Constructing ω-stable Structures: Model Completeness

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

Kitty Holland
Department of Mathematics
Northern Illinois University†

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Abstract

The projective plane of [2] is model complete in a language with additional constant symbols. The infinite rank bicolored field of [14] is not model complete. The finite rank bicolored fields of [4, 6] are model complete. More generally, the finite rank expansions of a strongly minimal set obtained by adding a 'random' unary predicate are almost strongly minimal and model complete provided the strongly minimal set is 'well-behaved' and admits 'exactly rank k formulas'. The last notion is a geometric condition on strongly minimal sets formalized in this paper.

There are a number of variants of the 'Hrushovski construction' [10] which produce ω-stable or even $\aleph_1$-categorical theories. All of them result in theories which are nearly model complete (all formulas are equivalent to a Boolean combination of existential formulas); some result in model complete theories (all formulas are equivalent to existential formulas). These 'quantifier elimination' results are closely connected to complexity of the axiomatization of the theory. Every model complete theory is $\forall\exists$-axiomatizable; a theorem of Lindstrom [11] asserts every $\forall\exists$-theory that is categorical in some infinite power is model complete. One of the intriguing features of the Hrushovski construction and the associated Shelah-Spencer random graph was that theories constructed for some other purposes naturally arose with

*e-mail: jbaldwin@uic.edu. Partially supported by NSF grant DMS-0100594.
†email: kholland@math.niu.edu. Partially supported by NSF grant DMS 0203747.

$\forall\exists\forall$-axiomatizations. Almost no natural theories are that complex. It was a surprise when Holland [9] proved that the strongly minimal Hrushovski constructions were in fact model complete (and so admit $\forall\exists$-axiomatizations). In this paper we investigate several variants on the construction and show that Holland's model completeness extends from the strongly minimal case to $\aleph_1$-categorical expansions by a unary predicates of strongly minimal sets which satisfy a certain geometric condition: the existence of exactly k-independent sequences. In fact we show that these theories are all almost strongly minimal. Ahlbrandt and Baldwin [1] had shown that every $\aleph_0$-categorical almost strongly minimal theory is axiomatizable using at most n alternations of quantifiers for some n. But Marker [12] has shown that the minimal such n could be arbitrary. The essence of Marker's counterexamples is to make the definition of the strongly minimal set complicated. Our strongly minimal sets are Σ_1-definable.

The infinite rank bicolored field of [14] is not model complete. Using an argument in the style of [9], we establish the model completeness of an expansion by constants of the projective plane in [2].

This paper builds on the notation and results in [4, 5, 2]. The general framework consists of a class $\overline{\boldsymbol{K}}_0$ of countable models which have hereditarily nonnegative rank with respect to a given predimension δ. A *strong substructure* relation is defined on $\overline{\boldsymbol{K}}_0$ by $A \leq B$ if for every finite B' contained in B, $\delta(B'/A) \geq 0$. In each case, the class $\overline{\boldsymbol{K}}_0$ has amalgamation with respect to $\leq$ and we are able to produce a countable generic structure $\mathbb{G}$ which is ω-saturated.

In the first section, we provide a sufficient condition for an almost strongly minimal theory to have a finite extension by constants that is model complete. The main goal of the paper is to prove certain constructions yield $\aleph_1$-categorical *model complete* theories. We exhibit two quite different proof methods. In the second section we deal with expansions of a 'well-behaved' strongly minimal set. Using the known $\aleph_1$-categoricity of the theories, we show they are model complete. In fact, no expansion by constants is needed. The result depends on the existence of exactly k-independent sequences. In the third section, we obtain directly a $\forall\exists$-axiomatization of the projective plane [2] and prove these axioms are $\aleph_1$-categorical. By Lindstrom, we conclude model completeness. And in the final section we review the significance of these results and suggest some further problems.

1 Some 'Ancient' model theory

We prove a sufficient condition for almost strong minimality that could have been proved long ago but the proper formulation was missed. For this, we must be careful about the adjunction of additional constants; we delineate this below. We will have to relativize certain notions to the domain in which they are computed; we indicate this by subscripting the ambient model.

We say the complete theory T' is a *principal extension* of T if the language of T' is obtained by adding a finite number of constants to that of T and if T' is axiomatized by adding one additional sentence to the axioms of T.

Notation 1.1 *$\phi(M, \mathbf{c})$ denotes the set of solutions of $\phi(v, \mathbf{c})$ in M.*

Definition 1.2 1. *For $X \subset M$, the* algebraic closure *of X in M, $\mathrm{acl}_M(X)$ is the set of elements a of M such that $M \models \phi(a, \mathbf{c}) \wedge (\exists^{<m} v)\phi(v, \mathbf{c})$ for some $\mathbf{c} \in X$ and some formula ϕ.*

2. *The theory T is* almost strongly minimal *(witnessed by T') if T' is a principal extension of T and there is a strongly minimal formula $\phi(x)$ (over the empty set in T') such that for every $M \models T'$, $M = \mathrm{acl}_M(\phi(M))$.*

We describe a technical condition on $M \subset N$ models of a theory T which guarantees that $M \prec N$. Recall that a definable subset X of N is minimal in N if every subset of X, that is definable with parameters in N is finite or cofinite. And X is strongly minimal if this condition remains true when parameters from an *elementary* extension of N are allowed.

