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REFLECTIONS ON THE DEZOLT AXIOM: MATHEMATICAL, PHILOSOPHICAL AND HISTORICAL

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Contents

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1. Introduction

Hartshorne's question

> We avoid the use of the words 'greater' and 'lesser' because these imply the existence of an order relation among figures which we have not yet established. In fact, the existence of an order relation for content depends on (Z) (Exercise 22.7). [We will also see that 'if squares are equal then their sides are equal' follows from (Z) (Exercise 22.6).]
>
> I do not know of any purely geometric proof of axiom(Z) *from the definition of content [area] we have given.* . . . (Z) holds however, whenever there is a measure of area function defined in the geometry.
>
> ([Har00, p.202])

He repeats the same sentiment in more detail on page 210 with additional detail. 'The proof [of De Zolt and of area function] is analytic in that it makes use of the field of segment arithmetic and similar triangles.' ([Har00, p.210]).

These passages raise several questions. What is De Zolt's axiom (Z)? What is De Zolt good for? What is a geometric proof? Hartshorne's answer to the first is:

Axiom 1.1. *(De Zolt (Z)* [Har00, p 201]*) If Q is a figure contained in another figure P, and if P - Q has non-empty interior then P and Q do not have equal content*[1]*.*

[1] We use *equidecomposable* for 'scissors congruence, dissection' and *equal content or equicomplementable* when subtraction is allowed as in Euclid I.35.

This formulation reflects the influence of the Grundlagen. De Zolt's version[2] is

> If a polygon is divided into parts in a given way, it is not possible, when one of these parts is omitted to recompose the remaining parts in such a way that they cover entirely the polygon.
> (De Zolt 1881 p. 12)

The underlying problem is this. It is quite clear that congruence classes of line segments are linearly ordered by $AB < CD$ if AB is congruent with a subsegment of CD. But a similar relation in two dimensions is complicated; there are certainly polygons P and Q where Q has lesser area than P (at least by a naive calculation) but P is not congruent with a subset of Q. De Zolt used the postulate above to argue for such an ordering [Gio21, §3.2]. But there is an important distinction; Hartshorne uses the property 'equal content' (so baptized by Hilbert but with the meaning of 'equal triangle' in Euclid I.35) as opposed to De Zolt's use of equidecomposition or scissors congruence[3] – a standard practice in the 19th century. As we see in Fact 4.2, for Archimedean planes these two notions define the same equivalence relation on pairs of figures, albeit witnessed differently. Hilbert aimed to establish a theory of plane area without use of the Archimedean axiom[4]. For that, he rigorously established another notion of equal area: the carpenter's area of a $2ft{\times}3ft$ rectangle is 6 sq ft. We take this up with the discussion of Definition 5.3.

We come to the fundamental dilemma posed by Hartshorne's question. Both De Zolt's axiom and the existence of a measure of area function hold in any model of $HP5$ (neutral geometry plus parallels (Definition 3.1). The extant proof uses a measure of area function. Is this technique necessary? More precisely, in any plane satisfying $HP5:$ DeZolt and a measure of area function exists. How can we possibly show they are independent (or dependent)? Frege faced the same problem in his dialog with Hilbert. He felt that the various axioms of geometry all were inherent in the fundamental concepts but nevertheless separately needed to expound geometry. But since they were all true, one could not apply Hilbert's 'formal method' [Bla07] to show independence. We are not in the same boat but a very similar one.

Our approach is similar to that hinted at in [**?**] of finding different geometrical interpretations (some where the hypothesis is true but the conclusion fails) of the same propositions. For this we formulate in Section 5 a general theory of equivalence relations on magnitude (e.g. 'equal area (or volume)'). This will allow us to compare properties of such equivalence relations that depend on a number of factors, including: the definition of figures, the background theory, the shape of the figures and the dimension of the space.

[Har00, Exercise 22.6] shows that in a Euclidean plane (Definition 3.1) that satisfies De Zolt for equicomplementation, also satisfy the 'squares property': equicomplementable squares are congruent. We are faced again with the dilemma. What does such a proof show, given that both the hypothesis and the conclusion are true (in EG). How can there be a non-trivial implication between two true assertions? By working in the general framework of Section 5, we identify the notion 'scaled'

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[2]As it appears in [Gio21]

[3]In [Hil62, §18] Hilbert writes 'equal area'; he later renamed the notion equidecomposable.

[4]It seems that much modern research, especially in higher dimensions, has discarded this generality. That is, such references as [Dup01] work over the reals without ever mentioning it. [Bol78, p 74] restricts to the Archimedean context in a sentence; it is not clear for how long the restriction is intended to hold.

