Philosophical implications of the paradigm shift in model theory

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Model Theory: Philosophy, Mathematics and Language
Jan. 10, 2017

May 5, 2017

Provocation

The announcement for a conference on Philosophy and Model Theory in 2010 began:

> *Model theory seems to have reached its zenith in the sixties and the seventies, when it was seen by many as virtually identical to mathematical logic. The works of Gödel and Cohen on the continuum hypothesis, though falling only indirectly within the domain of model theory, did bring to it some reflected glory. The works of Montague or Putnam bear witness to the profound impact of model theory, both on analytical philosophy and on the foundations of scientific linguistics.*

Response

My astonished reply to the organizers[1] began:

> *It seems that I have a very different notion of the history of model theory. As the paper at (Review of Badesa) points out, I would say that modern model theory begins around 1970 and the most profound mathematical results including applications in many other areas of mathematics have occurred since then, using various aspects of Shelah's paradigm shift. I must agree that, while in my view, there are significant philosophical implications of the new paradigm, they have not been conveyed to philosophers.*

[1] Letter to Halimi, September 20, 2009.

Forthcoming book

Model Theory and the Philosophy of Mathematical Practice:

Formalization without Foundationalism

This talk

Philosophical Issue: What is a paradigm shift?

Response today

I describe in some detail a specific 'paradigm shift in mathematics'
vaguely – a major change in the fundamental questions and techniques of a mathematical area

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Issues for later

What is a general definition of paradigm shift which encompasses this example and others?

Reference: Gillies, Revolutions in Mathematics

I raise other issues in philosophy and history of mathematics.

Two Theses

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

The Paradigm Shift

What paradigm shift?

Before

The paradigm around 1950 concerned the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

What paradigm shift?

After

After the paradigm shift there is a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

In this framework one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

What is the role of Logic?

Logic is the analysis of methods of reasoning

versus

Logic is a tool for doing mathematics.

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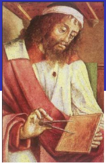
Logic is a tool for doing mathematics.

More precisely,
Mathematical logic is tool for solving not only its own problems but for organizing and do traditional mathematics.

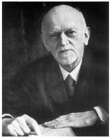
Euclid-Hilbert formalization 1900:



The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language.
For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1917-1956:


Hilbert


Gödel


Tarski


Vaught

In the Hilbert (the founder of proof theory)-Gödel-Tarski-Vaught framework, logic is a mathematical subject.

Vocabulary is chosen for the particular topic.
There are explicit rules for defining a formal language and proof.
Semantics is defined set-theoretically.

The completeness theorem establishes the equivalence between syntactic and semantic consequence.

Formalization

Definition

A *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
    1. Specify a class[2] of well formed formulas.
    2. Specify truth of a formula from this class in a structure.
    3. Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

This talk focuses on first order logic.

[2]For most logics there are only a set of formulas, but some infinitary languages have a proper class of formulas.

Theories

Contemporary model theory focuses on theories not logics.

Theories may be given by axioms (first order Peano) or as $\mathrm{Th}(M)$ (true arithmetic).

Examples

algebraically closed fields, dense linear order, the random graph, differentially closed fields, free groups, ZFC,

$\mathrm{Th}(Z, S)$
$\mathrm{Th}(Z, +)$
$\mathrm{Th}(Z, +, 1)$
$\mathrm{Th}(Z, +, 1, \times)$

pragmatic criterion

Properties of theories: complete, model complete, decidable, categorical, categorical in power, ω-stable, stable, $\pi_2 - axiomatizable$, finitely axiomatizable

Criterion

A property of a theory T is virtuous if it has significant mathematical consequences for T or its models.

Under this criteria

1. categoricity of an informative axiomatization of a 2nd order theory is virtuous.
2. Categoricity of $\mathrm{Th}^2(M)$ is not virtuous.
3. completeness of a first order theory is virtuous.
4. categoricity in uncountable power of a first theory (with infinite models) is virtuous.

Complete Theories

Complete theories are the main object of study.

Kazhdan:

> *On the other hand, the Model theory is concentrated on [the] gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model.*
>
> *One of the main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.*
>
> *Different models of complete theories are of different flavors and support different intuitions.*

Philosophical Issues

Philosophical

What are the criteria for choosing the logic, vocabulary, axioms?

In particular, are the properties of the deductive system relevant to this choice?

Historical Issues

First order logic and fixing vocabulary

Church (1956) still thinks of first order logic as a subsystem of a higher order functional calculus.