Theorem 1.3 *Let $T = \mathrm{Th}(N)$ and suppose there is an $\mathbf{e} \in N$ and an existential formula $\phi(x, \mathbf{e})$ such that $\phi(x, \mathbf{e})$ is minimal in N and $\mathrm{acl}_N(\phi(N, \mathbf{e})) = N$. If $M \subseteq N$ is a model of T that satisfies: a) $\mathrm{acl}_N(M) = M$, b) $\mathbf{e} \in M$ and c) $\phi(N, \mathbf{e}) \cap M$ is infinite then $M \prec N$.*

Proof. Applying the Tarski-Vaught test we fix a formula $\psi(x, \mathbf{c})$ with $\mathbf{c} \in M$ that has a solution in N and show it has a solution in M. Let $r \geq 0$ be least so that there is an r-tuple $\mathbf{d}$ from $\phi(N, \mathbf{e})$ such that $\psi(N, \mathbf{c}) \cap \mathrm{acl}_N(M\mathbf{d}) \neq \emptyset$. If $r = 0$, we are done. If not, choose $\theta(v, y_1, \ldots y_r)$ such that for some $n < \omega$,

$$T \models (\forall \mathbf{y})(\exists^{<m} v)\theta(v, \mathbf{y})$$

and

$$N \models (\exists v)(\exists \mathbf{y})(\theta(v, \mathbf{y}) \wedge \psi(v, \mathbf{c}) \wedge \bigwedge_{1 \leq i \leq r} \phi(y_i, \mathbf{e})).$$

Consider the formula

$$\theta'(y_1) := (\exists v)(\exists y_2, \ldots, y_r)[\theta(v, y_1, \ldots, y_r) \wedge \psi(v, \mathbf{c}) \wedge \phi(y_1, \mathbf{e})],$$

which has parameters in M. Then $\theta'(y_1)$ has a solution in M. (If θ' is algebraic, this follows by the definition of algebraic closure; if not $\theta'(N)$ is cofinite in $\phi(x, \mathbf{e})$ (by minimality). Since $\phi(N, \mathbf{e}) \cap M$ is infinite, θ' has a solution in M.) But if e_1 is a solution of θ' in M, for some $e_2, \ldots, e_r \in N$,

$$\psi(N, \mathbf{c}) \cap \mathrm{acl}_N(Me_2, \ldots, e_r) = \psi(N, \mathbf{c}) \cap \mathrm{acl}_N(Me_1, e_2, \ldots, e_r) \neq \emptyset,$$

contradicting the minimality of r. $\square_{1.3}$

Lemma 1.4 *Suppose a countable saturated model N and formula $\phi(\mathbf{x}, \mathbf{e})$ satisfies the hypothesis in the first sentence of Theorem 1.3 and for every $M \models T$ with $M \subseteq N$, $\mathrm{acl}_N(M) = M$. Suppose further that $\mathbf{e}$ realises a principal type. Then the theory $T' = \mathrm{Th}(M, \mathbf{e})$ is model complete and witnesses that T is almost strong minimal.*

Proof. Consider a pair M_1, M_2 of countable models, $M_1 \subseteq M_2 \models T'$. We want to show $M_1 \prec M_2$. Since N is saturated, M_2 can be elementarily embedded in N. So it suffices to show that any substructure M of $(N, \mathbf{e})$ that satisfies T' is an elementary submodel. Condition a) of Theorem 1.3 is part of our hypothesis; condition b) holds since $\mathbf{e}$ realises a principal type. For condition c) note that since T is complete and $\mathbf{e}$ realises a principal type, $\phi(M, \mathbf{e})$ is infinite in the sense of M; since $\phi(x, \mathbf{e})$ is existential, this implies $\phi(N, \mathbf{e}) \cap M$ is infinite. The almost strong minimality is automatic since N is saturated. $\square_{1.4}$

The most trivial example shows that in general it is necessary to pass to T'. Consider the theory of (ω, S). Of course, if the universe is the algebraic closure with no new parameters of the strongly minimal set, which itself is defined without parameters, the original theory T is model complete.

2 Expansions of strongly minimal sets

In this section we contrast the non-model completeness of the infinite rank bicolored fields with the model completeness of the finite rank case. In fact, our arguments show arbitrary unary expansions (by the Hrushovski construction with μ) of sufficiently well-behaved strongly minimal sets are almost strongly minimal without parameters. Thus, we extend the model completeness from bicolored fields to a more general setting.

We will consider expansions of a strongly minimal set (in a language denoted L_f) by a unary predicate P (to form the language L); we say the points satisfying P are black. The following notations summarize the notions used below; details and justifications for these definitions are in [4, 5, 6]. Recall that $\mathbb{G}$ denotes the generic model for the relevant case and is ω-saturated.

Notation 2.1

1. *The base theory T_f in the language L_f admits elimination of quantifiers, elimination of imaginaries and has the definable multiplicity property.*

2. $\delta(\mathbf{x}) = \delta_k(\mathbf{x}) = k \cdot R_M(\mathbf{x}) - |P(\mathbf{x})|$*; for* $X \subset N$, $d_N(X) = \inf\{\delta(X') : X \subseteq X' \subset_\omega N\}$, *where R_M is Morley rank.*

3. $\overline{\boldsymbol{K}}_0$ *is the class of models of T_0 that have nonnegative dimension relative to $\delta_k(\mathbf{x})$. We write T_k^ω for the theory of the generic of $\overline{\boldsymbol{K}}_0$.*

4. *$\overline{\boldsymbol{K}}_0^\mu$ is the members of $\overline{\boldsymbol{K}}_0$ in which the number of realizations of each primitive are bounded by a finite-to-one function μ as in Definition 2.9 of [4]. We write T_k^μ for the theory of the generic in rank k.*

We assume familiarity with the basic properties of these constructions and proceed directly to the new results. In particular, we frequently use

Fact 2.2 *Every member of $\overline{\boldsymbol{K}}_0$ (of $\overline{\boldsymbol{K}}_0^\mu$) can be strongly embedded in the generic model $\mathbb{G}$ for $\overline{\boldsymbol{K}}_0$ (for $\overline{\boldsymbol{K}}_0^\mu$).*

If M is in $\overline{\boldsymbol{K}}_0$ and $\mathbf{c}$ realizes a complete L_f-type over M, we denote by $M[\mathbf{c}]$ the structure whose universe is the L_f-algebraic closure of $M\mathbf{c}$ and whose only black points are in $M\mathbf{c}$.

