Variations on a theme of Makowsky

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Abstract

We distinguish between the axiomatic study of proofs *in geometry* and study *about geometry* from general axioms for mathemathematics. We briefly report on an abuse of that distinction and its unfortunate effect on US high school education. We review a number of 20th century approaches to synthetic geometry. In doing so, we disambiguate (in the Wikipedia sense) the terms: metric, orthogonal, isotropic and hyperbolic. With some of these systems we are able to axiomatize 'affine geometry' over the complex field[1]. We examine the general question of the connections between axioms for Affine geometries and the stability classification of associated complete first order theories of fields. We conclude with reminiscences of a half-century friendship with Janós.

Contents

1 Introduction

Our topic is inspired by Makowski's article 'Can one design a geometry engine?' [Mak19]. It introduced me to several ways to first order axiomatize 'Euclidean geometry' that were unusual because of very different choices of the fundamental notions. In Section 2 we contrast first order axiomatization of geometry (proofs in geometry) with arguments in ZFC or 2nd order logic (axioms about geometry, such as Birkhoff's [BB59]).

[1] The argument is trivial from [Wu94] or [Szm78], but not remarked by either of them.

[Szm78] carefully describes the *linear Cartesian plane* over a field F and the geometry on F^2 whose lines are the solutions of linear equations over F. Offending all algebraic geometers[2], we often abbreviate to 'plane over F' since the synthetic theory of such planes and straight lines is the target of this investigation. In particular, we will speak of the real and complex planes in this sense.

In the spirit of the clarification of the distinction in [Mak19] between mutual and bi-interpretability, we clarify several other terminological confusions. In fact a principal motive for Section 3 is to sort out for myself the rich diversity of first order approaches to coordinatizable plane geometry.

This analysis illuminates a deeper classification. Two of the prototypic structures in model theory are the (geometries over) the real and complex planes that lead to the 'twin' notions of strong minimality and o-minimality. The distinctions among the geometries discussed in section 3 reflect this dichotomy. But we note in Section 4 that the geometry over the p-adic numbers fall into quite a different location in the map of stable theories at forkinganddividing.com. This raises the question, 'What distinguishes the geometries?'. How are the different choices of fundamental notions and approaches to coordinatization reflected in this di-(tri)-x chotomy?

2 Proofs in or about

{inabout}

We compare the synthetic proof of Eucid *et al* with the 20th century study of geometry by distinguishing three species of the proof of a proposition P in a geometry. What is a geometric proof? Any proof requires assumptions, rules of inference, and definitions. The three species are

- **Approach 1** *proof in* a formal language for geometry[3];
- **Approach 2** *proof about* i.e., in a metatheory (e.g. ZFC), with geometry a defined notion.

  Whether such a proof in the second sense is 'geometric' is a purity issue.
- **Approach 3** We don't dwell here on a standard model theorerif technique: Use 2) to get 1). Using the completeness theorem [HA38] [Bal18, p 257] outline the *method of semantic proof*. If a proposition is stated in first order logic and show to be true by a proof *about geometry* then in every model of a specific first order theory T of geometry, then it is provable in T.

Approach 1. Both Euclid and Hilbert (1899) wrote in natural language and had no explicit rules of inference. A formal proof in geometry requires:

1. Choosing a vocabulary (after conceptual analysis) of the fundamental notions (basic concepts). Euclid uses point, line, circle, incidence, congruence of segments of segments and of angles. Hilbert adds betweeness and order but omits circle. In Section 3, we discuss such 20th century basic concepts as orthogonality, parallelism, and perpendicularity.
2. Choosing a logic (first order, $L_{\omega_1,\omega}$, second order)
3. Choosing the axioms that reflect the conceptual analysis.

Approach 2. Through the late 19th and twentieth century as geometry metastasized from Euclid to hyperbolic, to differential, algebraic, etc., etc. the most published proofs were informal proofs (nominally

[2] Von Staudt published a 3 volume study of complex projective geometry including higher dimensional curves in 1856/1860.

[3] We restrict to geometry only for uniformity; the analysis applies to any formalized topic.

reducible to ZFC for the last century) *about*, say, algebraic geometry. But they were not formalized in any specifically 'geometric' system. At best the appropriate geometry was defined in the (informal) metatheory.

