Entanglement of Set Theory and Model Theory
Eventual Behavior and Noise

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Methodological Approaches in the Study of Recent Mathematics
Mathematical Philosophy and Mathematical Practice
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Outline

The Paradigm Shift in Model Theory

What is the role of Logic?

Logic is the analysis of methods of reasoning

versus

Logic is a tool for doing mathematics.

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Logic is a tool for doing mathematics.

More precisely,
Mathematical logic is tool to solve not only its own problems but to organize and do traditional mathematics.

Formal Definability as the unifying thread.

Two theses from: Philosophy of Mathematical Practice: Formalization without Foundationalism

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

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2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Formalization

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   - a Specify a class of well formed formulas.
   - b Specify truth of a formula from this class in a structure.
   - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

The Paradigm Shift

The paradigm around 1950

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After the paradigm shift (1945-75)

There is a systematic search for a small set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

After the shift one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

I. By their fruit ye shall know them!

Pragmatic Approaches

We attempt to synthesize positions of Whitehead, Wilson, Wagner, and Maddy to provide a pragmatic approach to theory building.

Two types of fruit

1. applications/predictions (external)
2. coherence (internal)

Whitehead

Whitehead 

Science is the organization of thought. . . . It is an organization of a certain definite type which we will endeavor to determine.
. . .
Science is a river with two sources, the practical source and the theoretical source.
The practical source is the desire to direct our actions to achieve predetermined ends.

. . . The theoretical source is the desire to understand. . . .

Whitehead continued

I do not consider one source as in any sense nobler than the other, or intrinsically more interesting. . . .
The importance, even in practice, of the theoretical side of science arises from the fact that action must be immediate and takes place under circumstances which are excessively complicated. . . .

Success in practice depends on theorists who, led by other motives of exploration have been there before, and by some good chance have hit upon the relevant ideas.

page 19-20 A. N. Whitehead, The Interpretation of Science

Wilson's change of scale Wilson 

Mark Wilson (Wandering Significance, Physics Avoidance) expounds the difficulties of examining the same physical process at different scales and the difficulties of making transitions. E.g. the structure of ice.
Analogously we regard the study of mathematics as a family of interacting studies rather than attempting a single foundation of everything.

Thus, while the tools of ZFC are the best known for establishing a basis for clarifying connections between e.g. algebraic geometry and analytic number theory,
they are not appropriate for actually carrying out investigations in either area.

Roi Wagner: constraint based POM

Instead of asking foundational questions about the grounding of mathematics, its freedom, its unique position, its monsters, or its source of authority, we will ask some questions about mathematical practice. How are mathematical statements used? How do people get to agree on them? How are they interpreted?
The resulting observations will be integrated into a constraints-based philosophy of mathematics: instead of debating the reality of mathematical entities, we will think of mathematics as a field of knowledge that negotiates various kinds of real constraints. This contingent array of constraints and the various ways of juggling them lead to the formulation of various mathematical cultures.

page 58 of Making and Breaking Mathematical Sense

Pen Maddy: What do we want a foundation to do?

Maddy 

So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides Risk Assessment for mathematical theories, a Generous Arena where the branches of mathematics can be pursued in a unified setting with a Shared Standard of Proof, and a Meta-mathematical Corral so that formal techniques can be applied to all of mathematics at once.

'What do we want a foundation to do? Comparing set-theoretic, category-theoretic, and univalent approaches'

Four aspects of set theory

1. pre-Gödel set theory
2. Modern set theory
    1. ZFC interactions with traditional mathematics (generous arena)
    2. ZFC^+ interactions with traditional mathematics (meta-mathematical corral)
    3. Internal considerations
        1. combinatorial set theory, point set topology and cardinal arithmetic
        2. Seeking V or V's.

pre-Gödel set theory

Risk Assessment

ZFC successfully assesses/isolates the problems/paradoxes of the late 19th century.
Cantor, Dedekind, Noether, Artin, Landau, Zermelo

Shared Standard of Proof

Provides a common foundation for analysis, algebraic topology, and modern algebra.
Bourbaki.