We now show that the infinite rank bicolored field fails to be model complete. Thus, it provides a somewhat less contrived example than Lindstrom's original one [11] of a theory which is $\forall\exists$-axiomatizable but not model complete. In fact, the argument extends to unary expansions of any strongly minimal set in which singletons have infinite algebraic closure.

Theorem 2.3 *Let $\overline{\boldsymbol{K}}_0$ be the class of countable bicolored fields with nonnegative dimension relative to $\delta(\mathbf{x}) = k \times R_M(\mathbf{x}) - |P(\mathbf{x})|$. T_k^ω is not model complete.*

Proof. By algebraic we mean 'L_f-algebraic'. Let k_0 be the algebraic closure of a point a_1 that is not algebraic and satisfies $\{a_1\} \leq k_0$. Let M_0 be a countable model of T_k^ω with $k_0 \leq M_0$. Let $\phi(\mathbf{x}, y)$ be a formula which asserts that $k+1$ elements are pairwise algebraic over y and distinct without implying that any of the elements are algebraic over y. Let $\langle b_0, b_1, \ldots, b_k\rangle$ be a sequence of black points that satisfy $\phi(\mathbf{b}, a_1)$. So $\delta(\mathbf{b}/k_0) < 0$. Embed $M_0[b_0, b_1, \ldots, b_k]$ in $\mathbb{G} \models T_k^\omega$ by Fact 2.2. Then M_0 is a submodel of $\mathbb{G}$ but not an elementary submodel because there is no $k+1$-tuple of distinct black points in M_0 satisfying $\phi(x_0, x_1, \ldots, x_k; a_1)$ as $k_0 \leq M_0$. $\square_{2.3}$

In contrast, we show that for each k the theory T_k^μ of a rank k unary expansion of a strongly minimal set [6] that satisfies Assumption 2.11 is model complete. For this, we use freely the 'code notation' described in [4]; this includes formulas such as $\theta_{\boldsymbol{C}}$, $\phi_{\boldsymbol{C}}$.

Notation 2.4 *For $N \models T_k^\mu$ and $\mathbf{b} \in N$,*

1. $\mathrm{tp}_N(\mathbf{b})$ *denotes the set of parameter-free L-formulas $\psi(\mathbf{x})$ such that $N \models \psi(\mathbf{b})$.*

2. $\mathrm{Diag}(\mathbf{b})$ *denotes the set of parameter free L_f-formulas $\psi(\mathbf{x})$ such that $N \models \psi(\mathbf{b})$. We need no subscript N because we have assumed that T_f admits elimination of quantifiers.*

3. *Following [4], we denote by $I(\mathbf{y})$ a collection of universal L-formulas such that for any $N \in \overline{\boldsymbol{K}}_0$ and $\mathbf{b} \in N$, if $N \models I(\mathbf{b})$ then $\mathbf{b} \leq N$.*

4. *Let* $\boldsymbol{c}$ *be a primitive code and suppose* $N \models \theta_{\boldsymbol{c}}(\mathbf{b}')$ *and* $\mathbf{b}' \subset \mathbf{b} \subset N$. *We write* $\chi_{\boldsymbol{c}}(\mathbf{b}) = k$ *to denote a first order formula that holds in* N *if the cardinality of a maximal set in* N *of pairwise disjoint solutions (each disjoint from* $\mathbf{b}$*) of* $\phi_{\boldsymbol{c}}(\mathbf{x}, \mathbf{b}')$ *is* k.

The next few lemmas show that if $M \subset N$ are models of T_k^μ then M is strong in N; indeed, M is d-closed in N (i.e. if $a \in N, X \subset M$ and $d_N(a/X) = 0$ then $a \in M$).

Lemma 2.5 *Let* $\mathbf{b} \leq N \models T_k^\mu$. *Then*

$$T_k^\mu \cup I(\mathbf{y}) \cup \mathrm{Diag}(\mathbf{b}) \models \mathrm{tp_N}(\mathbf{b}).$$

Proof. By saturation of $\mathbb{G}$, $\mathrm{tp}_N(\mathbf{b})$ is realized in $\mathbb{G}$ by some $\mathbf{b}'$. For some $M \models T_k^\mu$ let $\mathbf{c} \in M \models T_k^\mu$ satisfy $I(\mathbf{y}) \cup \mathrm{Diag}(\mathbf{b})$ (in the sense of M). Then $\mathrm{tp}_M(\mathbf{c})$ is realized in $\mathbb{G}$ by some $\mathbf{c}'$. But $\mathbf{c}'$ and $\mathbf{b}'$ are automorphic in $\mathbb{G}$ by genericity. So $\mathrm{tp}_N(\mathbf{b}) = \mathrm{tp}_{\mathbb{G}}(\mathbf{b}') = \mathrm{tp}_{\mathbb{G}}(\mathbf{c}') = \mathrm{tp}_M(\mathbf{c})$, as required. $\square_{2.5}$

Lemma 2.6 *If* $M \leq N$ *are models of* T_k^μ *there is no* $\boldsymbol{a} \in N - M$ *which is primitive over* M. *Thus,* M *is* d*-closed in* N.

Proof. Fix a primitive code $\boldsymbol{c}$ and $\mathbf{b}' \subset M$ such that $\theta_{\boldsymbol{c}}(\mathbf{b}')$. Let $\mathbf{b}$ be the intrinsic closure of $\mathbf{b}'$ in M (hence in N, since $M \leq N$). There is a maximal r such that $M \models \chi_{\boldsymbol{c}}(\mathbf{b}) = r$, witnessed, say, by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$. By Lemma 2.5, $N \models \chi_{\boldsymbol{c}}(\mathbf{b}) = r$. Therefore $\phi(\mathbf{x}, \mathbf{b}')$ is not realized in $N - M$ since any realization $\boldsymbol{a}_{r+1}$ would be disjoint from $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \mathbf{b}$ and so contradict the definition of $\chi_{\boldsymbol{c}}$.