For example, the *global method: analytic/metric* method of assigning area to a figure is described in [Bol78]. Fix a unit; say, a square; tile the plane with congruent squares. Then to measure a figure, continually refine the measure by cutting the squares in quarters and count only those increasingly smaller squares which are contained in the figure. As one ponders this method, one realizes that it assumes a *real-valued* (to guarantee covergence) metric. This assumption is not mentioned but considered (correctly for most readers of the book) as a universally known assumption. Such is mathematics and I have no quarrel with it. But there is one hybrid which has had a disastrous impact on United States high school mathematics: Birkhoff's 'axioms'.

Our inspiration, Makowsky's article 'Can one design a geometry engine?', makes no mention of Birkhoff. Let us see why. Birkhoff [Bir32, BB59] works in a vocabulary of points, lines, distance ($d(A, B)$), and angle. Distance is a function from pairs of points to the real field (a topic assumed to be fully understood by students who survived one year of algebra.) (An angle is measured by a similar function from triples of points.) Postulate 1 (Ruler postulate) asserts that the points of any line can be put into 1-1 correspondence ($A \mapsto x_A$) with the reals so $d(A, B) = |x_A - x_B|$. The protractor postulate posits a similar measure for angles. In many texts [BCea05], an early proof shows 'equals distances subtracted from equal distances are equal'. The proof is to apply the ruler postulate twice along with their deep understanding of the axioms of the algebra of the real numbers. This is in the first week of geometry for 14-15 year old students.

Raimi [Rai05] presents Birkhoff's motivation for the high school text [BB59] as a reaction to shoddy treatment of limits in U.S. high schools during the first half of the 20th century. Unfortunately, the cure is as bad as the disease. And the School Mathematics Study Group[4] adopted this his system for high school geometry.

Contrary to Birkhoff, this is not a fully formalized axiom system. The properties of the reals are introduced as convenient oracles. Thus, as a proof about but not in geometry, it is not in the purview of [Mak19].

2.1 Proofs in Geometry: Choosing basic notions {vocabchoice}

In this subsection we survey several axiomatic approaches to the study of geometry. These systems are similar in that the initial axioms are first order and if/when Archimedes or Dedekind appears, it is explicitly mentioned. The distinction is in the choice of basic notions for geometry. We restrict to affine geometry as the translation (bi-interpretation) between projective and affine geometry is standard. In Section 3.1, we make a much finer distinction among six candidates for the title 'metric geometry'. The comparison between Hilbert style systems and the various orthogonal systems discussed there is the main concern of the paper.

2.2 Ordered Geometries

These are well-known; we just list them.

1. Hilbert/Euclid [Hil62, Hil71, Har00]: congruence is fundamental; two kinds of objects: point and lines

2. Tarski [Tar59, Szm78]: congruence is fundamental; one kind (sort) of object: a line is a set of collinear points (given by a ternary betweenness relation).

[4] These are the architects of the 'new math'. Much of their work especially in Algebra I is aimed at understanding but the SMSG postulates [SMS95, Ced01] remind one that a camel is a horse designed by a committee.

3. Various authors [BH07, Cla12, Lib08, Mar82, Wei97]: Transformations are central but in most cases developed in axiomtized Euclidean geometry[5].

4. Szmielew and Wu [Szm78, Wu94] add the order notion at the end of their development; see Remark 3.2.3c and 3.2.4.

3 3 homonyms in geometry: Is order essential?

{hom}

This section relates more directly to [Mak19]. We discuss three words which apply with apparently quite distinct meanings in developments of geometry from different choices of fundamental notions. We will then consider the relations of these developments with real and complex algebraic geometry. The three subsections address the three homonyms: metric, isotropic, and hyperbolic.

3.1 metric

What is a metric geometry? We describe here four very different notions of a metric geometry with many specific axiomatizations in various vocabularies. {metric}

Definition 3.1. *A* generalized *metric is a function f from $X \times X$ into an ordered field F, that is symmetric, $f(x,x) = 0$, other values are positive, and satisfies the triangle inequality. Normally, $F = \Re$.*

{classmet}

Remark 3.2 (Diverse notions of **'metric'**).

1. equipped with **congruence** (line segment/angle) This terminology is certainly inaccurate and likely only used when segment congruence is confused with the existence of a real valued distance metric. A congruence equivalence may not to be attached to a unit 'distance'. This is one of the crucial distinctions between Euclid and Hilbert. Euclid would not conceive of such a confusion because he viewed geometric and arithmetic magnitudes as incomparable (not merely incommensurable).

