# Strongly Minimal Steiner Systems
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# Zilber's hope

> *The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture), was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics.*
> *[PS98]. ([Zil05])*

We show some exotics are a bit more classical than expected.

# Overview

1. Classifying strongly minimal sets and their geometries
2. Coordinatization by varieties of algebras
3. Interactions with Combinatorics
    - Cyclic graphs in Steiner triple systems
    - Sparse Steiner systems

Thanks to Joel Berman, Gianluca Paolini, and Omer Mermelstein.

# Classifying strongly minimal sets and their geometries

# The trichotomy

## Zilber Conjecture

The $\mathrm{acl}$-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. non-locally modular : 'bi-interpretable' with an algebraically closed field.

Hrushovski's example showed there are non-locally modular which are far from being fields; the examples don't even admit a group structure.

# Classify non-locally modular geometries of SM sets

## Definition: Flat geometries

A geometry given by a dimension function $d$ is flat if the dimension of any set $E$ covered by $d$-closed sets $E_1, \ldots E_n$ is bounded by applying the inclusion exclusion principle to the $E_i$.

## Fact

If the geometry of a strongly minimal set $M$ is flat.

1. Forking on $M$ is not 2-ample. (I.e., is CM-trivial)
2. $M$ does not interpret an infinite group.
3. Thus, the geometry is not locally modular and so not disintegrated.

# Classifying Hrushovski Construction

## The $\mathrm{acl}$-geometry associated with Hrushovski constructions

Work of Evans, Ferreira, Hasson, Mermelstein suggests that up to arity or more precisely, purity, (and modulo some natural conditions) any two geometries associated with Hrushovski constructions are locally isomorphic.
Locally isomorphic means that after localizing one or both at a finite set, the geometries are isomorphic.
[EF11, EF12, HM18]

We are concerned not with the $\mathrm{acl}$-geometry but with the Object language geometry.

# ‘Object Language’ geometries

Strong minimality asserts the ‘rank’ of the universe is one and imposes a combinatorial geometry whose dimension varies with the model. We study here structures which are ‘geometries’ in the object language. E.g.

## Projective Planes: [Bal94]

There is an almost strongly minimal (rank 2) projective plane.
An example with the least possible structure in the Lenz-Barlotti class was constructed [Bal95].
In particular, the ternary function of the coordinatizing field cannot be decomposed into an ‘addition’ and a ‘multiplication’.

# Reformulating the problem

## What was the Zilber conjecture?

Conditions on the $\mathrm{acl}$-geometry imply conditions on the algebra of the structure.

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Conditions on the $\mathrm{acl}$-geometry imply conditions on the algebra of the structure.

Even if the flat $\mathrm{acl}$-geometries are all very similar here are some conditions that can distinguish them (or not)

1. Satisfy combinatorial conditions: Steiner systems, various types of designs;
2. Coordinatizability
3. Existence of a definable binary function.
4. Interprets a (variety of) quasigroups (or other universal algebra) A quasigroup is a structure $(A, *)$ such that specification of any two of $x, y, z$ in the equation $x * y = z$ determines the third uniquely. [Smi07]
5. properties arising in finite/infinite combinatorics.

# How is algebraic structure lost?

## Algebraic view

1. field
2. integral domain (lose inverses)
3. matrix ring (lose commutativity)
4. alternative ring (weaken associativity)

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## geometric view

1. field (Pappian plane)
2. division ring(Desarguesian plane: lose commutativity)
3. nearfield (lose left distributive)
4. quasifield( multiplication is a quasigroup with identity)
5. alternative algebra (Moufang plane: lose full associativity)
6. ternary ring (lose associativity and distributivity and even compatible binary functions, but still have inverse)

# Coordinatizability

## (Trial Definition)

A class of structures (specifically geometries) is coordinatizable if there is 1-1 correspondence between it and a well-behaved class of algebras. (Ganter-Werner) [GW75, GW80]

## impediments

1. How is the correspondence established? (Makowsky) [Mak18]
2. ‘well-behaved’?

# Coordinatizability

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## Definition

A class of structures (specifically object geometries) is coordinatizable if there is 1-1 definable correspondence between it and a *first order definable* class of algebras.