Definition 5.5 and show in Theorem 5.7 that scaled plus the squares property imply de Zolt. This general argument shows that squares property could be replaced by many similar scales (e.g. rectangles of the same height), witnessing the pivotal role of being well-scaled.

2. Common Notions

Euclid's Elements [Euc56] contains the following general properties of magnitudes.

Definition 2.1 (Common Notions)**.**

1. *Things which equal the same thing also equal one another.*
2. *If equals are added to equals, then the wholes are equal.*
3. *If equals are subtracted from equals, then the remainders are equal.*
4. *Things which coincide with one another equal one another.*
5. *The whole is greater than the part.*

Hilbert insists that Euclid errs in positing through the common notions general principles for the comparison of magnitudes [Hil62, p 62]. Rather, precise versions of each principle must be *proved* for each topic. Such a view is consistent with De Risi's [DR20] careful historical analysis arguing that the common notions were unlikely to be written by Euclid, but were added to clarify implicit hypotheses in his argument. Similarly, [Har00, p. 196] lists minor variants of the common notions and adds the first two of the following Euclidean assumption about 'equal area'

Definition 2.2 (Further properties of Area)**.**

1. *Congruent figures are 'equal'.*
2. *The* squares property *for 'equality': equal squares have equal (congruent) sides*[5]*.*
3. *The* rectangle property *for 'equality': 'equal' rectangles with corresponding (e.g. shorter) sides equal have the other side equal and are congruent figures*[6]*.*

We have written 'equal' to emphasize that the equivalence relations described in Definition 5.4 are possible substitutions. [Har00, Corollary 22.5] asserts

Fact 2.3. *If 'equals' is taken as equicomplementability the following hold.*

1. *In any Hilbert plane equicomplementability satisfies Definition 2.1 2 and 3.*
2. *And further, in any plane satisfying $HP5$ if equicomplementability satisfies Z,it also satisfies*[7] *4 and 5 of Definition 2.1 for area, the rectangle property (Definition 2.2.3) and the squares property (Definition 2.2.2).*

We prove a converse, Theorem 5.7.3, to the second statement: any model of HP5 (any Euclidean plane) satisfies the rectangle property (the squares property[8]). We

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[5] Equivalently, are congruent. De Zolt [DZ81] emphasizes Euclid ambiguous use of 'equal'.

[6] [Hil62, p. 61] states this problem and proves it holds in models of $HP5$ in the following section.

[7] [Har00, Corollary 22.5] asserts the second part only for Euclidean fields. But while circle-circle intersection is needed to convert arbitrary figures to rectangles or squares, it is not needed for the rectangle or square properties.

[8] Note that, using Hilbert's measure of area function, a model of HP5 for equicomplementation, satisfies de Zolt for equicomplementation.

prove the result in more generality to emphasize that the key property of equicomplementation is that is it well-scaled, if the squares property holds.

3. Logical Foundations

In [Gio21], Giovannini describes the context in which the issues discussed here arose: the need in the face of calculus to clarify rigorous foundations for geometry. These efforts concerned both the clarification of the axiomatic basis and an attempt to eliminate 'infinitary' methods such as limits.

The culmination of the effort[9] for a clear axiomatization of geometry was [Hil62]. As stressed in [Bal18b, §9], Hilbert and Euclid take different routes. Euclid first studies the theory of area by equicomplementability [Euc56, Books I, II] and then, invoking Archimedes and general properties of proportionality, studies similarity [Euc56, Book VI]. While, Hilbert develops segment arithmetic to justify proportion and uses a measure of area function to ground the comparison of areas. We take from [Har00] abbreviations for important subcollections of Hilbert's axioms.

Notation 3.1. Consider the following axiom sets[10].

(1) First-order axioms:
  - **HP, HP5:** We write HP (Hilbert plane) for Hilbert's incidence, betweenness[11], and congruence axioms. We write HP5 for HP plus the parallel postulate. HP is often known as absolute or neutral geometry. We will just say neutral geometry when discussing 3-space.
  - **EG:** The *axioms for Euclidean geometry*, denoted EG[12], consist of HP5 and in addition the circle-circle intersection postulate CCP.

(2) Hilbert's group continuity axioms, must be formalized in infinitary and second-order logic[13]
  - **Archimedes:** The sentence in the logic[14] $L_{\omega_1,\omega}$ expressing the Archimedean axiom.