{moise}

2. equipped with a **distance metric** [Moi90, p 137] carefully distinguishes between what he calls synthetic and metric approaches. Roughly speaking, his synthetic corresponds to Hilbert and metric to Birkhoff. Hilbert begins with congruence and, effectively but not explicitly[6], implicitly introduces a 'distance' measured on a field that varies with the model of the theory and with a unit distance in a model M as the congruence class of the segment 01.

   (a) in some ordered field [Hil62] or, more specifically,

   (b) equipped with a *real-valued* distance metric [Bir32].

   These are vastly different; the first is first-order axiomatized. As discussed in Section 2, the second is basically axiomatized in set theory and is really more describing a geometry from a global standpoint than giving axioms for geometry.

{orth}

3. orthogonal geometry: 'Throughout this paper metric will always refer to a structure with an orthogonality relation or in which one such relation[7]. *It is in no way related to metrics defined as distances*

[5] While these systems are ostensibly second order by quantifying over transformations as arbitrary functions satisfying certain conditions, one can adopt the standard first order trick of adding a sort for transformations θ and requiring that each such θ indexes a set of ordered pairs, the graph of a rigid motion.

[6] [Hil62] does not use the word distance in this sense or 'metric' at all.

[7] Line reflections are a basic concept in this system.

with real values.' [PSS07, p 419]. We describe four variants on 'metric' and associated versions of 'orthogonal'.

(a) [PSS07] describes two approaches: group theoretic and geometric. {Pamb}

i. [PSS07, p 423] axiomatize a group of rigid motions of a plane with a unary predicates for line reflections, an operation (composition), and a constant for the identity[8] The geometry is recovered by first order definitions [PSS07, §2.1] and one can distinguish the elliptic, euclidean and hyperbolic case. {Psyn}

ii. Alternatively, 'geometric' axioms [PSS07, §2.2] use the vocabulary of incidence, line orthogonality, and reflections in lines. {Art}

(b) Artin [Art57, p 51] calls the problem of defining a field from a two-sorted axiomatic geometry 'much more fascinating' than the familar Cartesian reduction of geometric problems to analytic geometry. Thus, unlike Birkhoff, he is explicitly working in set theory and perhaps (not his word) doing metamathematics. However, because of this clarity, lack of a linear order, and the his use of first order axiomatizations of some geometries, I consider Artin here rather than as 'about' in Section 2.

He writes [Art57, p 106] 'The study of bilinear forms is equivalent to the study of metric structures on V'. An orthogonality relation can be described as an 'inner product' possessing properties such as those imposed on real geometry by the inner product. The connection with 'metric' in the sense of Remark 3.2. 2) arises from the fact that the real inner product of vector with itself is the square of the length. I discuss this example in Remark 3.2.3.3) because the inner product of two vectors determines the angle between them and thus perpendicularity. But this approach is far more general than a real inner product space since it makes sense without any continuity hypothesis, for projective spaces, and for any vector space. {papdef}

Definition 3.3.

i. An incidence plane is collection of points and lines such that two points determine a line and there are three non-collinear points.

ii. An incidence plane is Pappian[9] *if for A_1, A_2, A_3 on line ℓ_1 and B_1, B_2, B_3 on line ℓ_2 (distinct points on distinct lines)*

$$(A_1B_2 \parallel A_2B_1 \wedge A_2B_3 \parallel A_3B_2) \rightarrow A_1B_3 \parallel A_3B_1.$$

Definition 3.4 (Wu's orthogonality axioms:)**.** *The orthogonality of two lines is denoted by $\ell_1 \perp \ell_2$ or $\mathrm{Or}(\ell_1, \ell_2)$. This is a basic concept for Wu. A line ℓ is* isotropic *if it is self-perpendicular.*

(O-1): $\ell_1 \perp \ell_2 \leftrightarrow \ell_2 \perp \ell_1$;

(O-2): For a point O and a line ℓ_1 there exists exactly one line ℓ_2 with $\ell_1 \perp \ell_2$and $I(0, \ell_2)$;

(O-3): $(\ell_1 \perp \ell_1 \wedge \ell_3 \perp \ell_3) \rightarrow \ell_2 \parallel \ell_3$.

(O-4): For every O there is an ℓ with $I(O, \ell)$ and $\ell \not\perp \ell$.

(O-5): The three heights of a triangle intersect in one point. {Wu}

(c) [Wu94, §2.2] axiomatizes in a vocabulary with points, lines, and perpendicular as basic concepts. He has four groups of axioms ordered by containment; the last two are metric.

