The Monster Model

John Baldwin

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Contemporary model theorists often begin papers by assuming 'we are working in a saturated model of cardinality kappa for sufficiently large kappa (a monster model). In every case I know such a declaration is not intended to convey a reliance on the existence of large cardinals. Rather, in Marker's phrase, it is declaration of laziness, 'If the stakes were high enough I could write down a ZFC proof'. As we note below, in standard cases the author isn't being very lazy; but formalizing a metatheorem expressing this intuition remains interesting.

The easiest way to find such a model is to choose kappa strongly inaccessible, thereby extending ZFC. I know of no first order example where this is necessary. In contrast there are uses of extensions of ZFC in infinitary model theory but they are explicitly addressed and do not arise through the monster model convention. In many cases the necessity of the extension is an open problem.

The fundamental unit of study is a particular first order theory. The need is for a monster model of the theory T. If M is a κ saturated model of T, then every model N of T with cardinality at most κ is elementarily embedded in M and every type over a set of size $< \kappa$ is realized in M. So every configuration of size less than κ that could occur in any model of T occurs in M.

Many use of this convention are to the study of ω-stable countable models (saturated models exist in every cardinal) or stable countable theories (there is a saturated model in λ if $\lambda^\omega = \lambda$. So there is no difficulty finding a monster. As model theory advanced to the detailed study of unstable theories, the choice of a monster model became more delicate.

In fact, the requirement that the monster model be saturated in its own cardinality is excessive. A more refined version of the 'monster model hypothesis' asserts: Any first order model theoretic properties of sets of size less than kappa can be proved in a κ-saturated strongly κ-homogenous model M (any two isomorphic submodels of card less than κ are conjugate by an automorphism of M). Such a model exists(provably in ZFC) in some κ' not too much bigger than κ. See Hodges (big models)[4] or or my new monograph on categoricity [1] for the refined version. (Hodges' condition is ostensibly stronger and slightly more complicated to state; but existence is also provable in ZFC.) Buechler [3], Shelah [7] Marker[5] expound harmless nature of the fully saturated version. Ziegler [8] adopts a class approach that could be formulated in Gödel Bernays set theory.

Replacing for all κ there exists κ' by 'there is one monster' is just a convenient shorthand for saying we can repeat the same proof for any given set of initial data.

There is of course a flaw in my description. What does 'any model theoretic property' mean? It would be valuable to formalize this notion but it has seemed unproblematic. Recently, however, there has been a concrete example of a property where finding the monster model is difficult.

Arising from problems is studying groups without the independence property, Newelski (in a preprint)[6] asked, what is the Hanf number for the property:

Let (T, T_1, p) be a triple of two countable first order theories in vocabularies $\tau \subset \tau_1$ and p be a τ_1-type over the empty set.

Specifically, Newelski asks, "What is the least cardinal kappa such that if there is a model N (of cardinality κ) of T_1 omitting p but such that the reduct of N to τ is saturated, then there are arbitrarily large such models?"

(Newelski saw computing this Hanf number (depending on the cardinality of τ_1) as an issue of computing the cardinality of the 'monster model').

Baldwin and Shelah show the Hanf number for this property is the same as the Löwenheim number for second order logic[2]. That is, 'as big as you want it be'.

http://www.math.uic.edu/ jbaldwin/pub/shnew8

This makes the meta-model theoretic problem more interesting. The formulation and proof of a general metatheorem is analogous to but seems much more tractable than the 'universes issue' in number theory and geometry.

References

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