rami Grossberg and Bradd Hart, *The classification of excellent classes*, The Journal of Symbolic Logic **54** (1989), no. 4, 1359–1381.

[GL02] Rami Grossberg and Olivier Lessmann, *Shelah's stability spectrum and homogeneity spectrum in finite diagrams*, Archive of Mathematical Logic **41** (2002), no. 1, 1–31.

[GV06] Rami Grossberg and Monica VanDieren, *Galois-stability in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[HH09] Åsa Hirvonen and Tapani Hyttinen, *Categoricity in homogeneous complete metric spaces*, Arch. Math. Logic **48** (2009), no. 3-4, 269–322. MR 2500987 (2010h:03044)

[HHM08] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson, *Stable domination and independence in algebraically closed valued fields*, Lecture notes in Logic, Cambridge University Press, 2008.

[HK] Tapani Hyttinen and Meeri Kesälä, *Categoricity transfer in simple finitary abstract elementary classes.*, To appear in the Journal of Symbolic Logic. Available at `http://mathstat.helsinki.fi/logic/people/meeri.kesala.html`.

[HK06] T. Hyttinen and M. Kesälä, *Independence in finitary abstract elementary classes*, Annals of Pure and Applied Logic **143** (2006), no. 1-3, 103–138. MR MR2258625

[HK07] Tapani Hyttinen and Meeri Kesälä, *Superstability in simple finitary AEC.*, Fundamenta Mathematicae **195** (2007), no. 3, 221–268.

[HK10] ———, *Interpreting Groups and Fields in Simple, Finitary AECs* , Institut Mittag-Leffler preprint series 2009/2010. Available at `http://www.mittag-leffler.se/preprints/`, 2010.

[HK11] ______, *Lascar types and Lascar automorphisms in Abstract Elementary Classes*, To appear in Notre Dame Journal of Formal Logic, 2011.

[HL06] Tapani Hyttinen and Olivier Lessmann, *Simplicity and uncountable categoricity in excellent classes*, Annals of Pure and Applied Logic **139** (2006), no. 1-3, 110–137.

[HLS05] Tapani Hyttinen, Olivier Lessmann, and Saharon Shelah, *Interpreting groups and fields in some nonelementary classes*, Journal of Mathematical Logic **5** (2005), no. 1, 1–47, Shelah [HLSh:821].

[Hru89] Ehud Hrushovski, *almost orthogonal regular types*, Annals of Pure and Applied Logic **45** (1989), no. 2, 139–155.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over P for first order T or categoricity for $\phi \in \mathrm{L}_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$*, Israel J Math **70** (1990), 219–235, Shelah [HaSh:323].

[HS00] Tapani Hyttinen and Saharon Shelah, *Strong splitting in stable homogeneous models*, Annals of Pure and Applied Logic **103** (2000), 201–228, Shelah [HySh:629].

[HS01] T. Hyttinen and S. Shelah, *Main gap for locally saturated elementary submodels of a homogeneous structure*, J Symbolic Logic **66** (2001), 1286–1302.

[J́60] B. Jónsson, *Homogeneous universal relational systems*, Mathematica Scandinavica **8** (1960), 137–142.

[Kei70] H. Jerome Keisler, *Logic with the quantifier "there exists uncountably many"*, Annals of Mathematical Logic **1** (1970), no. 1, 1–93.

[Kei71] ______, *Model theory for infinitary logic*, Studies in logic and the foundations of mathematics, North-Holland Publishing Company, 1971.

[Ken] Juliette Kennedy, *Gödel and formalism freeness*, preprint.

[Kes06] Meeri Kesälä, *Finitary Abstract Elementary Classes*, Ph.D. thesis, University of Helsinki, Department of Mathematics and Statistics, 2006.

[Kie80] Henry A. Kierstead, *Countable models of ω_1-categorical theories in admissible languages*, Annals of Mathematical Logic **19** (1980), no. 1-2, 127–175. MR 593031 (82c:03054)

[Kir08] Jonathan Kirby, *Abstract elementary categories*, Available at http://people.maths.ox.ac.uk/kirby/pdf/aecats.pdf, 2008.

[Kir10] Jonathan Kirby, *On quasiminimal excellent classes*, J. Symbolic Logic **75** (2010), no. 2, 551–564. MR 2648155

[KS06] Roman Kossak and James Schmerl, *The structure of models of peano arithmetic*, Oxford Logic Guides, no. 50, Oxford University Press, 2006.

[Kue08a] David W. Kueker, *Abstract elementary classes and infinitary logics*, Annals of Pure and Applied Logic **156** (2008), no. 2-3, 274–286. MR MR2484485

[Kue08b] ______, *Abstract elementary classes and infinitary logics*, Annals of Pure and Applied Logic **156** (2008), no. 2-3, 274–286. MR MR2484485

[Les05a] Olivier Lessmann, *An introduction to excellent classes*, Logic and its applications (Andreas Blass and Yi Zhang, eds.), Contemporary Mathematics, vol. 380, AMS, 2005, pp. 231–259.

[Les05b] ______, *Upward categoricity from a successor cardinal for tame abstract classes with amalgamation*, The Journal of Symbolic Logic **70** (2005), no. 2, 639–660.

[Lie09] Michael Lieberman, *Topological and category-theoretic aspects of abstract elementary classes*, Ph.D. thesis, University of Michigan, 2009.