[8] [Pam17] axiomatizes the 'same' geometry using only the relation symbol $\perp$ (with $\perp(abc)$ to be read as 'a, b, c are the vertices of a right triangle with right angle at a').

[9] Each of Szmielew and Wu discuss various refinements of the Pappian notion and relations with various forms of Desargues; they agree on the statement here as the decisive condition for obtaining a commutative field.

i. A Wu-orthogonal plane satisfies the usual (Hilbert) incidence axioms, five orthogonality axioms, asserts lines are infinite, unique parallels, and two forms of Desargues[10]. He concludes that a Wu-orthogonal plane satisfies Pappus and has a definable commutative coordinatizing field.
ii. An *unordered Wu-metric* plane arises by adding the symmetric axis axiom [Wu94, p 91]: Any two non-isotropic (See 3.2. 2.) lines have a symmetric axis[11] With these hypotheses, Wu [Wu94, p 92] *defines* a notion of congruence (called equidistance and added to the vocabulary in [Mak19]) and proves the Pythagorean (Kou-Ku) theorem.
iii. Adding Hilbert's order axioms gives an *ordered Wu-metric plane* [Wu94, §2.5].
This system defines an ordered coordinatizing field. Thus it is bi-interpretable with Hilbert's system ([Hil71, Har00]). Hilbert relies directly on what he calls Pascal's theorem, a variant of Desargues and Pappus; Hartshorne [Har00, §19] uses the cyclic quadrilateral theorem[12]
iv. Adding Hilbert (non-first order) continuity axiom Wu reaches his 'ordinary geometry' [Wu94, §2.6].

4. [Szm78] affine and parallelity planes {Szmielew}

A collinearity structure is a ternary relation (collinearity) such that two points determine a line. Such a structure is an *affine plane* if for any line ℓ and point A there exist a unique parallel to ℓ through A. Planarity is enforced by saying that if one line is parallel to two distinct lines then the two intersect.

By adding a constant to an affine plane we can fix a unit of distance. Since naming constants has no effect on interpretability, we will be careless about whether a point is named.

(Alternatively, [Szm78, §2] uses parallel as the only basic symbol and axiomatizes a two sorted system of points and lines, *parallelity planes* which are bi-interpretable with affine planes[13]. Moreover, {Szmielewbiint}

Fact 3.5. *(*)* [Szm78, 4.5.3.iii)], [Szm78, 4.5.7)] show commutative fields are binterpretable with Pappian parallelity planes.

Szmielew follows the 'projective geometry approach' of introducing ternary fields and gradually adding geometric conditions that strengthen the algebraic properties. This crucially distinguishes her approach from that of Hilbert, Hartshorne, and Wu. On the other hand, Wu and Szmielew differ from Hilbert/Hartshorne in applying Desargues/Pappus to find the field before introducing either order or congruence.

The particular affine geometry on $\mathbb{C}$ with 'lines' defined by linear equations is an affine plane and $(\mathbb{C}, +, \cdot, 0, 1)$ is definable in $(S, L, \|)$. Of course this structure is very different from the 'complex plane' in the sense of algebraic geometry. With the field, we can define algebraic curves in the plane.

It seems to me that 3(a)ii, 4, and 3c are very close together; each extends the orthogonality geometry to order to regain 'ordinary geometry' (although Wu equates 'ordinary with $\Re$-geometry and so requires Dedekind's axiom for that description). {aofields}

Definition 3.6. *A Pythagorean field is a field in which every sum of two squares is a square. A Euclidean field (is an* ordered *field in which all non-negative elements are squares).*

[10] [Wu94, Section 2.1] shows that the 'linear Pascalian axiom' a) allows the proof that the coordinatizing Skew field is commutative and b) follows from axioms for Wu-orthogonality. Thus, unlike [Szm78], there is not a separate Pappian field stage in his development.

[11] Let ℓ be the perpendicular bisector of (the segment between) two points A, B. Then ℓ is called the symmetric axis of (A, B).

[12] Thus, Hartshorne [Har00, p 173] differs from Hilbert in using circles, but does not use the intersection of circles postulate E.

[13] [Szm78, p85]; a predicate for parallel is needed for AE-axiomatizability.

A Euclidean field (axiom E: circle-circle intersection) is Pythagorean by the Pythagorean theorem and the use of Axiom E to construct a hypoteneuse for any pair of given lengths.