$\square_{2.6}$

Remark 2.7 *Let* $\mathbb{G}$ *be generic for* T_k^μ. *The definition of genericity yields immediately that for* $aX \subset \mathbb{G}$, $d_{\mathbb{G}}(a/X) = 0$ *if and only if* $a \in \mathrm{acl}_{\mathbb{G}}(X)$. *For this, recall that if* $d_{\mathbb{G}}(a/X) > 0$, *then* a *has infinitely many conjugates over* X *in* $\mathbb{G}$.

Lemma 2.8 *Let* $M \subset N$ *be countable models of* T_k^μ. *Then* $M \leq N$. *Moreover, if* $M \subset \mathbb{G}$ *then* $acl_{\mathbb{G}}(M) = M$.

Proof. Suppose for the sake of contradiction that $M \not\leq N$. Let $A \subset N - M$ be minimal with respect to inclusion so that $\delta(A/M) < 0$. Necessarily, all elements of A are black. Let ℓ be the transcendence degree of A over M. Since M is L_f-algebraically closed, $\ell > 0$. Then,

$$k \cdot \ell - |A| = \delta(A/M) < 0.$$

Choose $\boldsymbol{a}$ from A of length $k \cdot \ell$ that extends a transcendence basis for A over M. Then $\delta(\boldsymbol{a}/M) = 0$ and $M \leq M\boldsymbol{a}$ so $\boldsymbol{a}$ contains a primitive $\mathbf{b}$ over M. But $M[\mathbf{b}]$ imbeds strongly into $\mathbb{G}$, contradicting Lemma 2.6 (with $\mathbb{G}$ as N and M as M).

Now suppose $M \subset \mathbb{G}$. We have just seen $M \leq \mathbb{G}$; whence by Lemma 2.6, M is d-closed in $\mathbb{G}$. By Remark 2.7, $acl_{\mathbb{G}}(M) = M$. $\square_{2.8}$

Since it follows easily from Zilber's Irreducibility Lemma (Proposition 2.12 of [13]) that finite rank fields are almost strongly minimal, we could conclude that the relevant T' is model complete. However, we have a stronger result.

We generalize to strongly minimal sets a property that the formula $x_1 + \ldots + x_k = 0$ has in either vector spaces or algebraically closed fields.

Definition 2.9 *A formula $\phi(x_1, \ldots, x_k)$ has* exactly rank $k-1$ *if for every generic (over the parameters of ϕ) solution $\boldsymbol{a} = \langle a_1, \ldots, a_k \rangle$ of $\phi(x_1, \ldots, x_k)$, $R_M(\boldsymbol{a}) = k-1$ and any proper subsequence of $\boldsymbol{a}$ is independent.*

Note that ϕ is exactly rank k, just if for any generic solution $\boldsymbol{a}$ of ϕ, a subset of $\boldsymbol{a}$ is independent if and only if it has at most k-elements: i.e. the sequence is exactly k-independent.

Remark 2.10 *Note that the k-ary $\phi(\mathbf{x})$ has exactly rank $k-1$ if and only if for any subsequence $\mathbf{x}'$ of $\mathbf{x}$ with length $r < k$, and any $\psi(\mathbf{x}')$ with Morley rank less than r,*

$$R_M(\psi(\mathbf{x}') \wedge \phi(\mathbf{x})) < k-1.$$

That is, ψ is not satisfied by a generic solution of ϕ.

We augment our requirement on the underlying theory T_f of a strongly minimal set by requiring that exactly rank k formulas are dense in the following sense.

Assumption 2.11 *The underlying theory T_f satisfies the following condition: If $\psi_i(\mathbf{x}, y)$ for $i < m$ are a finite set of $k+1$-ary formulas such that for some g, for each $i < m$, $\psi_i(\mathbf{x}, g)$ has rank at most $k-1$, there is a formula $\phi(\mathbf{x}, y)$ such that $\phi(\mathbf{x}, g)$ has exactly rank $k-1$ and for each $i < m$,*

$$R_M(\phi(\mathbf{x}, g) \wedge \psi_i(\mathbf{x}, g)) < k-1.$$

Remark 2.12 *In either vector spaces or algebraically closed fields the formula $x_1 + \ldots + x_k + y + z = 0$ gives us (as we fix g for y and substitute various elements e of the prime model for z) a family of disjoint exactly rank k formulas. Any theory of a strongly minimal set with such a disjoint family $\phi(\mathbf{x}, y)$ (or more weakly with $R_M(\phi(\mathbf{x}, a_i) \wedge \phi(\mathbf{x}, a_j)) < k-1$ if $i \neq j$) satisfies Assumption 2.11. For example, in vector spaces, for any g and $\psi_i(\mathbf{x}, y)$ for $i < m$ there is an e in the prime model such that $R_M(\psi_i(\mathbf{x}, g) \wedge x_1 + \ldots + x_k + g + e = 0) < k-1$.*

Note that trivial strongly minimal sets do not have exactly rank k formulas for $k > 2$. It is not hard to check that some structures constructed by the Hrushovski method (e.g. the original strongly minimal set) do have exactly rank k formulas in $k+1$ variables for every k. The construction of a bicolored field that is not ω-stable (although built by the Hrushovski construction) in [4] used essentially the fact that in algebraically closed fields for $m \leq n$ there are exactly rank m formulas in n variables.

Theorem 2.13 *Let $\overline{\boldsymbol{K}}_0^\mu$ be the class of those models of a well-behaved L_f-strongly minimal theory as specified in paragraph 2.1 and satisfying Assumption 2.11. There is an existential L-strongly minimal formula $\lambda(x)$ over the empty set such that the generic model satisfies $\mathbb{G} = \mathrm{acl}_{\mathbb{G}}(\lambda(\mathbb{G}))$.*

Thus, since $\mathbb{G}$ is satuated, the theory, T_k^μ, of the generic model is almost strongly minimal without naming parameters, i.e. with itself as the required T'.

