The Amalgamation Property

Excellence

Covers

First Order and Infinitary Model Theory: Obtaining Amalgamation

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Ecclesiates 1:1-2, 9

1 The words of the Preacher, the son of David, king in
Jerusalem.
2 Vanity of vanities, saith the Preacher, vanity of vanities; all is
vanity.
9 The thing that hath been, it is that which shall be; and that
which is done is that which shall be done: and there is no new
thing under the sun.

Dialectical Model Theory

Thesis Model theory is a technique for solving problems in mainstream mathematics, usually involving structures of small cardinality.

Antithesis Model theory is a branch of mathematics in its own right. In particular, it explores the 'higher infinite'.

Synthesis The insights developed in exploring the Antithesis repeatedly enrich the Thesis. This leads to new 'pure' explorations.

The 70's

Stability theory developed

1. abstractly with the stability classification
2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

The absoluteness of fundamental notions such as $\aleph_1$-categoricity and stability liberated first order model theory from set theory.

In the Background

At the same time and largely unnoticed, Shelah developed the fundamentals of stability theory for infinitary logic.

It was not until Zilber's exploration of complex exponentiation in the 1990's that the significance of this work for mainstream mathematics was realized.

Submitted by 1978

1. Sh:3 Shelah, Finite diagrams stable in power (1970)
2. Sh:10 Shelah, Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory – (1971)
3. Sh:48 Shelah, Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$ – (1975)
4. Sh:54 Shelah, The lazy model-theoretician's guide to stability – (1975)
5. Sh:72 Shelah, Models with second-order properties. I. Boolean algebras with no definable automorphisms
6. Sh:87a, 87b Shelah, Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A, Part B (1983)

The Higher Infinite

Large enough models so that pathologies have disappeared and the essential uniformity of classes of models are apparent.

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Large enough models so that pathologies have disappeared and the essential uniformity of classes of models are apparent.

Today the higher infinite means:
greater than $\aleph_\omega$ or even,

Greater than $\aleph_1$!

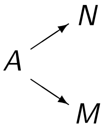
Generalized Amalgamation

The crucial notion is generalized amalgamation.

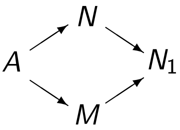
In this talk, we discuss developments of generalized amalgamation and connections between the first order and infinitary versions.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

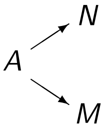


there exists an $N_1 \in \mathbf{K}$ such that



SET AMALGAMATION PROPERTY

The class **K** satisfies the *set amalgamation property* if for any situation with $M, N \in \mathbf{K}$ and $A \subset M, A \subset N$:



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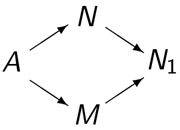

Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

Is there a difference?

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There is no difference.

Tweak the language and we obtain set amalgamation.

(Tweak: put predicates for every definable set in the language)

There is a difference!

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting examples do not have set amalgamation.

A simple example

There is a first order theory T with a prime model M such that:

1. M has no proper elementary submodel
2. but M contains an infinite set of indiscernibles.

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Inhomogeneity

Construct an $L_{\omega_1,\omega}$-sentence ψ whose models are partitioned into two sets; on one side is an atomic model of T, on the other is an infinite set.

ψ is categorical in all infinite cardinalities but no model is $\aleph_1$-homogeneous because there is a countably infinite maximal indiscernible set.

J. Knight's construction

Three unary predicates, W, F, I partition the set. Let W and I be countably infinite sets and fix an isomorphism f_0 between them.

F is the collection of all bijections between W and I that differ from f_0 on only finitely many points.

Add also a successor function on W so that (W, S) is isomorphic to ω under successor and the evaluation predicate $E(n, f, i)$ which holds if and only if $n \in W, f \in F, i \in I$ and $f(n) = i$.

The resulting structure M is atomic and minimal. Since every permutation of I with finite support extends to an automorphism of M, I is a set of indiscernibles.

An orthogonal notion

Definition (R. Knight)

A sentence σ of $L_{\omega_1,\omega}$ is amalgamative in a fragment Δ if for any complete Δ-types, $p(\mathbf{x}, y), q(\mathbf{x}, z), r(\mathbf{x})$ such that r is the common restriction of p and q, there is an $s(\mathbf{x}, y, z)$ that restricts to both p and q.

For any sentence σ, let Δ^α $(\alpha < \omega_1)$ be the Morley/Scott expansions of Δ.