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[9][Hil62] and all further editions are written in natural language. Hilbert's 1918 lectures [Hil13] made clear the first order (restricted predicate calculus) nature of the essential axioms in the Grundlagen; Tarski [Tar59, GT99] provides a full account in a single sorted language. This was done much earlier than the published accounts.

[10]The names HP, HP5, and EG come from [Har00]. Tarski also studies EG under the name $\mathcal{E}_2''$ in [Tar59].

[11]These include Pasch's axiom (B4 of [Har00]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[12]There is a natural translation of 'Euclid's axioms' into first-order statements in a vocabulary with ternary betweenness and collinearity predicates, 4-ary (6-ary) congruence relations on pairs of segments (on triples representing angles) (e.g. [Bal18b, BM12]). The construction axioms have to be viewed as 'for all – there exist' sentences. The axiom of Archimedes is of course not first-order. We write Euclid's axioms for those in the original as opposed to modernized (first-order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [ADM09].

[13]Hilbert's completeness axiom is not even in second order [Vää14]; we use the more familiar Dedekind axiom.

[14] The Archimedean axiom is a property of an ordered semigroup. In the logic, $L_{\omega_1,\omega}$, quantification is still over individuals but now countable conjunctions are permitted so it is easy to formulate Archimedes axiom: $\forall x, y (\bigvee_{m\in\omega} mx > y)$. By switching the roles of x and y we see each is reached by a finite multiple of the other. In proving Fact 3.3, we use the equivalent version expressing that for every m for every x, there is a y, $y < \frac{s}{m}$.

Dedekind: Dedekind's *second-order* axiom[15] that there is a point in each irrational cut in the line.

The distinction between the first order and the infinitary/second order axioms is crucial. The collection of all the axioms has a unique model which is the Euclidean plane over the real numbers. But Hilbert was laying out a much more flexible and general arena. $HP5$ is an incomplete first order theory. While the distinction between first and second order was not fully grasped in 1899, in Hilbert's lectures of 1918, where the restricted predicate calculus is clearly seen a restriction of higher order logic, he observes the significance of the first order axioms of the Grundlagen.

Remark 3.2. *A note on definitions*From the standpoint of modern logic an *explicit definition* is an abbreviation for a first order formula; an *implicit definition* of a predicate P is given by a first order formula in the given vocabulary with a symbol for P such that in every model of the theory P designates a unique relation. Note that concepts like equidecomposable meet neither of these conditions. They metamathematical notions that are proved to behave well.

In the Grundlagen, fundamentally geometric propositions are proved in HP5, while the axiom of Archimedes is used primarily to construct counterexamples. The actual use is a few pages that show the only model of all the axioms, including Hilbert's version of the Dedekind postulate, is the Cartesian plane over $\Re$.

Hilbert proves the axiom of Archimedes is independent from various axiom groups of HP5. He shows the independence of the first order axioms from Archimedes using standard techniques from number theory. These proofs might suggest that coordinatization and knowledge of polynomial and power series rings are central for the following result. However, this is not the case. A simple argument is:

Fact 3.3. *There is a non-Archimedean plane satisfying $HP5$.*

Proof. Fix points A and C on a plane satisfying $HP5$. Consider the set of sentences[16] $\phi_n(A,B,C)$ where $\phi_n(A,B,C)$ asserts the B_n are decreasing, $B_1C \approx AB_1$ and for each i, $2 \leqslant i \leqslant n-1$ $B_iB_{i-1} \approx AB_i$. Since each ϕ_n is true for an appropriate choice of B_{n+1} to witness B, the compactness theorem[17] for first order logic implies there is a B_∞ such that $\phi_n(A, B_\infty, C)$ for every n. Then B_∞ is an infinitessimal. $\blacksquare_{3.3}$

We turn now to the methodological problems. The main result, Theorem 5.7 is true in (every model of) the first order theory $HP5$. It would not makes sense to say it is proved in $HP5$. 'Proved in first order theory' means there is a deduction in the first order predicate calculus[18] of a statement of first order logic in a fixed

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[15]Hilbert added his *Vollstandigkeitsaxiom* to the French translation of the 1st edition and it appears from then on.

[16]In symbols, using B as an antisymmetric (to avoid extra notation for order) strict betweenness predicate:

$$\exists B_1, \dots B_n (B(C, B_1, B_2) \wedge \bigwedge_{2 \leqslant i \leqslant n-2} B(B_i, B_{i+1}, B_{i+2}) \wedge (B(B_{n-1}, B_n, A) \wedge B(B_n, B, A)$$

[17]This theorem asserts that if each finite subset of an infinite set of sentences is consistent then the entire set is here. In the application here we consider the axioms of HP5 and the ϕ_n.