Internal Set Theory

Are set theorist's model theorists in disguise?

Consider this introduction from Roitman's set theory text,
'These concepts (of first-order theory and of model) have had profound effects on set theory. . . . One can make a case for the statement that modern set theory is largely the study of models of set theory.'
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Woodin's creed

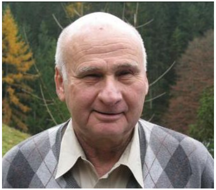
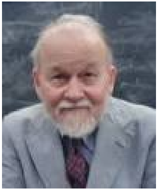
The correct axioms for set theory will be apparent: they will provide a coherent beautiful description that accounts for all forcing phenomena.

Internal Set Theory

1. cardinal arithmetic, pcf theory
2. solving the problems of 1920's topology with forcing
3. combinatorial set theory
4. inner model program

II. Why model theory AND set theory

Entanglement


Such authors as Kennedy, Magidor, Parsons, and Väänänen have spoken of the entanglement of logic and set theory.

It depends on the logic

There is a deep entanglement between (first-order) model theory and cardinality.

There is No such entanglement between (first-order) model theory and cardinal arithmetic.
At least for stable theories; more entanglement in neo-stability theory.

There is however such an entanglement between infinitary model theory and cardinal arithmetic and therefore with extensions of ZFC.

Shelah: Set theory and first order model theory

Shelah  Jensen 

During the 1960s, two cardinal theorems were popular among model theorists. . . . Later the subject becomes less popular; Jensen complained when I start to deal with gap n 2-cardinal theorems, they were the epitome of model theory and as I finished, it stopped to be of interest to model theorists.

Two Questions

I. Why in 1970 did there seem to be strong links of even first order model theory with cardinal arithmetic and axiomatic set theory?

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II. Why by the mid-70's had those apparent links evaporated for first order logic?

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

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Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$. And we write $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ if *every theory* that admits (κ, λ) also admits (κ', λ').

Set Theory Intrudes Morley 

Theorem: Vaught

$(\beth_\omega(\lambda), \lambda) \to (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Theorem: Morley's Method

Suppose the predicate is defined not by a single formula but by a type: $(\beth_{\omega_1}(\lambda), \lambda) \to (\mu_1, \mu_2)$ when $\mu_1 \geq \mu_2$.

Both of these results need replacement; the second depends of iterative use of Erdös-Rado to obtain countable sets of indiscernibles.

In the other direction, the notion of indiscernibles is imported into Set Theory by Silver to define $O^{\#}$.

Set Theory Becomes Central

Vaught asked a ‘big question’, ‘For what quadruples of cardinals does $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ hold?’

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Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

Why did it stop? Lachlan  Bays 

Revised Theorem: solved in ZFC

Suppose

1. [Shelah, Lachlan $\approx$ 1972] T is stable
2. or [Bays 1998] T is o-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if T admits (κ, λ) then T also admits (κ', λ').

Ask the right question

Reversing the question

set theorist:
For which cardinals (κ, λ) does T have a (κ, λ) model for all theories T?

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set theorist:

For which cardinals (κ, λ) does T have a (κ, λ) model for all theories T?

model theorist:

For which theories T does T have a (κ, λ) model for all pairs of *cardinals* (κ, λ)?

Really, Why did it stop?

Definition

[The Stability Hierarchy:] Fix a countable complete first order theory T.

1. T is stable in χ if $A \subset M \models T$ and $|A| = \chi$ then $|S(A)| = |A|$.
2. T is
    1. ω-stable[a] if T is stable in all χ;
    2. superstable if T is stable in all $\chi \geq 2^{\aleph_0}$;
       That is, for every A with $A \subset M \models T$, and $|A| \geq 2^{\aleph_0}$, $|S(A)| = |A|$
    3. stable if T is stable in all χ with $\chi^{\aleph_0} = \chi$;
    4. unstable if none of the above happen.

[a] This 'definition' hides a deep theorem of Morley that T is ω-stable if and only if it stable in every infinite cardinal.