Theorem (R. Knight)

If σ has uncountably many models then for a cub C, for any $\alpha \in C$, σ is amalgamative in Δ^α.

Quasiminimal Excellence

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the 'excellence condition' which follows.

Conditions 1 and 2 are sufficient for $\aleph_1$-categoricity.

Necessary Notation

In the following definition it is essential that $\subset$ be understood as proper subset.

Definition

1. For any Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.
2. We call C (the union of) *an n-dimensional* cl*-independent system* if $C = \mathrm{cl}^{-}(Z)$ and Z is an independent set of cardinality n.

Essence of Excellence

There is a primary model over any finite independent system.

Let $C \subseteq H \in \mathbf{K}$ and let X be a finite subset of H. We say $\mathrm{tp_{qf}}(X/C)$ is *defined over* the finite C_0 contained in C if it is determined by its restriction to C_0.

Quasiminimal Excellence

Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or in $\mathbf{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^-(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp_{qf}}(X/C)$ is defined over C_0.

QM EXCELLENCE IMPLIES CATEGORICITY

QM Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

Methodology

Amalgamation for independent families of models is a key to Zilber's algebraic proofs of quasiminimal excellence.

But Shelah had already seen a bigger picture.

General Setting

An amalgamation problem is a functor $\mathfrak{a}$ taking $\mathcal{P}(n)^-$ into a family $\mathcal{A}$ of subsets of models in a class. Some notion of independence is assumed.

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An amalgamation solution is an extension $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ to map $\mathcal{P}(n)$ into $\mathcal{A}$F such that:

$$\{\overline{\mathfrak{a}}(u) : u \subset n\}$$

is an independent system contained in $\overline{\mathfrak{a}}(n)$.

The formulation and examples are derived from Hrushovski and Kolesnikov.

3 properties

n-existence

Every $\mathcal{a}$ from $\mathcal{P}(n)^-$ has a solution.

Weak n-Uniqueness

Any two solutions for any $\mathcal{a}$ are compatible over $\{\overline{\mathcal{a}}(u) : u \subset n\}$.

n-Uniqueness

Any two solutions for any $\mathcal{a}$ are isomorphic over $\{\overline{\mathcal{a}}(u) : u \subset n\}$.

Choices of $\mathcal{A}$

1. Zilber: **QM-excellence**: closed subsets of a combinatorial geometry
2. Shelah: **Excellence 1**: full atomic models of a first order theory;
3. Shelah: **Excellence 2**: atomic models of a first order theory;
4. Shelah: **NOTOP**: models of a superstable theory with NDOP;
5. Kim-Pillay: Independence property;
6. Hrushovski: **groupoid paper**: algebraically closed sets;
7. Kolesnikov, Kim, Tsuboi: ***n*-simple**: boundedly closed sets

Increasing n

n-existence and n-uniqueness imply $n+1$ existence.

The excellent case shows a close connection between uniqueness and goodness.

Consequences of n-amalgamation

Shelah : existence and classification of models

Kolesnikov: Hierarchy of n-simplicity

Hrushovski: Elimination of Imaginaries, . . .

Kim,Depiro, Millar: Group configurations

$L_{\omega_1,\omega}$ and 'first order' model theory

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\mathbf{K}$ is the class of atomic models (realize only principal types) of a first order theory.

We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

Two Examples

T_1 is the theory of an infinite set under equality. $M \models T$. p asserts $x \neq m$ for every $m \in M$. Then $p \in S_{at}(A)$.

(Marcus/J. Knight): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Every $p \in S_{at}(M)$ is realized in M.

$L_{\omega_1,\omega}$: The General Case

Quasiminimality is the rank one case

Any geometry has a notion of independent n-system.

In the more general setting

Splitting gives an analogous notion of independent n-system.

Goodness

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A, this is the same as $|S(A)| = \aleph_0$.

Are there prime models over good sets?

Goodness

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A, this is the same as $|S(A)| = \aleph_0$.

Are there prime models over good sets?
YES, In $\aleph_0$ and $\aleph_1$
but not generally above $\aleph_1$ (J. Knight, Kueker, Laskowski-Shelah).

Yes, if excellent.

Excellence

Definition

1. **K** is (λ, n)-*good* if for any independent n-system $\mathcal{S}$ (of models of size λ), the union of the nodes is good.

   That is, there is a prime model over any countable independent n-system.

2. **K** is *excellent* if it is $(\aleph_0, n)$-good for every $n < \omega$.