[18]Using any of the many, equivalent by the completeness theorem, different sets of axioms and rules of inference for first order

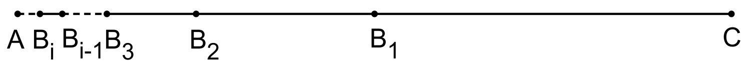

FIGURE 1. Euclid I.35

vocabulary[19] for the subject axiomatized. But notions such as equidecomposable (or equicomplementable) are not first order; 'equidecomposable' is equivalent to an infinite disjunction of first order formulas: equidecomposable by n-triangles for each n. Normal (unformalized) mathematics proves (in set theory) that such statements as De Zolt (1.1) are true in every model of $HP5$ [Har00, Proposition 23.1.d]; neither the measure of area function nor the equicomplementability are first order definable notions.

4. The role of Archimedes

While Euclid (or his successors) had postulated through the common notions, that there was general theory of magnitudes that in particular applied to both area and volume, there was divergence in his treatment of the two.

Remark 4.1. While Euclid implicitly[20] gives a formula for the area of triangle, he uses the method of exhaustion to show 'triangular pyramids of the same angle are to each other as to their bases.' Removing such limit processes is one of the goals of 19th century rigorizing.

In the last decades of the 19th century such geometers as Stolz, Schur, and De Zolt were well aware of using the Archimedean axiom[21] One of Hilbert's achievements was to clarify exactly where the Axiom of Archimedes is needed in geometry. His third problem asked whether this role changed in passing from two to three dimensions.

Fact 4.2 (Wallace-Bolyai-Gerwien Theorem)**.** *Two polygons* in an Archimedean plane *are equidecomposable (scissors congruent) if and only if they have the same measure of area.*

The analog in three dimensions would drop the hypothesis of same Dehn invariant. But Hilbert's third problem suggested and Dehn proved that that is impossible.

Fact 4.3 (Dehn-Snyder Theorem)**.** *Two polyhedra in $\Re^3$ are scissors congruent iff they have the same volume and the same Dehn invariant.*

Dehn [D] proved in 1901 that equality of the Dehn invariant is necessary for scissors congruence. Snyder proved the converse forty years later. The cube has Dehn invariant 0 (and tiles the 3-space). The other Platonic solids have non-zero invariant and do not tile[22] 3-space.

Dehn's theorem is a counterexample to the linear ordering consequence of De Zolt for polyhedra in three dimensions. While Archimedes is needed in two dimensions, it does not suffice in three.

[19]For a suitable vocabulary see [Bal18b, 9.3.5] or [Bal18a].

[20]He doesn't measure area so he shows only area is proportional to height and base.

[21]Stolz coined the name in 1881. See [Gio21, Ehr06].

[22]3-space can be tiled by tetrahedra if they are not required to be regular.

As we formalize in Definition 5.5 there are two distinct issues. 1) De Zolt's axiom can be seen as a clarification of Euclid's common notion 5, the whole is greater than the part[23]. That hypothesis provides a strict *partial* order of the polygons. 2) De Zolt explictly asks for a *linear order* (trichotomy property) of equivalence classes by decomposition. He claimed to prove this from his axiom. We note in Corollary 5.10 that without the axiom of Archimedes this is just false.

5. General Framework

As promised in the introduction, we establish a general framework for the study of 'equivalent geometric magnitudes'.

Definition 5.1 (General framework)**.** *Let $n = 2$ or 3.*

(1) An atom *is an n-dimensional convex hull of $n+1$ points in n-space.*
(2) A figure *in n-space is a non-overlapping (intersection cannot have an interior) union of atoms.*
(3) A figure $P$is contained *is another Q if Q is a non-overlapping union of P and a figure R.*
(4) An equivalence relation E on figures is admissible *if*
  (a) Congruent atoms are E-equivalent (i.e. CN4);
  (b) For disjoint figures P, P' and Q, Q', if $E(P, P')$ and $E(Q, Q')$, then $E(P \cup Q, P' \cup Q')$ (i.e. CN2).
(5) The De Zolt order *with respect to an admissible equivalence relation on figures with non-empty interior is defined by $[P] < [Q]$ if $(\exists R)[Q] = [P+R]$.*

We have written 'atom' to have a common term for triangle in the plane and tetrahedron in three space. Thus a figure will be a non-overlapping union of triangles (tetrahedra) in the plane (3-space). Thus each figure has non-empty interior. Whether the De Zolt order is strict or satisfies trichotomy depends very much on the choice of equivalence relation.