Proof. Choose any L_f-formula $\rho(\mathbf{x})$ which asserts that $k-1$ elements are pairwise algebraic and distinct without implying that any of the elements are algebraic; thus $\rho(\mathbf{x})$ has infinitely many solutions. Then any $\boldsymbol{a}$ that is L_f-generic for $\rho(\mathbf{x}) \wedge \bigwedge_{i<k} P(x_i)$ is minimal strong over the empty set with $\delta(\boldsymbol{a}/\emptyset) = 1$. Now, $\lambda(x) := (\exists \mathbf{x}')\rho(x, \mathbf{x}')$ is L-strongly minimal (where $\mathbf{x} = x\mathbf{x}'$). To see this just note that in $\mathbb{G}$ for any $\boldsymbol{a} \in \mathbb{G}$, each $b \in \lambda(\mathbb{G})$ is either algebraic over $\boldsymbol{a}$ (if $d_{\mathbb{G}}(b/\boldsymbol{a}) = 0$) or is in the infinite orbit of those elements in $\lambda(\mathbb{G})$ with $d_{\mathbb{G}}(b/\boldsymbol{a}) = 1$. This gives minimality; strong minimality follows since $\mathbb{G}$ is saturated.

Now we show each element of $\mathbb{G}$ is in $\mathrm{acl}(\lambda(\mathbb{G}))$. Let g be an arbitrary element of $\mathbb{G}$ and let $G = \mathrm{icl}_{\mathbb{G}}(\mathrm{g})$. (Recall [4] that $\mathrm{icl}_{\mathbb{G}}(\mathrm{g})$ is the least subset of $\mathbb{G}$ containing g which is strong in $\mathbb{G}$.) Let $\mathbf{b}_i$ for $i < k$ be k independent (over G) black realizations of ρ. Write a_i for the first element of $\mathbf{b}_i$. Now each $\mathbf{b}_i$ is contained in a member of $\overline{\boldsymbol{K}}_0^\mu$ since $\delta(\mathbf{b}_i) = 1$ and $\mathbf{b}_i$ contains no primitive pairs. The only primitive pairs in the free amalgam of two finitely generated structures X_1, X_2 are those in one of the X_i. Let B denote the free union of the $\mathbf{b}_i$, $B \in \overline{\boldsymbol{K}}_0^\mu$; write BG for the free amalgam of B and G. Since no primitive pair is embeddable in any of the $\mathbf{b}_i$ and G is embeddable in a member of $\overline{\boldsymbol{K}}_0^\mu$, BG is embeddable in a member of $\overline{\boldsymbol{K}}_0^\mu$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\mathbf{v} = \mathbf{w}\mathbf{v}$ be an enumeration of variables for the elements of BG and also let u_i (as the first element of the sequence $\mathbf{w}_i$) stand for a_i. Let $\boldsymbol{a} = \langle a_1, \ldots a_{k-1}\rangle$ and $\mathbf{u} = \langle u_1, \ldots u_{k-1}\rangle$. Since each $\mathbf{b}_i$ is in $\mathrm{acl}(a_i)$ and since we will be concerned only with realizations of the diagram of $\boldsymbol{a}g$ that are independent from $\mathbb{G}$ over g, we need only to control the L_f-dimensions of subsets of $\boldsymbol{a}g$. Note that B has transcendence degree k (over both G and $\emptyset$) and has $(k-1)\cdot k$ elements which are all black; so $\delta(B/G) = k \cdot k - k \cdot (k-1) = k$. We want to choose an additional formula $\phi(\mathbf{u}, y)$ so that if B' is a solution of $\phi(\mathbf{u}, g) \wedge \bigwedge_{i<k} \rho_i(\mathbf{w}_i)$ containing $\boldsymbol{a}$, and B' is independent from $\mathbb{G}$ over g, then $\mathbb{G}[B']$ is a primitive extension of $\mathbb{G}$ that is in $\overline{\boldsymbol{K}}_0^\mu$. If we choose $\phi(\mathbf{u}, y)$ to be exactly rank k then $a'_0, \ldots, a'_{k-1}$ is exactly $k-1$ independent over g and this guarantees the primitivity. In particular, $\delta(B'/G) = k \cdot (k-1) - k \cdot (k-1) = 0$.

Now, possibly refining our choice of ϕ, we ensure $\mathbb{G}[B']$ is in $\overline{\boldsymbol{K}}_0^{\mu}$. Since μ is finite-to-one and BG is finite, there are only finitely many primitive codes $\boldsymbol{d}$ so that more than $\mu(\boldsymbol{d})$ realizations of $\boldsymbol{d}$ can occur in some set with the same cardinality as BG. Each possible arrangement σ of $\mu(\boldsymbol{d})+1$ realizations of $\boldsymbol{d}$ in BG is described by a formula $\psi_{\boldsymbol{d},\sigma}$ on a subset of the variables $\mathbf{wv}$. Since we are concerned only with solutions B' which are independent from $\mathbb{G}$ over g we may restrict to formulas of the form: $\psi_{\boldsymbol{d},\sigma}(\mathbf{u}', v)$ with $\mathbf{u}'$ a subsequence of $\mathbf{u}$. For each $\boldsymbol{d}, \sigma$, $R_{\mathbb{G}}(\psi_{\boldsymbol{d},\sigma}(\mathbf{u}', g)) < \lg(\mathbf{u}')$, since if a realization of $\psi_{\boldsymbol{d},\sigma}(\mathbf{u}', g))$ were independent from g and violated μ, BG would violate μ, contrary to hypothesis. If for some $\boldsymbol{d}, \sigma$, $\lg(\mathbf{u}') < k-1$, $\psi_{\boldsymbol{d},\sigma}(\mathbf{u}', g)$ is not satisfied by a generic solution of $\phi(\mathbf{u}, b)$, lest the fact that $\phi(\mathbf{u}, g)$ has exactly rank $k-1$ be violated. (See Remark 2.10.) So without loss of generality, each $\psi_{\boldsymbol{d},\sigma}(\mathbf{u}, y)$ is $k+1$-ary. It suffices to choose $\phi(\mathbf{u}, y)$ so that $\phi(\mathbf{u}, y)$ is exactly rank k and for each $\boldsymbol{d}, \sigma$,

$$R_{\mathbb{G}}(\phi(\mathbf{u}, g) \wedge \psi_{\boldsymbol{d},\sigma}(\mathbf{u}, g)) < k-1.$$