# Linear Spaces

**Definition**

A linear space [BB93] is a collection of points and lines such that

2 points determine a line;
consequently two lines intersect in at most one point.

Let the vocabulary $\tau$ contain a single ternary predicate $R$, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems. $\boldsymbol{K}^*$ includes infinite linear spaces.

1. $R$ is a predicate of sets (hypergraph)
2. Two points determine a line

There are natural generalizations:

1. $k$-points determine a line.
2. allow a finite number of line lengths

Mermelstein and Hasson [HM17] have investigations along these lines.

# 2-sorted vrs 1-sorted

In a two-sorted formulation, i.e. points and lines, clearly no strongly minimal theory has both infinitely many points and infinitely many lines.

Even in 1-sort, there cannot be two lines with infinitely many points.

Note that this does not preclude bi-interpretability between 1-sorted and 2-sorted descriptions. Because, interpretations do not need to preserve Morley rank.
In this case the universe of the two-sorted structure is interpreted as a set of pairs in the 1-sorted structure.

**Theorem [BP18]**

$\boldsymbol{k}_0^*$ and the class of two-sorted linear spaces are biinterpretable.

# Strongly minimal linear spaces I

**Fact**

Suppose $(M, R)$ is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

Suppose $\ell$ is an infinite line. Choose $A$ not on $\ell$. For each $B_i, B_j$ on $\ell$ the lines $AB_i$ and $AB_j$ intersect only in $A$. But each has a point not on $\ell$ and not equal to $A$. Thus $\ell$ has an infinite definable complement, contradicting strong minimality.

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## Corollary

There can be no strongly minimal affine or projective plane, since in such planes the number of lines must equal the number of planes.

# Strongly minimal linear spaces II

An easy compactness argument establishes

## The fundamental corollary of strong minimality

If $M$ is strongly minimal, then for every formula $\varphi(x, \overline{y})$, there is an integer $k = k_\phi$ such that for any $\overline{a} \in M$, $(\exists^{>k_\phi} x)\varphi(x, \overline{a})$ implies there are infinitely many solutions of $\varphi(x, \overline{a})$ and thus finitely many solutions of $\neg\varphi(x, \overline{a})$.

## Corollary

If $(M, R)$ is a strongly minimal linear system, for some $k$, all lines have length at most $k$. So it is a $K$-Steiner system.
$K = \{3, 4 \ldots k\}$.

# Specific Strongly minimal Steiner Systems

## Definition

A *Steiner* $(v, 2, k)$*-system* is a linear system with $v$ points such that each line has $k$ points.

## Theorem (Baldwin-Paolini)[BP18]

For each $k \geq 3$, there are an uncountable family $T_\mu$ of strongly minimal $(\infty, k, 2)$ Steiner-systems.
The theory is 1-ample (not locally modular) and CM-trivial (not 2-ample).

## IN ENGLISH

There is no infinite group definable in any $T_\mu$. More strongly, Associativity is forbidden.

# Not quite standard Hrushovski construction

$\mathbf{K}_0^*$ denotes the collection of finite linear systems in the vocabulary $\tau = \{R\}$.
A line in $M$ is a maximal $R$-clique
$L(A)$, the lines based in $A$, is the collections of lines in $(M, R)$ that contain 2 points from $A$.

## Definition: Paolini's $\delta$

[Pao] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

# Amalgamation and Generic model

## Definition

Let $A \cap B = C$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_C B$ as follows:

1. the domain of $D$ is $A \cup B$;
2. a pair of $a \in A - C$ and $b \in B - C$ are on a line $\ell'$ in $D$ if and only if there is a line $\ell \subseteq D$ based in $C$ such that $a \in \ell$ (in $A$) and $b \in \ell$ (in $B$). Thus $\ell' = \ell$ (in $D$).

## Definition

The countable model $M \in \hat{\boldsymbol{K}}_0$ is $(\boldsymbol{K}_0, \leq)$-*generic* if

1. If $A \leq M, A \leq B \in \boldsymbol{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$,
2. $M$ is a union of finite closed subsets ($A_i \leq M$).