III.a: Generous Arena: Set Theory

Not just foundations: uniqueness sets

The study of sets of uniqueness was intensively developed between 1910 and 1930. Luzin's school distinguished within the theory of real functions:

1. metric theory (differentiation, integration, trigonometric series, etc.)
2. descriptive theory (called today descriptive set theory).

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After 50 years as 'logic' not 'analysis' DST made significant contributions to the theory of sets of uniqueness after 1985.

Theorem: (A.Mazurkiewicz, 1936)

In the space $C(\mathbb{T})$, the set of functions that can be expanded in a trigonometric series is co-analytic but not Borel.

The extensive analysis of definability class solved a 1927 problem.

Theorem: (Ajtai-Kechris, 1987)

Every Borel set of extended uniqueness is of the first category.

Not just foundations: Ergodic Theory

Theorem: Dye[59,63], Ornstein-Weiss [80]

Every two free, ergodic, measure-preserving actions of amenable groups are orbit equivalent.

Turbulence is a method for establishing complexity of equivalence relations by Hjorth. Work of (Epstein-Ioana-Kechris-Tsankov) culminate in

Theorem: Epstein-Ioana-Kechris-Tsankov [2008/9]

Every non-amenable group admits continuum many non-orbit equivalent free, ergodic, measure-preserving actions.

Moreover, Orbit equivalence of free, ergodic, measure-preserving actions of any non-amenable group is not classifiable by countable structures.

The crux in both cases is the logical notion of definability – the Borel and analytic hierarchies.

III.b: Generous Arena: Model Theory

How does model theory fit in

Model theory relies on set theory for Risk Assessment and Shared Standard of proof.
But classification theory provides a Generous Arena and a Meta-mathematical Corral for finding patterns across mathematics.

1. Risk Assessment and Shared Standard of proof are ‘above MT pay grade’.
   That is, model theory like other areas of mathematics, leaves to ZFC the justification of it basic tools.
2. But model theory via classification theory provides a different organization of mathematical topics which better preserves the methods and ethos of various areas than set theory does.
3. The meta-mathematical corral of extensions of ZFC is particulary germane to the model theory of infinitary logic.

Model Theory and Analysis

I omit the well-known applications to algebraic geometry and number theory to discuss applications to analysis.

First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis

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First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis
2. Definable analysis
   Functions are defined implicitly:
   real exponentiation, number theory
   2014, 2018 Karp Prize

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Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

$\aleph_1$-categorical theories

Morley  **Lachlan**  **Zilber** 

Theorem

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.

Strongly minimal sets are the building blocks of structures whose first order theories are categorical in uncountable power.

$\aleph_1$-categorical theories

Definition

A model M of a complete theory T is prime over a subset X if every morphism from X into a model N of T extends to a morphism of M into N.

Theorem (Baldwin-Lachlan)

If T is categorical in some uncountable power, there is a definable strongly minimal set D such that every model M of T is prime over $D(M)$.

Thus, the dimension of $D(M)$ determines the isomorphism type of M.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

The Zilber Trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. non-locally modular : conjecturally field-like

Hrushovski's example showed there are non-locally modular which are far from being fields; the examples don't even admit a group structure.

Axiomatic analysis

Axiomatic analysis studies behavior of fields of functions with operators but *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions. The function symbols of the vocabulary act on the functions being studied; the functions are elements of the domain of the model.

Differential Algebra

The axioms for *differentially closed fields* are a first order sentences in the vocabulary $(+, \times, 0, 1, \partial)$ (where ∂f is interpreted as the derivative The first order formulation is particularly appropriate because many of the fields involved are non-Archimedean.

Differentially closed fields

Blum provided the first axiomatizations and she showed the characteristic 0 theory was ω-stable.

By Shelah's uniqueness theorem for prime models over sets for ω-stable theories, differential closures are unique up to isomorphism.

But Poizat, Rosenlicht and Shelah independently showed they are not minimal (by constructing in different ways a strongly minimal set with trivial geometry).