Assumption 2.11 guarantees we can choose such a ϕ. So if B' is a generic solution of $\phi(\mathbf{u}, g)$, $\mathbb{G}[B']$ is a primitive extension of $\mathbb{G}$, which is in $\overline{\boldsymbol{K}}_0^{\mu}$. Since $\mathbb{G} \in \overline{\boldsymbol{K}}_0^{\mu}$, $\mathbb{G} \leq \mathbb{G}[B']$ and $\mathbb{G}$ is d-closed in all strong extensions, $\phi(\mathbf{u}, g) \wedge \wedge_{i<k}\rho_i(\mathbf{w}_i)$ can be realized in $\mathbb{G}$ by some a'_i. The a'_i are k independent solutions of λ and $g \in \mathrm{acl}_{L_f}(a'_0, \ldots, a'_{k-1})$ as required. $\square_{2.13}$

Theorem 2.14 *If T_f satisfies 2.1 and satisfies Assumption 2.11 then T_k^{μ} is model complete. In particular, the rank k bicolored fields are model complete.*

Proof. In the general case apply Theorem 1.3, Lemma 2.8, and Lemma 2.13. Remark 2.12 guarantees the application to bicolored fields. Note that we don't need T' since we showed almost strong minimality without adding parameters. $\square_{2.14}$

3 The almost strongly minimal projective plane

In this section we show that the almost strongly minimal projective plane of [2] has a principal extension which is model complete. The argument is an elaboration of Holland's argument in [9] that the *ab initio* strongly minimal Hrushovski examples are model complete. The two innovations are the extension from rank 1 to rank 2 and the use of an extension by constants.

The first part of the argument, through Lemma 3.8, will work for any *ab initio* example. For concreteness, we work with the rank 2 case.

Fix a finite relational language L. If X and Y are L-structures, we write $X \subseteq Y$ to indicate X is a substructure of Y. In agreement with current terminology we replace the y of [2] by δ. We depend heavily on the development in [2] and [9]. In particular, for the projective plane, L contains a single binary relation which is constrained to be symmetric and irreflexive.

Definition 3.1 *Let M be an L-structure. For $X \subseteq_\omega M$, $r(X)$ is the number of (unordered) tuples $\overline{a}$ from X such that $M \models R(\overline{a})$ for some R in L. Let*

$$\delta(X) = 2|X| - r(X).$$

Primitive pairs are easy to define in this *ab initio* context.

Definition 3.2 1. *X is primitive over Y if $\delta(X/Y) = 0$ and for every proper subset X' of X, $\delta(X'/Y) > 0$.*

2. *The pair (X,Y) is a* minimal primitive pair *if X is primitive over Y and X is not primitive over any proper subset of Y.*

If X is primitive over M then there is a unique smallest $Y \subseteq_\omega M$ such that the pair (X,Y) is minimal primitive (take $Y = \{x \in M : \; R \cap ((XM)^n - M^n) \not\subseteq R \cap (XM - \{x\})^n$ for some n-ary $R \in L\}$).

Notation 3.3 *On page 701 of [2] we constructed arbitrarily large primitives over the empty set such that each primitive A contains three discrete points and four points so that no three are connected to a common point in A. Fix such a primitive A and let n be its cardinality.*

For any finite set G, $At_G(\mathbf{x})$ is the formula expressing the atomic diagram of G. We write $AT_{(P,G)}(\mathbf{v}, \mathbf{x})$ for the atomic diagram of PG where (P,G) is a primitive pair.

Definition 3.4 *Let T^μ be a collection of (universal) sentences, in the vocabulary of one binary relation symbol R and constant symbols $\{a_0, \ldots, a_{n-1}\}$, ensuring the following.*

1. *$\delta(X) \geq 0$ for all finite X.*

2. *Fix a set $A = \{a_0, \ldots, a_{n-1}\}$ (whose existence is guaranteed by the previous paragraph) that is primitive over the empty set and such that a_0, a_1, a_2 have no edges between them and no three of $a_3, \ldots, a_{n-1}$ are connected to a common point in A. Add an axiom saying the constants $\{a_0, \ldots, a_{n-1}\}$ form a structure isomorphic to this primitive structure A.*

3. *There is no 4-cycle in any model of T^μ.*

4. *Fix a function μ from (isomorphism types of) minimal primitive pairs (X,Y) into $\mathbb{N}$ with $\mu((X,Y)) \geq \delta(Y)$. For each pair (X,Y), the sentence*

$$\forall Y' \forall X_1, \ldots, X_{\mu(X,Y)+1} \bigwedge_i At_{(X,Y)}(X_i, Y') \rightarrow \bigvee_{i \neq j} X_i \cap X_j \neq \emptyset.$$

We write $\boldsymbol{K} = Mod(T^\mu)$; $\overline{\boldsymbol{K}}_0^\mu$ would closer to our usage in Section 2 but since we are dealing only with the finite rank case here, we suppress the parameters.

We use the formulation of the 'strong amalgamation lemma' from [9].