After introducing the idea of equicomplementability (equal content) and proving Euclid I.35 (triangles with same base and equal height are equicomplementable), [Hil62, p. 61] writes 'the discussion so far leaves it still in doubt whether all polygons are not, perhaps, of equal content.' He elaborates by asking for a proof that a rectangle is 'definitely determined by means of one side and its area'. The answering proof [Hil62, §18] requires defining a measure of area function. For this, Hilbert [Hil62, §15] introduces an algebra of segments[24] where *both* addition and multiplication are associative and commutative and multiplication distributes over addition. There are quotients but no additive inverse, 0 or 1. Contrast the properties of a field discussed in §13 and the weaker conditions in §15 of [Hil62]. These differences (except not fixing 1) are essential since the domain is non-empty segments. Then he introduces the following *measure of area function*.

He now defines the *measure of area* of a triangle with base a segment b and height a segment h to be $\frac{bh}{2}$. He proves[25] any polygon can be decomposed into a finite number of disjoint triangles and assigning the measure of the polygon to be

[23]But De Zolt did not see it that way [Gio21, §3.2].

[24]This was first done in [Sto85]. Clearly Stolz didn't call the result a field since the term was only coined by Weber in 1893 `https://en.wikipedia.org/wiki/Field_(mathematics)#History`.

[25]This proof is only sketched; Hartshorne provides a full proof in [Har00, §17,§22, §23]. This proof uses rigid motions which he proves to exist under weaker hypotheses than Hilbert plane.

the sum of the measures of the triangles is well-defined. [Hil62, §20,§21] proves, *without applying Archimedes axiom* that triangles have equal content if and only if they have equal measures of area and that this result extends to polygons.

Hilbert's claim[26] that this is a *geometric proof* is bolstered by his not defining a field or even saying 'multiplication is a cancellative semigroup' but just laying out the essential geometric machinery.

Remark 5.2 (Three ways to assign measure to area).

global method: Fix a unit; say, a square; tile the plane with congruent squares. Then to measure a figure, continually refines the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure.

local method: (Hilbert) Triangulate a figure with finitely many triangles, which are each assigned area

- **Euclidean Geometry:** $\frac{bh}{2}$
- **Hyperbolic Geometry:** $(0, \delta)$ or $(1, \delta)$ depending on the size of the defect δ

and the area of the figure is the sum of the areas of the triangles.

representative method: Fix a representative of each equivalence class.

The first two are described in [Bol78, §5](See [Har00, §36] for the hyperbolic case.); we introduce the third – choose a specific congruence class that is determined by the length of one side –in Theorem 5.7. The methods extend to volume in three space.

We slightly vary Hartshorne's general definition:

Definition 5.3. *A* measure of area function *on figures in n-space is a function α with values in* an ordered semigroup G[27] *satisfying:*

(1) Congruent atoms *have the same value.*
(2) For disjoint figures P, Q, $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.

Like Hilbert in[Hil62, §13] and unlike [Har00, 205], who posits a linearly ordered group[28], we require only that G is a linearly ordered semigroup (associative binary operation), with no notion[29] of positive and negative or even a 0, and no multiplication.

That is, the algebra of segments depends only on concatenation of intervals to define addition. We do not use Hilbert's definition of multiplication.

Returning to the general case we define some relevant equivalence relations.

Definition 5.4. *(1) Two figures P, P' are* equidecomposable *if they each can be written as a non-overlapping union of the same number of pairwise congruent atoms.*

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[26]Hilbert makes much of avoiding Archimedes and using only the first order axioms of plane geometry (Hilbert's first four groups.) at [Hil62, pg 69-70] and he objects to Euclid's appeal to general axioms about magnitudes on [Hil62, pg 62]. Perhaps his aim is not to avoid 'algebra' but to avoid any principles far from geometry – like common notions.

[27][Hil62] does not introduce the term semigroup. This is unsurprising since the term only came into use in the next decade [Hol14]. But Hilbert was avoiding such an algebraic slant. An ordered semigroup is a structure $(*, <)$ such that $*$ is associative and satisfies $(\forall x, y, z) x < y \to (xz < yz \wedge zx < yz)$.

[28]This is needed for the hyperbolic case considered later.

[29]If one considers a semigroup with 0, we would demand that α map into the positive cone.