Excellence implies large models

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose $\mathbf{K}$ has $(<\lambda, \leq n+1)$-existence and is $(\aleph_0, n)$-good. Then $\mathbf{K}$ has (λ, n)-existence.

Excellence implies large models

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose **K** has $(< \lambda, \leq n+1)$-existence and is $(\aleph_0, n)$-good. Then **K** has (λ, n)-existence.

This yields:

Theorem (ZFC)

If an atomic class **K** is excellent and has an uncountable model then

1. it has models of arbitrarily large cardinality;
2. if it is categorical in one uncountable power it is categorical in all uncountable powers.

Existence only

What can one conclude from existence only?

‘Universal’ Classes -propagation

Let **K** be defined by universal sentences and the omission of types.

Theorem B-Kolesnikov-Shelah

For any $s < \omega$, if $\mathbf{K}_k$ has the $(\lambda, \leq s+1)$ disjoint amalgamation property, then it has the $(\lambda^+, \leq s)$-disjoint amalgamation property.

Thus $(\aleph_0, < \aleph_0)$ disjoint amalgamation implies **K** has arbitrarily large models and disjoint amalgamation (of pairs).

'Universal' Classes -Limitations

Theorem B-Kolesnikov-Shelah

There is a 'universal' $\mathbf{K}_k$ such that:

1. $\mathbf{K}_k$ has no models of cardinality greater than $\beth_k$.
2. $\mathbf{K}_k$ has the disjoint amalgamation property on models of cardinality less than or equal to $\aleph_{(k-3)}$.
3. $\mathbf{K}_k$ has models of cardinality $\aleph_{(k-1)}$.

Theorem B-Kolesnikov-Shelah

For every countable ordinal α, there is a 'universal' class $\mathbf{K}_\alpha$ that has disjoint amalgamation over pairs of models of size $\leq \aleph_\alpha$. But $\mathbf{K}$ has no model of power $\beth_{(2^{\aleph_0})^+}$.

And consistently, $\aleph_\alpha$ can be replaced by $\beth_\alpha$.

Abstract Elementary Classes

Consider a class of L-structures and a notion of 'strong submodel' $\prec_{\mathbf{K}}$.

$(\prec_{\mathbf{K}}, (\mathbf{K}, \prec_{\mathbf{K}})$ is an *abstract elementary class* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism and satisfy a collection of conditions generalizing those of Jónnson for constructing homogeneous universal models.

In particular, the class must be closed under $\prec_{\mathbf{K}}$-increasing chains.

A further crucial requirement is the existence of a Löwenheim number for the class.

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property. Then there is a monster model $\mathcal{M}$.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Tameness-Algebraic form

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:
Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

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Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If $\lambda > \mathrm{LS}(\mathbf{K})$, **K** *is* λ^+*-categorical and* $(\lambda, < \infty)$*-tame then* **K** *is categorical in all* $\theta \geq \lambda^+$.

Theorem (Lessmann)

If **K** *with* $\mathrm{LS}(\mathbf{K}) = \aleph_0$ *is* $\aleph_1$*-categorical and* $(\aleph_0, \infty)$*-tame then* **K** *is categorical in all uncountable cardinals*

Tameness is essential!

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.

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3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
4. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
5. ϕ_k is not $\aleph_{k-2}$-Galois stable;
6. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;
7. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Two Problems

1. Is there a finitely axiomatizable strongly minimal set?
2. Is there an $\aleph_0$-homogeneous (over models) quasiminimal class which is not excellent?

Enough categoricity implies Excellence

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

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VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and is categorical in $\aleph_n$ for each $n < \omega$ is excellent.

Show by induction:

Very few models in $\aleph_n$ implies $(\aleph_0, n-2)$-goodness.

Covers of Algebraic Groups

Definition A cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \rightarrow Z^N \rightarrow V \stackrel{\exp}{\rightarrow} \mathbb{A}(\mathcal{C}) \rightarrow 1. \quad (1)$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$.

Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \ln(\mathrm{acl}(\exp(X)))$$

Covers

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If $(A)(\mathcal{C}) = (\mathcal{C}, \cdot)$, Σ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.

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Theorem

If $(A)(\mathcal{C}) = (\mathcal{C}, \cdot)$, Σ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.

But, the uncountable model H contains a maximal infinite indiscernible set. Thus, H is not $\aleph_1$-homogeneous.

Axiomatizing Covers

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.
Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q\in\mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each F-definable variety.
3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

$T_A + \Lambda = \mathcal{Z}^N$ asserts the kernel of ex is standard.

In general, VWGCH implies categoricity is equivalent to 'arithmetic' properties of $\mathcal{A}$.