Differentially closed fields II

Hrushovski and Itai lay out as an application of 'Shelah's philosophy' the following model theoretic fact (based on Buechler's Dichotomy) fundamental to the study of differential fields:

'an *algebraically* closed differential field K is *differentially* closed if every strongly minimal formula over K has a solution in K'.

Even more, by the general theory of superstability, their study reduces to the study of strongly minimal sets and definable simple FMR groups that are associated with strongly minimal sets.
Consider the theory of differentially closed fields with constant field the complex numbers $\mathcal{C}$.

Painlevè equations

In 1900 Painlevè began the study of nonlinear second order ordinary differential equations (ODE) satisfying the Painlevè property (no movable singularities). In general such an equation has the form

$$y'' = f(y, y')$$

with f a rational function (i.e. in $\mathbb{C}(t_1, t_2)$).
He classified such equations into 50 canonical forms and showed that 44 of these were solvable in terms of 'previously known' functions.
Here is a canonical form for the third of the remaining classes; the Greek letters are the constant coefficients; t is the independent variable satisfying $t' = 1$ and the goal is to solve for y.

$$P_{III}(\alpha, \beta, \gamma, \delta): \qquad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

Problem 1

Show that a generic equation (i.e. the constant coefficients are algebraically independent) of each of the six forms is irreducible.

For this, one must take on the logicians task: ‘What does *not reducible* mean’?

By reducible Painlevè meant, solvable from ‘known functions’.

The Japanese school clarified ‘solvable’ to mean, roughly speaking: generated from solutions to order one ordinary differential equations (ODE) and algebraic functions through a fixed family of constructions (integration, exponentiation, etc.).

In the formal setting, this is equivalent to showing that:
If an order two differential equation is strongly minimal; then there can be no *classical solutions*.

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This problem was solved (without the formalization) in each of the six cases by the Japanese school (led by Umemura) in the late 1980’s.

Problem II

Conjecture

If there are *n* algebraically independent solutions of a generic strongly minimal Painlevè equation then that set along with its first derivatives is also algebraically independent.

The model theoretic step is to invoke the Zilber trichotomy which holds for differentially closed fields.
To reduce to a disintegrated strongly minimal set. Pillay and Nagloo show the other alternatives are impossible in this situation and indeed that the strongly minimal set is $\aleph_0$-categorical. Using the geometric triviality (from the Zilber trichotomy) heavily and tools from the Japanese analysts

Theorem: Nagloo-Pillay

The conjecture is true.

Problem II - a contrast

Freitag and Scanlon show the order three algebraic differential equation over $\mathbb{Q}$ satisfied by the analytic Weierstrass j-function defines a non-$\aleph_0$-categorical strongly minimal set with trivial forking geometry.

This result requires 'hard analysis' and so the Archimedean basis of Dedekind and Weierstrass.
In contrast the functional transcendence result of Pillay and Nagloo use 'algebra/ model theory'
Not higher order objects.

Contrasting arenas

Wilson's perspective

The main role of set theory in providing a generous arena for analysis is providing a firm Archimidean basis for the limiting process for the real numbers and associated function spaces.

Model theoretic perspective

The main role of set theory in providing a generous arena for 'model theoretic' analysis is providing a firm basis for first order logic.
This allows the study of 'algebraic properties of function spaces' without being concerned with the details of convergence and continuity.
Wilson's role remains in place; but the use of model theory is a tool to systematize abstract analysis.
The number of steps in the 'Well, it can all be formalized' is reduced and made more conceptual.

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2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Perhaps Descriptive set theory is a local formalization?

Combining these viewpoints

The path from experimental science leads through increasing levels of abstraction and generalization.
Successive levels should be judged on

1. coherence of its systematization of the previous levels;
2. its applications to and connections with earlier levels.

As in Wilson's pragmatic analysis, the view passes between stages.
But rather than
between different scales of the empirical data
to different perspectives and levels of abstraction.

IV.a Meta-Mathematical Corral: Set Theory

Absoluteness

Definition

Two Banach spaces are said to be incomparable if neither of them embed into the other, and an infinite-dimensional space is minimal if it embeds into all of its infinite-dimensional subspaces.