(2) *Two figures P, P' are* equicomplementable *if there are other figures Q, Q' such that:*
   (a) *P and Q are non-overlapping;*
   (b) *P' and Q' are non-overlapping;*
   (c) *Q and Q' are equidecomposable*
   (d) *$P \cup Q$ and $P' \cup Q'$ are equidecomposable.*

(3) *Two figures P, P' are* equimeasured *(by α) if there is a measure of content function α into a semigroup such that $\alpha(P) = \alpha(P')$*

(4) *For a subgroup G of the group of rigid motions of the space, two figures P, P' are G-*equivalent *if there is $g \in G$ with $g(P) = P'$.*

The proof of [Har00, 22.4], points out that in 2) Q and Q' can be taken disjoint.

We explore various admissible equivalence relations below to understand the interactions between the following properties. We introduce new terminology to give a general description of one of the main results of the first two books of Euclid. In particular, we describe as a scale a figure whose congruence class is determined by the length of either one (e.g. a regular polygon) or two (rectangles with fixed height) sides.

Definition 5.5. (1) *A* figure *type*[30] $\boldsymbol{A}$ *is an explicit description of a figure given by a first order formula; e.g. square, triangle, regular pentagon, rectangle with a fixed height.*

(2) *A figure type $\boldsymbol{A}$ is a* scale[31] *if whenever P, P' each satisfy $\boldsymbol{A}$ and*
   1-parameter case: *If one side*[32] *of P is congruent to one side of P'*
   or 2-parameter case: *If a designated segment (e.g. height) of P and of P' is congruent to a segment AB, fixed by naming constants, and separate distinguished segments* [33] *of P (e.g. the base) and of P' are congruent*

   then P and P' are congruent.

(3) *An equivalence relation on figures is* scaled *(by a scale $\boldsymbol{A}$) if every equivalence class contains at least an instance of $\boldsymbol{A}$.*

(4) *A scaled equivalence relation E on figures is* well-scaled *(by a scale $\boldsymbol{A}$) if P, P' satisfy $\boldsymbol{A}$ and $E(P, P')$ implies $P \approx P'$. (Here and below $P \approx P'$ means P and P' are congruent.)*

By Definition 5.5.1.a, $inHP5$ rectangle with a fixed height is a scale for equicomplementability ([Euc56, Proposition I.44]) and in EG so is square ([Euc56, Proposition II.14]). Note that the distinguished edge determines the congruence class but the equivalence class of the figure may not determine its congruence class. For example, [Euc56, Proposition II.14] does not guarantee that all squares 'equal to a given regular figure' are congruent but only that there is at least one such square. So well-scaled is stronger than scaled.

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[30] In Euclid's Data, he calls a similar notion of figures given in species.

[31] The definition of scale does NOT depend on an equivalence relation but is a property of a figure type.

[32] Perhaps the only examples are regular polygons and regular polyhedra.

[33] Example: Let the figure type be: rectangle of fixed height (designated segment). Take the distinguished segment as base. The defining formula $\phi(A, B, x_1, y_1, y_2, x_2)$ asserts that the x_i, y_i are the vertices of a rectangle, $x_1x_2 = AB$ and y_1, y_2 the distinguished segment. If the figure is a square, there is no need to distinguish.

Definition 5.6. *(1) De Zolt's axiom* holds with respect to an admissible equivalence relation, E, on figures *means: If Q is a figure contained in another figure P, and if $P - Q$ has non-empty interior then P and Q are not E-equivalent.*

(2) We write[34] $[P] \leqslant [Q]$ *if there exists a figure R with non-empty interior such that $E(P + R, Q)$.*

Now the main theorem is the following two statements, which are essentially converses.

Theorem 5.7. *Work in neutral geometry.*

(1) If E is well-scaled by $\boldsymbol{A}$ then there is measure of area function α for E, the equivalence classes are linearly ordered, and E satisfies De Zolt.
(2) If E is scaled by a scale $\boldsymbol{A}$ and E satisfies De Zolt, E is well-scaled.

Proof. 1) Since E is scaled, there is a representative of $\mathbf{A}$ in each equivalence class. Since E is also well-scaled we can define a function α' from figures to the congruence class of figures of type $\mathbf{A}$: $\alpha'(P)$ is the unique congruence class of $\mathbf{A}$ in $[P]_E$ and thus determines the length of the distinguished segment. But, by Definition 5.5.2) the correspondence between length of the distinguished segment and congruence class is a bijection. So by setting the measure of area function $\alpha(P)$ equal to the length of the distinguished side of $\alpha'(P)$, the ordering of line segments gives a linear ordering of (equivalence classes of) polygons which satisfies De Zolt.