Notation

Definition

A *multiplicatively closed divisible subgroup* associated with $a \in \mathcal{C}^*$, $a^{\mathbb{Q}}$, is a **choice** of a multiplicative subgroup isomorphic to $\mathbb{Q}$ containing a .

$\mathbb{A}_n(F) = \{a \in \mathbb{A}(F) : a^n = 1\}$

Fact. For any acf F, $\mathbb{A}_n(F) \approx (Z/nZ)^N$.

For any $a_1, \ldots a_n$, let $k_a = k_0(\mathbb{A}_{\mathrm{tors}}, a_1^{\mathbb{Q}}, \ldots a_n^{\mathbb{Q}})$.

$\aleph_1$-categorical iff

Algebraic Conditions for ω-stability and homogeneity

1. there is a d such that for any $m \in \mathbb{N}$, $\mathrm{Gal}(\tilde{k_0} : k_0)$ has at most d orbits on

$$\{\langle a_1, \ldots a_N \rangle : a_1, \ldots a_N \text{ generate } \mathbb{A}_m\}.$$

2. Let $F_0 \subset F(\mathbb{A})$ be a countable acf, $b_1, \ldots b_n$ be multiplicatively independent over $\mathbb{A}(F_0)$. There is an $\ell \in \mathbb{N}$ such that:

$$\mathrm{Gal}(F_0(b_1^{\frac{1}{m\ell}}, \ldots, b_k^{\frac{1}{m\ell}}) : F_0(b_1^{\frac{1}{\ell}} \ldots, b_k^{\frac{1}{\ell}})) \approx (\mathcal{Z}/m\ \mathcal{Z})^{Nk}.$$

Algebraic Formulations of Model theoretic properties

Let $\mathcal{S} = \{F_s : s \subset n\}$ be an independent n-system of algebraically closed fields contained in a suitable monster $\mathcal{M}$. Denote the subfield of $\mathcal{M}$ generated by $(\bigcup_{s \subset n} F_s)$ as k.

Canonical completions

$$\mathcal{A}(k) = A^n \oplus \prod_{s \subset n} \mathcal{A}(F_s)$$

where A^n is a free Abelian group.

Excellence

The following are equivalent under VWGCH

1. The cover of $\mathbb{A}$ is categorical in all uncountable κ.
2. The cover of $\mathbb{A}$ is categorical in all $\aleph_n$ for $n < \omega$.
3. The cover of $\mathbb{A}$ is quasiminimal excellent.
4. $\mathbb{A}$ satisfies algebraic conditions for ω-stability and homogeneity and has canonical completions.

Refining the Model theoretic analysis

Consider $T_A + \Lambda = Z^N$:

1) all $\mathbb{A}$
   a) has 2-amalgamation;
   b) has arbitrarily large models;
2) Identify algebraic properties corresponding exactly to small, $\omega - stable$, $\aleph_0$-homogeneity over models, excellence.
3) Do these properties differentiate algebraic groups in a natural way?

Will more model theory help?

Application of Theorem of Grossberg-Kolesnikov

If an ω-stable atomic class has $(\aleph_0, n)$-weak uniqueness for all n, then it is $(\aleph_0, \infty)$-tame.

Still more model theory

Recall

Theorem (Lessmann)

If $\mathbf{K}$ *with* $\mathrm{LS}(\mathbf{K}) = \aleph_0$ *is* $\aleph_1$*-categorical and* $(\aleph_0, \infty)$*-tame then* $\mathbf{K}$ *is categorical in all uncountable cardinals*

So if we establish weak-uniqueness in $(\aleph_0, n)$ for all n for an algebraic group $\mathcal{A}$, we can conclude the 'arithmetic' consequences on $\mathcal{A}(k)$ for all k
(modulo Shelah's theorem and VWGCH)

Is WCH necessary?

Does MA + $\neg$ CH imply there is a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$ categorical but

a) is not ω-stable

b) does not satisfy amalgamation even for countable models.

There is such an example in $L_{\omega_1,\omega}(Q)$ but Laskowski showed the example proposed for $L_{\omega_1,\omega}$ by Shelah (and me) fails.

Is $\aleph_1$-categoricity of a sentence of $L_{\omega_1,\omega}$ absolute?

Concluding the reading

10 Is there any thing whereof it may be said, See, this is new?
it hath been already of old time, which was before us.
11 There is no remembrance of former things; neither shall
there be any remembrance of things that are to come with
those that shall come after.