Lemma 3.5 *Let B_1B_2 be a free amalgam of B_1 and B_2 over $B_0 = B_1 \cap B_2$, where $B_1, B_2 \in \boldsymbol{K}$ and $B_1 - B_0$ is primitive over B_0. Suppose that (Q, F) is a minimal primitive pair with $F \subseteq B_1B_2$ and that $\mathbb{P}$ is a collection of $\mu(Q, F) + 1$ pairwise disjoint copies of Q over F in B_1B_2. Then the following hold.*

1. *$F \subseteq B_1$ and some element P of $\mathbb{P}$ is contained in $B_1 - B_0$.*
2. *If $B_0 \leq B_2$, then $F \subseteq B_0$ and $B_1 - B_0 = P$.*

For each $N \in \boldsymbol{K}$, we modify δ by defining for any finite $X \subset N$, $d_N(X) = \min\{\delta(X') : X \subseteq X' \subset_\omega N\}$. As usual [7] for any finite $X \subset Y \subset N$, $d_N(X) \leq d_N(Y)$ and d_N is lower semimodular: $d_N(XY) + d_N(X \cap Y) \leq d_N(X) + d_N(Y)$. Denote $d_N(XY) - d_N(Y)$ by $d_N(X/Y)$ for finite X, Y and $d_N(X/Y) = \inf\{d_N(X/Y_0) : Y_0 \subset_\omega Y\}$ for finite X and infinite Y. Then we can define a closure relation by $a \in \mathrm{cl_N(X)}$ if $d_N(a/X) = 0$. We call this relation d-closure. Note that the relation does not define a geometry as exchange fails.

Notation 3.6 *Let $\boldsymbol{K} = Mod(T^\mu)$ and let $\boldsymbol{K}^*$ denote the collection of $M \in \boldsymbol{K}$ such that if $M \leq N \in \boldsymbol{K}$, then for all $a \in N - M$, $d_N(a/M) > 0$. If $M \in \boldsymbol{K}^*$, we say M is d-closed in every strong extension in $\boldsymbol{K}$.*

Now, using Lemma 3.5, we produce a $\forall\exists$ axiomatization of $Th(\boldsymbol{K}^*)$; we show this theory is consistent and $\aleph_1$-categorical. Thus, it is complete by the Loś-Vaught test and model complete by Lindstrom's theorem. Let $\varphi_{(P,G)}(\overline{x})$ be a formula such that if $M \models At_G(\mathbf{g})$ and if $M[P']$ is an extension obtained by realizing $AT_{(P,G)}(\mathbf{v}, \mathbf{g})$ by a P' free from M over $\mathbf{g}$ then $M \models \varphi_{(P,G)}(\mathbf{g})$ if and only if for some minimal primitive pair (Q, F) with $QF \subseteq P'\mathbf{g}$, Q is realized too many times (i.e. at least $\mu(Q, F) + 1$ times) over F in $M[P']$. More formally, fix (P, G) and (Q, F), minimal primitive pairs, with $QF \subseteq PG$. Let $\overline{g}, \overline{q}, \overline{p}$ and $\overline{f}$ enumerate G, Q, P and F, respectively. Suppose that $\overline{q}_1, \ldots, \overline{q}_r$ are $r = \mu(Q, F) + 1$ pairwise disjoint copies of $\overline{q}$ over $\overline{f}$ in some free amalgam PM of PG with an element M of $\boldsymbol{K}$ over G. Let $\overline{s}$ enumerate $\bigcup rng\{\overline{q}_i\} - P$ and let $C(\overline{v}; \overline{y})$ be the atomic type of $\overline{sg}$. Note that there are only finitely many possibilities for C. Let $\varphi_{(P,G),(Q,F)}(\overline{x})$ be the disjunction over all such C of the formulas $\exists \overline{v} C(\overline{v}; \overline{x})$. For each minimally primitive pair (P, G) whose atomic type is realized in an element of $\boldsymbol{K}$, $\varphi_{(P,G)}(\mathbf{x})$ denotes the formula:

$$At_G(\overline{x}) \rightarrow \bigvee \varphi_{(P,G),(Q,F)}(\mathbf{x}),$$

where for fixed (P, G), the disjunction ranges over all minimally primitive pairs (Q, F) contained in PG. Note these axioms are $\forall\exists$.

Definition 3.7 *Let T^* denote the union of T^μ with the collection of all sentences $(\forall \mathbf{x})\varphi_{(P,G)}(\mathbf{x})$.*

Lemma 3.8 *T^* axiomatizes $\boldsymbol{K}^*$.*

Proof. If $M \models T^*$ then $M \in \boldsymbol{K}^*$ since if M has a proper primitive extension $M[P] \in \boldsymbol{K}$ (for some minimal primitive pair (P, G)) then some (Q, F) violates μ in $M[P]$ by the definition of T^*, contradicting $M[P] \in \boldsymbol{K}$. Conversely, if $M \not\models T^*$, for some (P, G) and $\mathbf{g} \in M$, $M \models \neg\varphi_{(P,G)}(\mathbf{g})$. Then if $M \models AT_G(\mathbf{g})$ and P' is a free realization of (P, G) over M, $M[P'] \in \boldsymbol{K}$. But $M \leq M[P']$ so $M \notin \boldsymbol{K}^*$. $\square_{3.8}$

Now we have two further properties of d-closure. Recall that we have constants for the elements of A in the language.

Lemma 3.9 *1. If $X \leq M \in \boldsymbol{K}^*$ and $Y \leq N \in \boldsymbol{K}^*$ and f is an isomorphism from X onto Y then f extends to an isomorphism from $\mathrm{cl_M(X)}$ onto $\mathrm{cl_N(Y)}$.*

2. If $X \leq M \in \boldsymbol{K}^*$, $\mathrm{cl_M(X)} \subseteq \mathrm{acl_M(X)}$.

Proof. 1) First consider P contained in M which is primitive over X. So $XP \in \boldsymbol{K}$. There is a copy $P'Y$ of PX, so that P' is primitive over Y and $YP' \in \boldsymbol{K}$. Now by Lemma 3.5, either there is an embedding of P' into N over Y or $NP' \in \boldsymbol{K}$. But the second case is impossible since $N \in \boldsymbol{K}^*$. With this base step in mind it is easy to construct the isomorphism by a back and forth.

2) The use of μ shows that if $X \leq M$, $XP \subset M$ and $d_M(P/X) = 0$, there are only finitely many copies of P over X in M so $P \subseteq \mathrm{acl}_M(X)$. $\square_{3.9}$

In this context, we require a slight variant on Fact 2.2, which can be proved by amalgamation and taking unions of chains. Note that the existence of a function μ guaranteeing that there are only a finite number of solutions for each primitive is essential for this version.