{Alperin}

5. [Alp00, p 121] studies coordinatization of origami geometries given by first order axioms and writes, 'Our main contribution here is to show that with all six axioms we get precisely the field obtained from intersections of conics, the field obtained from the rationals by adjoining arbitrary square roots and cube roots and conjugates'. He provides six axioms for construction (which can be done by paper folding) and *working within the complex numbers* shows that his first three axioms allow the construction from $0, 1, \alpha$, where α is not real, a subfield of $\mathbb{C}$. His fourth and fifth axioms extend the result to Pythagorean and Euclidean fields (Definition 3.6); with the sixth axiom, solutions to cubics can be constructed.

Note that Pythagorean fields need not be ordered; [Alp00, p 121] studies some as subfields of $\mathbb{C}$. However, the minimal Pythagorean field Ω is orderable and is the minimal field satisfying Hilbert's betweenness and congruence axioms [Har00, 16.3.1].

The crucial distinction between items 1) or 2) and items 3)-iii),3-iv) or 4) of Remark 3.2 is that the systems in the latter pair, while called 'metric' do not *require a notion of length or ordering of segments. They coordinatize with* unordered *fields. Item 3c .iii defines congruence but remains unordered. Alperin's field do not admit a linear order.*

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A key feature of (axiomatic) orthogonal geometries is that the existence of a field is either assumed (Artin) or arises directly from assumed Pappian configurations rather than Desarguesian/Pappus being derived from the parallel postulate using segment congruence as in Hilbert.

3.2 Isotropic

{isotrop}

1. Artin says a subspace of an orthogonal space in the sense of item 3b is isotropic if it is annihilated by the form.

{wuisotrop}

2. Wu says a line is isotropic if it is self-perpendicular. An example of an isotropic line through the origin in the complex plane is $x_2 = ix_1$.

3. Schwartz (`https://www.math.brown.edu/reschwar/INF/handout10.pdf`) says a geometry is isotropic if for any point and any angle can find a symmetry (distance preserving bijection) which fixes that point and rotates by that angle around the point.

   The first two notions are closely related; the third distinct.

3.3 hyperbolic space

1. The standard notion in non-euclidean geometry:

2. [Art57, Def 3.8] A non-singular plane which contains an isotropic vector is called hyperbolic.

It seems pretty clear that these notions of hyperbolic and isotropic are really distinct. The question is whether, as in my comment in item 3.2.3b, there is some etymological explanation for the overlap in terminology.

4 Classifying Geometries Model Theoretically

{class}

By 3.5 we know the (linear cartesian) plane π over any commutative field (constructed as in e.g. [Har00, §14] satisfies the parallelity axioms. So π is bi-interpretable with its coordinatizing field. The bi-interpretability, indeed interdefinability, is particulary easy to see for the orthogonality case.

{easybi}

Remark 4.1 (Biinterpretability)**.** Given the plane. Fix two orthogonal lines and interpret the field on one line ℓ_1 using Pappus. By fixing a family of lines of the same slope define a bijection f (and field isomorphism) between the lines. Formally define over that field the plane on $\ell_1 \times \ell_2$. Now it is definably isomorphic to the original plane) by mapping $\langle a_1, a_2 \rangle$ to the intersection in the plane of the line parallel to ℓ_1 meeting ℓ_2 in a_2 and the line parallel to ℓ_2 meeting ℓ_1 in a_1.

So if the coordinatizing field has a recursively axiomatizable complete first order theory, the first order theory of a particular plane is a complete decidable theory; for example, the real and complex planes.

{biint}

Fact 4.2. *[Biinterpretations] The following classes of geometries and fields are quantifier-free bi-interpretable*[14].

1. *Pappian geometries (Wu – unordered metric planes and Szmielew– Paffian affine planes) and fields;*
2. *Infinite Pappian geometries with linearly ordered lines (Hilbert planes, Wu-ordered metric geometries, ordered affine planes [Szm78, §8]) and ordered fields;*
3. *Hyperbolic geometries with limiting parallels and ordered Euclidean fields.*

The following is immediate from the existence of a suitable biinterpretation as in Fact 4.2.