## Theorem: Paolini [Pao]

There is a generic model for $\boldsymbol{K}_0^*$; it is $\omega$-stable with Morley rank $\omega$.

# The strongly minimal case

## Definition: A is strong in B

$A \leq B$ if $A \subseteq B$ and there is no $B_0$, $A \subsetneq B_0 \subsetneq B$ with $\delta(B_0/A) < 0$.

## Goal

Force that if $\delta(B/A) \leq 0$ then $B \subseteq \mathrm{acl}(A)$.
Automatic if $\delta(B) < \delta(A)$.

Focus on simplest case: $(A, B)$ is a good pair if it is minimal example $\delta(A) = \delta(B)$

Example: $\alpha$ is the iso type of $(\{a, b\}, \{c\})$ where $R(a, b, c)$.

# Overview of construction

1. $\boldsymbol{K}_0^*$: all finite linear $\tau$-spaces.
2. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily $\geq 0$.
3. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: $\mu$ bounds number of ‘good pairs’.
4. $\boldsymbol{K}_{\mu,d} = \mathrm{mod}(T_\mu)$ strongly minimal.

# Primitive Extensions and Good Pairs

## Definition

Let $A, B, C \in \mathbf{K}_0$.

1. $A \leq B$ if $A \subseteq B$ and there is no $B_0$, $A \subsetneq B_0 \subsetneq B$ with $\delta(B_0/A) < 0$.
2. $B$ is a 0-*primitive extension* of $A$ if $A \leq B$ and there is no $A \subsetneq B_0 \subsetneq B$ such that $A \leq B_0 \leq B$ and $\delta(B/A) = 0$.
3. We say that the 0-primitive pair $B/A$ is *good* if for every $A' \subsetneq A$ we have that $\delta(B'/A) > 0$.
4. For any good pair $(A, B)$, $\chi_M(A, B)$ is the number of copies of $B$ over $A$ appearing in $M$.

# $\boldsymbol{K}_\mu$

## Definition

1. $\mu$ is a function from isomorphism types of good pairs into natural numbers such that
   1. $\mu(\boldsymbol{\alpha}) \geq 1$
   2. if $\beta$ is a good pair $C/B$ in $\boldsymbol{K}_0$ with $|C - B| \geq 2$, $\mu(B, C) = \mu(\beta) \geq \delta(B)$ then $\mu \in \mathcal{U}$.
2. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair $(A, B)$.
3. $X$ is $d$-closed in $M$ if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0$.).
4. Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \boldsymbol{K}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ implies $M$ is $d$-closed in $N$.
   Moreover, if $M \in \boldsymbol{K}_d^\mu$, and $B \leq M$, for any good pair $(A, B)$, $\chi_M(A, B) = \mu(A, B)$.

# Main existence theorem

**Theorem (Baldwin-Paolini)[BP18]**

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory $T_\mu$.
If $\mu(\alpha) = k$, all lines in any model of $T_\mu$ have cardinality $k + 2$.
Thus each model of $T_\mu$ is a Steiner $k$-system and $\mu(\alpha)$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. $\alpha$).

# Tools

1. *Tool*$_0$: Study theories not ad hoc structures
   Techniques for varying the theory.
   1. *Tool*$_1$: Constrict the class $\boldsymbol{K}_0$. [Hru93, Bal95]
   2. *Tool*$_2$: Impose conditions on $\mu$ to require or avoid certain configurations.[Bal18]
   3. *Tool*$_3$: expand the vocabulary [Bal18]

# 2-transitivity of models of $T_\mu$: Consequences

## Lemma ($Tool_0$)

*For any $\mu \in \mathcal{U}$, if $(M, R) \models T_\mu$, $A \subset \mathcal{G}_\mu$ and $|A| = 2$ implies $A \leq \mathcal{G}_\mu$ then the automorphism group of $(M, R)$ acts 2-transitively on $(M, R)$.*

## Proof.

Since all pairs $(a, b)$ are isomorphic and each sits strongly in the generic $\mathcal{G}$, the result is immediate for $\mathcal{G}$. But this property extends to all models since if one model of a complete theory has a single 2-type, all models do. And each model of a strongly minimal theory is finitely homogeneous. □