Lemma 3.10 *Each member of $\boldsymbol{K}$ can be strongly embedded in a member of $\boldsymbol{K}^*$.*

To interpret the models of T^* as planes, regard each point of M as both a point and a line. Interpret $R(a, b)$ to mean both the point a is on the line b and the point b is on the line a. Thus there is a built in polarity mapping a point a to the line Rxa. The following lemma implies that any model M of T^* is a projective plane under this interpretation. (The third projective plane axiom, that every pair of points lie on a unique line, follows from the first by the duality in the definition of the plane.)

Lemma 3.11 *The theory T^* implies the following.*

1. Any two lines intersect in a unique point.

2. There are four points with no three lying on a line.

Proof. If two lines intersected in more than one point there would be a square. But for any two points $a, b \in M$, adding a point c related to both is a primitive extension. By Lemma 3.5, either such c exists in M or in some strong extension. But M is d-closed in any strong extension, so there is such a $c \in M$. Every $N \models T^*$ is a strong extension of a copy of the primitive A, described in Definition 3.4. Condition 2) is witnessed in N by elements of this copy of A. $\square_{3.11}$

In any projective plane, given any line ℓ and two points a, b not on ℓ, the entire plane is in $\mathrm{dcl}(\ell, a, b)$. Thus, in any infinite plane, any line has the same cardinality as the plane.

In this next lemma we write Rxa_0 for the set of points in the ambient structure N which are R-related to a_0. We complete the proof by showing:

Theorem 3.12 *T^* is $\aleph_1$-categorical; thus complete and model complete.*

Proof. Let $N \models T^*$. The formula Rxa_0 defines a line in the projective plane. Since Rba_0 implies $d_N(b/a_0) \leq 1$, taking for $X \subset Rxa_0$ the closure of X as $\mathrm{cl_N(XA) \cap Rxa_0} = \mathrm{cl_N(Xa_0) \cap Rxa_0}$ defines a geometry on the line Rxa_0. I.e., exchange holds on the line. More specifically, the restriction satisfies all the axioms of Fact 2 of [9]. Suppose $(M, A), (M', A')$ are models of T^* with cardinality $\aleph_1$. Since each line has the same cardinality as the universe, and any infinite set X has the same cardinality as its d-closure, there is a 1-1 correspondence between the bases X, Y for the lines Rxa_0 and Rxa'_0. Note that the only edges between X and A (respectively Y and A') are the edges between a_0 (a'_0) and each point of X (Y). Moreover, A and A' are isomorphic, so there is an isomorphism f between AX and $A'Y$. Note that in general if $A \leq M$ and X is independent over A, $AX \leq M$. In particular, $AX \leq M, AY \leq M'$. By Lemma 3.9 1) f extends to an isomorphism of $\mathrm{cl_M(XA)}$ and $\mathrm{cl_{M'}(YA')}$. But $\mathrm{cl_M(XA) = M}$. To see this suppose $b \in M - \mathrm{cl_M(XA)}$. Note first that since X is a basis for the geometry on Rxa_0, every element of the line $\{x : Rxa_0\}$ is in $\mathrm{cl_M(XA)}$. If b is not on the line through a_1a_2, there is a line through a_1b which intersects the line a_0 in b' and a line through a_2b which intersects the line a_0 in b''. Now these lines are $\{x : Rxc_1\}$ and $\{x : Rxc_2\}$ for some c_1, c_2, so $d_M(b/c_1c_2) = 0$. But c_1, c_2 must be in $\mathrm{cl_M(XA)}$ since the line through a_0b' (respectively a_0b'') is fixed setwise by every automorphism which fixes AX. Then $b \in \mathrm{cl_M(XA)}$ as required. (A slight elaboration of this argument handles the case where b is on the line a_1a_2.)

Completeness is immediate by the Łos-Vaught test and model completeness follows since the axioms for T^* are $\forall\exists$. $\square_{3.12}$

4 Context and Further Problems

In the wake of Lindstrom's proof that a categorical $\forall\exists$ theory is model complete, there was speculation that categoricity would imply some approximation to model completeness. The results of Ahrbrandt-Baldwin [1] and [12], mentioned in the introduction, describe the totally categorical case. We have taken two different approaches here to show that a large family of examples are model complete. For the bicolored field case, we have used the known complete theory and proved it is model complete. In the projective plane case, we extended the language by constants, introduced a new $\forall\exists$ axiomatization, and proved it was categorical.

Recent work shows the importance of geometrical considerations when considering the relation between model completeness and $\aleph_1$-categoricity. Goncharov, Harizanov, Laskowski, Lempp, and McCoy [8] show that if a strongly minimal set has a trivial geometry then an extension by naming (possibly) infinitely many constants is model complete.

Our work is different in several respects. First, we are dealing with the family of examples generated by the Hrushovski construction. In particular, we are looking at expansions of strongly minimal sets. Second, the geometric condition we require is one of sufficient complexity, not of triviality. Further investigation is needed on the following questions.

Question 4.1 *What strongly minimal sets have (uniform families of) formulas with exactly rank k?*

We have some preliminary results on this issue. We used formulas with exactly rank k earlier [4] to show that the "finite-to-one" requirement is necessary to construct $\aleph_1$-categorical bicolored fields. In particular one might ask.

Question 4.2 *Must every nontrivial strongly minimal set have formulas of exact rank k for each k?*

In Section 3 we added constants to find a model complete theory.

Question 4.3 *Is some (any) projective plane constructed in this way model complete in the language of projective planes* without *constants?*

The theory of the projective place described above is almost strongly minimal via the constants a_0, a_1, a_2. Note however that a_1, a_2 are algebraic over the empty set (since A was chosen primitive). Thus the plane is actually algebraic over the line Rxa_0. In [3], we constructed a plane which is the definable closure of any line. But in that case there were no algebraic points.

Question 4.4 *Is every projective plane with a built-in polarity as described after Lemma 3.10 or in [3] necessarily the definable closure of a line?*

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