2) Suppose P and P' are E-equivalent and both satisfy $\boldsymbol{A}$ but they are not congruent. Then by Definition 5.5.2 the designated sides are not congruent and without loss of generality the designated side AB is congruent to a subsegment of $A'B'$. But then $P' - P$ has non-empty interior and by De Zolt $P' \not\approx P$. That is, each equivalence class contains (up to congruence) a unique figure satisfying $\boldsymbol{A}$. $\blacksquare_{5.7}$

Corollary 5.8. *If the plane π satisfies*

(1) $HP5$ and the rectangle property (Definition 2.2) for equicomplementation
(2) or EG and the squares property (Definition 2.2) for equicomplementation

then equicomplemenation satisfies De Zolt and there is a measure of area function on π.

Proof. By I.44 (II.14) of [Euc56], equicomplementation is scaled by rectangles with fixed height (squares[35]). The rectangle (squares) property says each equivalence class contain only congruent rectangles (squares) and so the equivalence relation is well-scaled. and the result follows by Theorem 5.7.1. $\blacksquare_{5.7}$

We now illustrate the necessity of the *two* conditions in the theorem.

Lemma 5.9. *There is a model of EG where Equidecomposition is not scaled by squares.*

Proof. We show there is a plane π satisfying $HP5$ and indeed EG with a parallelogram $EBCF$ that is not equidecomposable with a square. Consider a Cartesian plane π over a non-Archimedean elementary[36] extension F of the reals $\Re$. We say a point A in π is standard or finite, if A is in $\Re$.

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[34] For any F, $[F]$ indicates the E-equivalence class of F.
[35] [Har77, p. 205] notes that circle-line intersection is used in the proof.
[36] F satisfies the same first order sentences as $\Re$. In particular F is a Euclidean field.

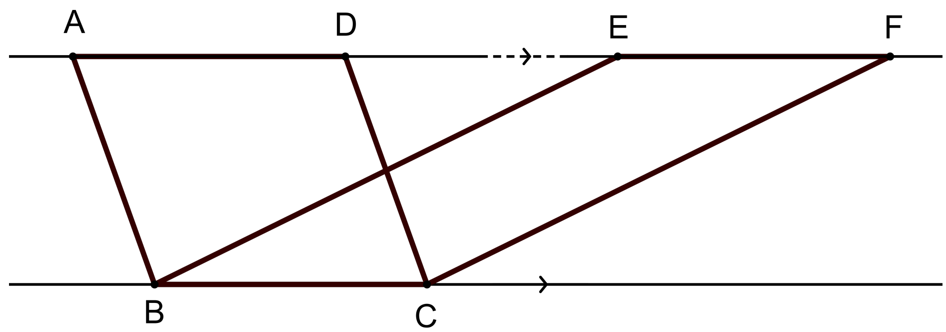

FIGURE 2. Euclid I.35

Using the diagram for Euclid I.35, suppose A, B, C, D are finite (standard) but E and F are not; although the length of EF must be standard since $EF \approx BC$. Now we know $ABCD$ is equidecomposable with a (standard) square, since all its sides are in the real field and Theorem 4.2 applies.

Since $\pi \models HP5$, Hilbert's measure of area function gives the same finite area to both $ABCD$ and $EBCF$. But $EBCF$ is not *equidecomposable* with any finite square because it has a side of infinite length so finitely many triangles cannot exhaust the perimeter. But then $EBCF$ is not equidecomposable with any square. This holds because de Zolt implies that if there is a square representing an equivalence class, it is unique up to congruence. $\blacksquare_{5.9}$

Corollary 5.10. *The proof of Theorem 5.7.2 requires the 'scaled' hypothesis.*

Proof. In every model of $HP5$, Equidecomposition satisfies De Zolt, (since it refines equicomplementability which does. But Lemma 5.9 shows not every figure is equidecomposable with a square and so there is no measure of area function that witnesses equidecomposition (unless we assume Archimedes). So equidecompostion is not scaled. Consider the two parallelograms in Lemma 5.9. They have the same area by Hilbert's measure of area function, but are not equidecomposable. Thus, they are incomparable under the De Zolt order, Definition 5.1.4, as DeZolt comparable classes cannot have the same measure of area $\blacksquare_{5.10}$

Remark 5.11. *We give two further examples*

1) Declaring all figures equivalent is an admissible equivalence relation that fails De Zolt and does not have a non-trivial measure of area function.