Theorem 4.3. *The complete theory of the complex affine plane is axiomatized by adding the axioms of ACF_0 to the incomplete theory of fields given by the bi-interpretation with either i) theory of Pappian parallelity planes Fact[Szm78, 4.5.iii)] or ii) the theory of Wu-orthogonal planes.*

{Zieg}

Fact 4.4. *[Zie82, Bee] If T is finitely axiomatized subtheory of RCF or ACF_0 then T is undecidable.*

{foralldec}

Fact 4.5. *[Mak19, Thm 17 pg 26; Prop 6 pg 10]. The universal first order consequences of a) any extension of (the orthogonal geometries in Remark 3c, 4 or 5) or b) HP5 whose interpretation with consistent a ACF_0 or b) RCF_0 is decidable.*

The proof uses heavily the quantifier-free interpretations laid out in [Mak19]. Recalling Ziegler, Fact 4.4 and noticing that the axioms of the various geometries described in Remark 3.2 are $\forall\exists$-axiomatizable[15] Thus, decidability of universal sentences is most that can be hoped for in any general geometry; Fact 4.5 is optimal.

We have described a family of different axiomatizations in different vocabularies that have some claim to 'axiomatizing geometry'. Many are bi-interpretable. Such theories are often regarded as 'the same'. But 'same' is far from true here. The orthogonal geometries are not ordered; Hilbert's are. Tarski's first order completion is the first order theory of the reals – real closed fields while the orthogonal geometries

[14] The first two are proved with an argument emphasizing the quantifier eliminability are summarised in [Mak19, Theorems 5-7] and the third in [Har00, §43].

[15] As described (e.g. [ADM09, 707]) the propositions of Euclid fall into i) *theorems* which are universally quantification of an implication of two diagram (conjunction of atomic and neg-atomic formulas) and ii) constructions: π_2 sentences: For any instance of a diagram there are witness to an extended diagram.

are exemplified by the Complex affine plane. Note these interpretations are 2-dimensional. Is 1-dimensional any better?

What do we know about the fields? A Hilbert field is ordered using betweenness : [Szm78, 7.1.9]. But orthogonality geometries don't have betweenness. Alperin's origami are subfields of the complexes.

What are axioms for linear Cartesian planes over p-adic fields? Fix p and consider the affine plane over Q_p (or perhaps a countable elementary submodel?). We include in the vocabulary of Q_p a predicate for the valuation since the topological information is central to the notion. Let T' be the theory of Q_p. By [DGL11], T' can be formalized in a one-sorted language as a theory that is NIP but neither distal nor o-minimal but is dp-minimal. It is easy to see Q_p is not linearly ordered as for various p, there are negative integers that are perfect squares[16].

But (linear cartesian) geometry over Q_p is bi-interpretable with the field (without the valuation) Q_p (since the geometry is Pappian). What (if anything) needs to be added to the geometric vocabulary to define the valuation? It is not clear that dp-minimality is preserved by a 2-dimensional interpretation. If it were, we would know from [DGL11] that its complete first order theory has the same place in the stability geography. Which formalism is most useful for axiomatizing the geometry?

5 Reminiscences

I met Janos in the summer of 1972 during the International Congress of Mathematics in Vancouver. A group of us traveled to Banff and Calgary. I recall two small episodes: his insisting on swimming in his underwear in Shuswap Lake and refusing a bottle of wine in a fancy restaurant in Calgary. The second was a lesson I was able to apply a couple of times later. Much more memorable was his spelling me in carrying my daughter in a back carrier up a mountain near Banff. (My wife thinks this happened not in Banff but closer to Vancouver. But an ancient CV shows I gave a talk in Calgary that summer.) Sometime in the late 70's, my wife Sharon, daughter Katie, and I joined Janos and Eritt in a tour of Switzerland. The highlight was pre-school Katie directing us, 'Follow the D-car'. (Janos was working in Berlin.) I returned the child-on-back favor in 1980, carrying Amichai during our excursion from the Patras Conference to Delphi. We have no joint papers yet; our closest 'collaboration' was extended discussions about his contribution [Mak85] to the Model-theoretic logics book. A later adventure whose date escapes me was following up a swank dinner in Kolmar , Strasbourg? by smuggling (details may vary) a computer into West Germany. Maybe it was that the computer was smuggled out and then reimported to establish 'legality'. The fine dining stories continued with a visit to Perroquet in Chicago where Janos won an argument with the maitre'd by insisting that any reasonable high class restaurant would recognize his cardigan as a 'jacket' or provide jackets to traveling guests. We have exchanged visit over the years. Perhaps our long and highly-valued friendship can continue with another visit to Chicago by Janos and Misha.

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