Theorem: Rosendal (ZFC)

Let X be an infinite-dimensional Banach space. Then X contains either a minimal subspace or a continuum of pairwise incomparable subspaces.

Relies heavily on metamathematical techniques: determinacy of games defined by constraints on definability (in particular the 'Gowers game'), Shoenfield absoluteness
Rosendal first proves the result under (MA + not CH) and then shows the property of having a minimal subspace is Σ^1_2.

Forcing

One striking example of the use of forcing to answer a question in Banach space theory is Farah's 2011 proof that under the open covering axiom (OCA) every automorphism of the Calkin algebra is inner. This shows a 1977 question is independent in ZFC.

As, Douglas-Weaver (2007) had shown that under CH the Calkin algebra has an outer automorphism.

Note that Farah's result applies substantial development by Todorcevic and Shelah on the open covering axiom and the proper forcing axiom.

IV.b: Meta-mathematical Corral: Eventual Behavior and Noise

Eventual Behavior

A viewpoint

There are certain anomalies that occur at small cardinals but every thing comes out well in the end!

Eventual Behavior

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Set theoretic examples

1. Strongly compact cardinals predict compactness phenomena on all larger cardinals.
2. (Woodin) If there is an inner model, that is
    1. built using extenders in the only way we know,
    2. that contains a supercompact,

   then any large cardinal witnessed by extenders (which is basically all of them) will be in the inner model.

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Model theoretic examples

1. Morley's conjecture/Shelah's (ZFC) theorem: The spectrum (model counting) function of a first order theory can decrease only on its values from $\aleph_0$ to $\aleph_1$. This is a ZFC result.
   In particular, the existence of theories that are categorical in $\aleph_1$ but not $\aleph_0$-categorical illustrates the exceptional character (noise) of $\aleph_0$,
2. Shelah's eventual categoricity conjecture: Every AEC is either eventually categorical or eventually has at least 2 models in every cardinality.
   Proved recently assuming a strongly compact by Shelah-Vasey.

Eventual Behavior as Tameness

While set theory is the protypically wild area of mathematics, set theorists, in fact, try to tame the wilderness.
And such eventual behavior assertions are a kind of tameness.

Other avatars of eventual behavior

1. The probabilistic method of Erdös
   more generally 0-1 laws
2. Theorems that distinguish the finite from the infinite.
   All finite division rings are commutative.
3. Combinatorial results that are true for sufficiently large size.
   Latin square results

These distinctions are all made uniformly for all uncountable cardinals.

Who cares about uncountable cardinals?

Most mathematical theorems are either about structures of size at most the continuum or are oblivious to cardinality.

Model theory is an exception

1. $\aleph_0$-categoricity is a completely different phenomena than $\aleph_1$-categoricity.
2. A first order (countable) theory is stable (an absolute syntactically defined property)
   iff there are at most κ types over a set of power κ UNLESS $\kappa^\omega > \kappa$.
3. A type is omitted in every model if it is omitted in every model of cardinality $\leq \beth_{\omega_1}$.

The types of objects considered

We analyzes properties of three kinds of objects:

1. Properties of cardinal numbers
2. Properties of structures
3. Properties of classes of structures

Classification theory

Shelah's classification methodology

A *dividing line* is a property such that both it and its negation are virtuous i.e., have strong mathematical consequences.
In effect, the negation of a tame property is a chaotic one. Now there are three possibilities:

1. the virtuous property is eventually true or
2. The chaotic one is eventually true
3. Neither is eventually true.

Case 1 establishes an eventual classification;
2) and 3 deny one exists.

One hopes tame properties will be eventually true. But if a certain property is eventually true, there are several attitudes about behavior below the threshhold.

What is noise?

The notion of ‘noise’ arises for eventually determined properties.

The counterexamples below the lower bound on paradise may be called noise.

‘Noise’ or ‘change of scale’

Shelah remarks that $\aleph_0$ is an ‘exceptional’ cardinal.