Tarski, Robinson, and Henkin (based on the 1935 definition of class of algebras by Garrett Birkhoff) are moving towards the modern concept fully stated in

Tarski-Vaught 1956 (but also Tarski-Mostowski-R. Robinson 1953 and A. Robinson, 1951)

The role of definition

How do apparently minor technical shifts in terminology reflect major changes in viewpoint?

other examples: truth in a model, q.e. by fiat, T^{eq}

The Role of Set Theory

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

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Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$.

   We write $(\kappa, \lambda) \to (\kappa', \lambda')$

   if *every theory* that admits (κ, λ) also admits (κ', λ').

Set Theory Becomes Central in the 60's

Vaught asked a 'big question', 'For what quadruples of cardinals does $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ hold?'

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Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

Revised Theorem: solved in ZFC

Suppose

1. [Shelah, Lachlan $\approx$ 1972] T is stable
2. or [Bays 1998] T is *o*-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if T admits (κ, λ)

then T also admits (κ', λ').

Ask the right question

$P(\kappa, \lambda, T)$ means, 'there is a (κ, λ)-model of T.'

Reversing the question

before the shift:
For which cardinals does $P(\kappa, \lambda, T)$ hold for all theories ?

after the shift:
For which theories does $P(\kappa, \lambda, T)$ hold for all cardinals ?

Why does it matter?

Morley's categoricity theorem

A countable first order theory is categorical in $\aleph_1$ if and only if it is categorical in every uncountable cardinal.

B-Lachlan characterization

A countable first order theory is categorical in $\aleph_1$ if and only if it is

1. ω-stable
2. has no two-cardinal model

That is, Each model is determined by the dimension of a strongly minimal set.

From Boole to Shelah

Analogy to Theorem to method

Analogy

Schlimm explains, successive analogies led to theorems:

1. (Boole) propositional logic with algebra
2. (Stone) Boolean algebras with rings
3. (Tarski) deductive systems with Boolean algebra

Theorem

Theorem

[Stone representation theorem] There is a 1-1 correspondence between Boolean algebras and totally disconnected Hausdorff spaces.

Reinterpretation: Lindenbaum-Tarski theorem

1. There is a duality between the Boolean algebra of sentences (up to T-equivalence) and the totally disconnected Hausdorff space $S_0(T)$ of all completions of T.
2. There is a duality between the Boolean algebra of formulas with n-free variables (up to T-equivalence) and the totally disconnected Hausdorff space $S_n(T)$ of all complete n-types of T.

Reinterpretation: The model theoretic view

A complete n-type over the empty set is a description of an n-tuple (over the empty set).

Replace T by $\mathrm{Th}(M, A)$ where $M \models T$ and $A \subset M$.
A complete n-type in $S_n(\mathrm{Th}(M, A))$ is a description of an n-tuple over A.

Definition

Write $S_n(M, A)$ for $S_n(\mathrm{Th}(M, A))$.

The complete theory T is λ-stable if for every $M \models T$ and every $A \subset M$,

$$|A| \leq \lambda \Rightarrow S_n(M, A) \leq \lambda.$$

Semantic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T is stable in no λ.
2. (strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$
3. (superstable) T is stable in those $\lambda \geq 2^{\aleph_0}$.
4. (ω-stable) T is stable in all infinite λ.

Syntactic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T has the order property; some formula $\phi(\mathbf{x}, \mathbf{y})$ defines a linear order on M^n.
2. (stable) For every formula ϕ, there is a rank R_ϕ so that for every formula ψ, $R_\phi(\psi) < \omega$.
3. (superstable) There is a global rank R_C (with respect to n-inconsistency) such that $R_C(\psi) < \infty$ for all ψ.
4. (ω-stable) There is a global rank R_M (with respect to inconsistency) such that $R_M(\psi) < \infty$ for all ψ

From all theories towards classification

Theorem

1. the (strict) hierarchies on the last two slides are the same.
2. The defining conditions are either arithmetic or Π^1_1, so absolute in ZFC.

Historical Consequence

After the paradigm shift first order model theory is no longer entangled with set theory.

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Question

What constitutes syntax?

Shelah classification strategy

A property P is a dividing line if
both P and $\neg P$ are virtuous.

Stable and superstable are dividing lines

ω-stable and $\aleph_1$-categorical are virtuous but not dividing lines.

Two further Theses

3. The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.

4. The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself (through the medium of geometric stability theory) plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics.

Three kinds of geometry

1. First order Euclidean geometry
2. first order formalizations of real and complex algebraic geometry
3. combinatorial geometry

Dimension: the essence of geometry

Dimension is a natural generalization of the notion of two and three dimensional space.