[Lin69] Per Lindström, *On extensions of elementary logic*, Theoria **35** (1969), 1–11. MR MR0244013 (39 #5330)

[Mar00] David Marker, *Model theory of differential fields*, Model theory, algebra, and geometry, Math. Sci. Res. Inst. Publ., vol. 39, Cambridge Univ. Press, Cambridge, 2000, pp. 53–63. MR 1773702 (2001j:12005)

[Mar07] ______, *The number of countable differentially closed fields*, Notre Dame Journal of Formal Logic **48** (2007), no. 1, 99–113 (electronic). MR 2289900 (2007m:03076)

[MMP96] D. Marker, M. Messmer, and A. Pillay, *Model theory of fields*, Springer-Verlag, 1996.

[Mor65] Michael D Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[Mor67] M. Morley, *Countable models of* $\aleph_1$ *-categorical theories*, Israel J. Math **5** (1967), 65–72.

[Mor70] ______, *The number of countable models*, The Journal of Symbolic Logic **35** (1970), 14–18.

[Mos57] A. Mostowski, *On a generalization of quantifiers*, Polska Akademia Nauk. Fundamenta Mathematicae **44** (1957), 12–36. MR MR0089816 (19,724f)

[MS89] Alan H. Mekler and Saharon Shelah, $L_{\infty\omega}$*-free algebras*, Algebra Universalis **26** (1989), 351–366.

[Pil96] Anand Pillay, *Geometric stability theory*, Oxford University Press, 1996.

[She70] Saharon Shelah, *Finite diagrams stable in power*, Annals Math Logic **2** (1970), 293–300, Shelah [Sh:3].

[She74] Saharon Shelah, *Categoricity of uncountable theories*, Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. of California, Berkeley, Calif., 1971) (Providence, R.I.), Amer. Math. Soc., 1974, pp. 187–203. MR MR0373874 (51 #10074)

[She75] ______, *The lazy model-theoretician's guide to stability*, Logique et Analyse. Nouvelle Série **18** (1975), no. 71-72, 241–308, Comptes Rendus de la Semaine d'Étude en Théorie des Modèles (Inst. Math., Univ. Catholique Louvain, Louvain-la-Neuve, 1975). MR 0539969 (58 #27447)

[She78] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, North-Holland, 1978, Second Revised Edition 1990.

[She83] ______, *Classification theory for for nonelementary classes, I. The number of uncountable models of* $\psi \in L_{\omega_1,\omega}$*. Parts A and B*, Israel J Math **46** (1983), 212–273, Shelah [Sh:87a] and [Sh:87b].

[She87] ______, *Classification of Non Elementary classes II, Abstract elementary classes*, Classification theory, Proceedings, Chicago 1985 (John T. Baldwin, ed.), Lecture notes in Mathematics, vol. 1292, Springer-Verlag, Berlin, 1987, Shelah [Sh:88], pp. 419–497.

[She99] ______, *Categoricity of abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), 261–294, Shelah [Sh:394].

[She09] ______, *Classification theory for abstract elementary classes*, Studies in Logic, Mathematical Logic and Foundations, vol. 18, College Publications, 2009.

[SHM84] S. Shelah, L. Harrington, and M. Makkai, *A proof of Vaught's conjecture for ω-stable theories*, Israel Journal of Mathematics **49** (1984), no. 1-3, 259–280. MR 788270 (86j:03029b)

[Smo81] C. Smoryński, *Elementary extensions of recursively saturated models of arithmetic*, Notre Dame Journal of Formal Logic **22** (1981), no. 3, 193–203. MR MR614117 (82g:03064)

[Trl09] Jan Trlifaj, *Abstract elementary classes induced by tilting and cotilting modules have finite character*, Proceedings of the American Mathematical Society **137** (2009), no. 3, 1127–1133. MR MR2457454 (2010c:03025)

[Vau61] R. L. Vaught, *Denumerable models of complete theories*, Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959), Pergamon, Oxford, 1961, pp. 303–321. MR 0186552 (32 #4011)

[Wag00] Frank O. Wagner, *Simple theories*, Springer, 2000.

[Yaa09] Itaï Ben Yaacov, *Model theoretic properties of metric valued fields*, Available in Math Archive arXiv:0907.4560, 2009.

[Zil84] B.I. Zil'ber, *Strongly minimal countably categorical theories III*, Siberian Mathematics Journal **25** (1984), 559–571.

[Zil93] Boris Zilber, *Uncountably categorical theories*, Translations of Mathematical Monographs, vol. 117, American Mathematical Society, Providence, RI, 1993, Translated from the Russian by D. Louvish. MR MR1206477 (94h:03059)

[Zil04] B.I. Zilber, *Pseudo-exponentiation on algebraically closed fields of characteristic 0*, Annals of Pure and Applied Logic **132** (2004), 67–95.

[Zil05] Boris Zilber, *A categoricity theorem for quasi-minimal excellent classes*, Logic and its applications, Contemp. Math., vol. 380, Amer. Math. Soc., Providence, RI, 2005, pp. 297–306. MR MR2167585 (2006m:03061)

[Zil06] ______, *Covers of the multiplicative group of an algebraically closed field of characteristic zero*, Journal of the London Mathematical Society. Second Series **74** (2006), no. 1, 41–58. MR MR2254551