3) Let T be the group of translations of real 3-space. $E_{\boldsymbol{T}}$ where $\boldsymbol{T}$ is the group of translations satisfies De Zolt but not represented and refines equidecomposition [Bol78, §10]*. This gives an Archimedean example where De Zolt does not imply linearly ordered (or existence of measure) showing both De Zolt and scaled are essential for Theorem 5.7.1 even in the Archimedean case.*

6. GEOMETRIC PROOF AND PURITY

Hilbert is careful to describe his argument in geometric terms. Multiplication in an 'algebra of segments' defines the area of a triangle as $\frac{bh}{2}$. The most 'algebraic' component of the proof is the use of proportionality in both [Hil62, §20] and [Har00, Lemma 23.3]. But this is not essential.

Theorem 6.1. *Any of the three choices of base for a triangle give the same value for the product of the base and the height.*

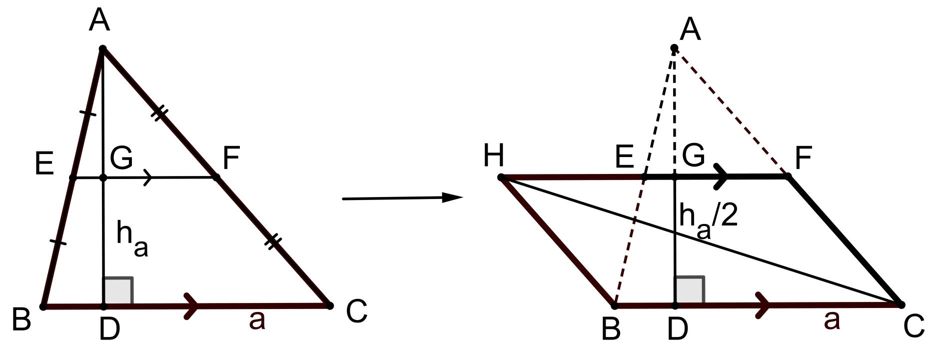

FIGURE 3. Area of Triangle Independent of Choice of Base

Proof. Take triangle ABC. Choose an arbitrary side, say a. Draw the midline EF parallel to side a. The height h_a perpendicular to a is bisected by the midline at G (or by its extension).

Rotate triangle AEF about point E to get parallelogram $BCFH$ with base a and height $\frac{h_a}{2}$. HC divides the parallelogram into two congruent triangles (BCH and FHC) with base a and height $\frac{h_a}{2}$ and therefore area $\frac{1}{2} \times \frac{ah_a}{2}$, so the area of the parallelogram $BCFH$ is $\frac{ah_a}{2}$.

Since that is done without preference of base, the argument is valid for the other two sides and corresponding heights of the triangle, giving the second and third area formulas as $\frac{bh_b}{2}$ and $\frac{ch_c}{2}$. All three measure areas of parallelograms with equal content as triangle ABC.

This shows that the triangle area formula $\frac{1}{2}(base)(height)$ is independent of choice of base.

The scale function in Theorem 5.7 is still more geometric as it avoids the definition of multiplication altogether. The argument of Theorem 5.7 requires only the structure of an ordered additive semigroup. Moreover, it does not require Archimedes.

The Hartshorne problem can also be regarded as an issue of purity. Here is a similar example of a 'more geometric' proof. Hilbert proved that plane is Desarguesian if and only if it is embedded in three space. And argued that this showed that '3-space' was an impure component of the traditional proof. But his argument, [Hil62, §29], was scarcely 'geometric' since he defines a 'Desarguesian number system' on a line and performs two pages of algebraic calculations with linear equations in three variables. Baldwin and Howard [Bal13, appendix] gave a direct geometric argument that a Desarguesian projective plane π can be embedded in a three dimensional geometry.

Remark 6.2 (Conclusion)**.** Hartshorne's question asks about the significance of the existence of a measure of a area function in establishing the theory of area. Equidecomposablity and thus equicomplementability are described by infinite *disjunctions* of first order formulas. Thus, (as grasped by De Zolt), non-equidecomposablity is a treacherous notion; to establish it requires checking infinitely many possibilities. Moreover, these possibilities are too wild to support an induction. Hilbert's computation of the measure of area also has infinitely many cases depending on the number of vertices of the figure and the number of triangle in the triangulation. But, regardless of the number of triangles, the measure does not depend on the triangulation[37]. That is exactly what it is summarised by saying measure of area is a function. Some such uniformity is needed to prove De Zolt.

Any well-scaled equivalence relation will work. But, like Hartshorne, we see a complete proof only for Hilbert's.

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[37]([Hil62, p. 68]) gives a procedure showing that any figure, that has area g with respect to some triangulation is equicomplementable with a triangle of height 1 and base g. Thus, two triangles are equicomplementable if and only if any calculation of their areas give the same value.

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