# [$Tool_1$]:Getting $n$-transitivity of models of $T_\mu$:

**Lemma ($Tool_1$)**

*If for any $B \in \boldsymbol{K}_0$ with $|B| \geq n$, $\delta(B) \geq n$ then, for any $\mu$ the number of $n$-types in $\mathcal{G}_\mu$ is bounded by the number of quantifier free $n$-types.*

**Corollary: [Hru93]**

If every $B \in \boldsymbol{K}_0^-$ (don't assume two point determine a line) satisfies $|B| \geq 3$, $\delta(B) \geq 3$, then every model of $T_\mu$ is a 2-transitive Steiner triple system.

**Corollary: [Bal18]**

If for any $B \in \boldsymbol{K}_0$ with $|B| \geq 2$, $\delta(B) \geq 2$, every model of $T$ is 2-transitive and every line is a set of indiscernible over $\emptyset$.

See earlier variant for projective planes in [Bal95].

# Coordinatization by varieties of algebras

# Coordinatizing Steiner Systems

## Definition

A collection of algebras $V$ "weakly coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in $V$ expands definably to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in $V$.

2 VARIABLE IDENTITIES: [Eva82]

## Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

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Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct $a, b, c$:

$$R(a, b, c) \text{ if and only if } a * b = c$$

# Coordinatizing Steiner Systems II

## Example

A Stein quasigroup is a groupoid (one binary function) which satisfies the equations: $x \circ x = x, (x \circ y) \circ y = y \circ x, (y \circ x) \circ y = x$.

[GW75, GW80]

A Stein quasigroup can be imposed on any Steiner $(v, 2, 4)$-system
Proof: For distinct $a, b, c, d$:
A set $A$ with $|A| = 4$ is a maximal $R$-clique (line) if and only if for some enumeration $a_1, a_2, a_3, a_4$ of A

$$a_1 \circ a_2 = a_3 \wedge a_2 \circ a_1 = a_4.$$

Clearly each Stein quasigroup gives a Steiner $(v, 2, 4)$-system.

But, Steiner quadruple systems do not (in an obvious way) determine the Stein quasigroup definably.

# More Precisely

**Theorem**

Every strongly minimal Steiner (2,3)-system given by $T_\mu$ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroups definable in the system.

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**Theorem**

1. ($Tool_1$,$Tool_2$) There is a strongly minimal theory of Steiner $(2,4)$-systems where the Stein quasigroup is not definable.
2. ($Tool_1$, $Tool_2$) There are strongly minimal theories of structures $(M, R, F)$ where
    1. $(M, R)$ is a Steiner $(2,4)$-system, and
    2. $(M, F)$ is a Stein quasigroup.

    [S.K64]

Some theories are coordinatizable – some only weakly coordinatizable.

# Coordinatizing Steiner Systems: Summary

**Fact**

Let $q$ be a prime power.
Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality $q$ and an element $a \in F$, define a multiplication $*$ of $F$ by $x * y = y + (x - y)a$. An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

The $(2, q)$ Steiner systems are weakly coordinatized by block algebras. [GW80, page 6]

**Question**

For which $k$, $\mu$ are the models of $T_\mu$ definably coordinatized?

# Universal Algebra facts and questions

## Fact

Steiner quasigroups are congruence permutable, regular, and uniform. The variety of Steiner quasigroups is not residually small. Finite members are directly decomposable.

We showed above that models of the $T_\mu$ are not locally finite.

## Question

1. Are these $\aleph_1$-categorical Steiner quasigroups subdirectly irreducible or even simple? Surely they are not free?!
2. How does the variety associated with $T_\mu$ depend on $\mu$?
3. There is a close correspondence between subgroups and congruences for quasigroups that suggest something along the descending chain condition for $\omega$-stable groups might be developed.