With coordinatization, the dimension tells us how many coordinates are needed to specify a point.

unidimensionality and categoricity in power

This dimension (for a countable language) and uncountable strongly minimal (more generally $\aleph_1$-categorical) structure is the same as the cardinality of the model.

The role of geometry

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

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Bourbaki's 3 great mother structures

order, groups, topology

ADD geometry

Geometric Stability Theory

Classification

The geometries of strongly minimal sets (regular types) fall into 4 classes:

1. discrete (trivial) $(\mathrm{cl}(ab) = \mathrm{cl}(a) \cup cl(b))$
2. modular or vector space like: (the lattice of closed subsets of the geometry is a modular lattice).
3. field-like (somehow bi-interpretable with a field).
4. none of the above: non-desarguesian but not vector space like.

This classification has immense consequences in both pure and applied model theory.

Geometry and Algebra are inevitable

Zilber / Hrushovski

Abstract model theoretic conditions
e.g. $\aleph_1$ categorical theories that are not almost strongly minimal
imply the existence of definable group and fields

and (more technical hypothesis)
imply the group is an abelian or a matrix group over an ACF of rank at most 3.

Why does this matter?

Why does this matter to model theorists and philosophers?

The Main Gap: No Structure or structure

Let T be a countable complete first order theory.

1. Either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ or
2. T is superstable without the *omitting types order property* or the *dimensional order property* and is shallow whence
   1. each model of cardinality λ is decomposed into countable models indexed by a tree of countable height and width λ.
   2. and thus, for any ordinal $\alpha > 0$, $I(T, \aleph_\alpha) < \beth_\delta(|\alpha|)$ (for a countable ordinal δ depending on T);

Either there is uniform way to assign invariants or there is the maximal number of models in every uncountable power.

Why does this matter to mathematicians?

1. (unstable)
   linear order, Boolean algebras, set theory, Peano Arithmetic
2. (strictly stable)
   separably closed fields, $(Z, +, 1)^{\omega}$, DCF_p, free non-abelian groups, any abelian group
3. superstable
   $(Z, +, 1)$, (Z_p^n, H_i), finitely refining sequences of equivalence relations
4. (ω-stable)
   ACF_0, ACF_p, matrix rings over ω-stable fields, $((Z_4)^{\omega}, +)$, DCF_0, complex compact manifolds,

Does this matter to mathematicians?

First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis

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1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis
2. Definable analysis
   Functions are defined implicitly:
   real exponentiation, number theory

Does this matter to mathematicians?

Substantial Applications

1. number theory and Diophantine geometry
2. real algebraic geometry
3. compact convex manifolds
4. real exponentiation
5. complex exponentiation
6. differential algebra
7. motivic integration
8. asymptotic analysis
9. combinatorial graph theory

Why might this matter to philosophers?

Martin Davis wrote:

Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents.

I disagree

Contemporary model theory provides several methods for taming mathematical problems

1. formalize the topic as a stable or o-minimal first order theory.
2. imbed the problem in a 'stable piece' of a natural structure.

It's inevitable: Abstract Model theory to algebra

Hart, Hrushovski, Laskowski

Any model of a complete theory, whose uncountable spectrum is

$$I(\aleph_\alpha, T) = \min(2^{\aleph_\alpha}, \beth_{d-1}(|\alpha + \omega| + \beth_2))$$

for some finite $d > 1$,

interprets an infinite group.

Reliability or Clarity

. . . a long-term look at achievements in mathematics shows that genuine mathematical achievement consists primarily in making clear by using new concepts *. . .*

We look for uses of mathematical logic in bringing out these roles of of concepts in mathematics. Representations and methods from the reliability programs are not always appropriate.

We need to be able to emphasize special *features of a given mathematical area and its relationship to others, rather than how it fits into an absolutely general pattern. (Manders 1987)*

Summary

1. Contemporary model theory makes formalization of *specific mathematical areas* a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
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Dedekind's Structuralism

Theorem: Dedekind

The sentence ϕ asserting a) f is a 1-1 function and 0 is the only element not in the range

$$b) \quad \forall X[((\forall x)x \in X \rightarrow f(x) \in X) \rightarrow (\forall x)x \in X]$$

is categorical.

First order version

a) plus omit the type $\{x \neq f^n(0) : n < \omega\}$ is categorical

Dedekind's Structuralism: II

In either case, *the natural numbers* is seen as the *isomorphism type* of such structures (N, f).

What is an isomophism type?

possible answers

1. mathematician: 'Who cares?' -any such structure is representative
2. Choose representative canonically in set theory
3. Dedekind: concept
4. A formula (or class) of GB defining the class of such structures
5. A second order sentence describing the class (inadequate answer for mathematics)