In particular, the stable/unstable dichotomy has little to say about the number of countable models of a first order theory while it offers a clear break for uncountable cardinals.

An unstable first order theory has 2^κ models of cardinality κ for every uncountable κ.

The technology for proving this theorem is powerless in the countable because the notion of stationary set is meaningless there.

Noise or sporadic

The noise in this example differs from the notion of a *sporadic* simple group. There, the problem is to analyze, among all finite groups, those which are simple.

While the sporadics are those that don't fit into the main families, they aren't just the groups that are left over; we didn't have a list of simple finite groups until it was shown there were 26, not 24 or 25 sporadicss. In contrast, Shelah' theorem provides an explanation for a property emerging – the cardinality was just too small to exhibit the property. The combinatorial complexity reflected by different linear orders is not available in the countable.

So 'just noise' disappears' when there is some explanation given for the exceptions.

Hanf numbers: idea

Definition

A property P is *downward closed* if there is a κ_0 such that if $P(\boldsymbol{K}, \lambda)$ holds with $\lambda > \kappa_0$, then $P(\boldsymbol{K}, \mu)$ holds if $\kappa_0 < \mu \leq \lambda$.

Hanf observed:

Theorem: Hanf

If a property P is downward closed then for any κ there is a cardinal μ such for any AEC $\boldsymbol{K}$ with $\kappa_{\boldsymbol{K}} = \kappa$, if some model in $\boldsymbol{K}$ with cardinality $> \mu$ has property P, then there is a model with property in all cardinals greater than μ.

Note the κ_0 is important; else, the unstable implies many models example would not fit into the picture.

Hanf numbers exist

That is, if AEC are downward closed for a property *P* there is a Hanf Number for *P* in the following sense.

Definition: Hanf Numbers

The *Hanf number* for *P*, among AEC $\boldsymbol{K}$ with $\kappa_{\boldsymbol{K}} = \kappa$, is μ if: there is a model in $\boldsymbol{K}$ with cardinality $> \mu$ that has property *P*, then there is a model with property *P* in all cardinals greater than μ.

Theorem Hanf

If there is a set of classes $\boldsymbol{K}$ of given kind (e.g. defined by sentences of $L_{\mu,\nu}$ for some fixed μ, ν) of a given similarity type then for any property $P(\boldsymbol{K}, \lambda)$ there is a cardinal κ such that if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda > \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ.

Downward Closure implies eventual behavior

Rewriting Hanf

If a property or its negation is downward closed the property is eventually determined.

We introduced the tree property to illustrate how a 'structural property' can induce a cardinal property.
It has consequences for cardinal arithmetic since tree property implies regular.
Since the tree property always fails on singular cardinals and consistently fails arbitrarily large regular cardinals (e.g. all successors if $V = L$),
it requires extensions of set theory to make it eventually determined.
And it isn't in $V = L +$ *no inaccessible*.

Large and Small Hanf numbers

Example (Various Properties)

1. **Small:** existence. lb =ub: Hanf number is eventually determined by $\beth_{(2^\kappa)^+}$ of the Löwenheim number (Morley)
2. **Big:** tameness. lb =ub: equivalent to strong compact (Boney-Unger)
3. **Middling:** every model extendible. lb =ub: measurable cardinal (Baldwin-Shelah)
4. **Open:** amalgamation
    1. lb: eventually determined by $\beth_{(2^\kappa)^+}$ of the Löwenheim number. (Kolesnikov/Lambie-Hansen)
    2. ub: (Baldwin-Boney) true above a strongly compact

Question

Can one find a principled distinction among the kinds of properties that explains the size of the Hanf number?

A specific question

Can one code a small large cardinal κ (e.g Erdös) in such a way that there is a (complete) sentence of $L_{\omega_1,\omega}$ that

1. There are maximal models in cofinality up to κ
2. But, never again

Not small - calculable

Recall that the continuum can be made weakly inaccessible. The Hanf number for existence now seems bigger.
But in the presence Martin's axiom, the cardinals below the continuum have of the properties of $\aleph_0$.
Is existence noise?