# III. Interactions with Combinatorics

# Infinite linear spaces

*There is no theory of infinite linear spaces comparable to the enormous amount known about finite linear spaces. This is due to two contrasting factors. First, techniques which are crucial in the finite case (notably counting) are not available. Second, infinite linear spaces are too easy to construct; instead of having to force our configurations to close up, we just continue adding points and lines infinitely often! The result is a proliferation of examples without any set of tools to deal with them.*
*Cameron, [Cam94]*

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1. Find families (namely the models of a complete first order theory) of infinite linear spaces that are similar both combinatorially and model theoretically.
2. Import non-trivial constructions from model theory to construct interesting linear spaces.

## III.1 Cyclic graphs in Steiner triple systems

# The graph of a Steiner triple system

This is a standard topic in finite combinatorics, extended to infinite system by e.g. Cameron and Webb. [CW12]

## Definition

1. Fix any two points $a, b$ of a Steiner triple system $\mathcal{S} = (P, L)$. The cycle graph $G(a, b)$ has vertex set $P - \{a, b, c\}$ where $(a, b, c)$ is a block. There is an edge coloured $a$ (resp., $b$) joining $x$ to $y$ if and only if $axy$ is a block (resp., $bxy$ is a block).

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2. a proper initial segment of an $(a, b)$-cycle is called an $(a, b)$-chain.
3. It is uniform if the graphs $G(a, b)$ are pair-wise isomorphic.

# From combinatorics to model theory

## Lemma

*There are infinitely many mutually non-imbeddible primitives in $\boldsymbol{K}_0$ over a two element set. In fact, there are infinitely many mutually non-embeddible primitives in $\boldsymbol{K}_0$ over the empty set and similarly over a 1-element set.*

## Proof.

Over any $a, b$ for each $k$ build an $(a, b)$-cycle $C_k$ , $c_1, c_2, \ldots c_{4k}$ of length $4k$ with $c_1bc_{4k}$ and $c_1ac_2$. $C_k$ has $4k$ points and $(\{a, b\}, C_k) \in \boldsymbol{K}_0$ has $4k$ 3-element lines. So $\delta(\{a, b\}, C_k)) = 2 = \delta(\{a, b\})$. Primitivity easily follows since if the cycle is broken, the $\delta$-rank goes up.
Minor variants for $\emptyset$ and singletons. $\square$

# What lengths of cycles are possible?

## Fact

In any Steiner triple system, $\mathcal{S}$, each $G(a, b)$ is a disjoint union of cycles. Finite cycles have order divisible by 4. The cycles partition $G(a, b)$. Each cycle is completely determined by $a, b$ and the first element $c_1$.

## Notation

In any Steiner triple system $(M, R)$, for every finite $k$ there is a $\tau$-formula $\gamma_k(a, b, c_1)$ which holds exactly when $ac_1x$ starts a cycle in $G(a, b)$ that returns to $c_1$ as a $4k$-cycle.

Note that when $\gamma_k(a, b, c_1)$ holds, so does $\gamma_k(b, a, c_1)$; the cycle is traced in the other direction.

# The length of cycles

## Definition

We denote the isomorphism type of $(\{a, b\}, C_n)$ by $\gamma_{\boldsymbol{n}}$.

Since for any $n$, $\mu(\gamma_{\boldsymbol{n}})$ is finite, we have

## Lemma

For any $\mu \in \mathcal{U}$ and any $M \models T_\mu$, for every $n$, and every $(a, b)$ there are only finitely many $(a, b)$-cycles of length $n$. Since $G(a, b)$ is infinite, there must be arbitrarily long finite $(a, b)$-chains. Since $\mathcal{G}_\mu$ is saturated there is also an infinite cycle.

More careful analysis shows only the prime model can omit infinite chains.

# Avoiding Finite Cycles I

## Definition

($Tool_1$, $Tool_2$) Let $\mathcal{B}$ denote the set of $\mu$ such that for every $n$, $\mu(\gamma_n) = 0$ and $\mu(\alpha) = 1$.
We denote by $\boldsymbol{K}_{\mathcal{B},\mu}$ the class of finite structures such that for all $B$:

$$(*)\ |B| > 1 \text{ implies } \delta(B) > 1 \text{ and } \mu \in \mathcal{B}.$$

When $\boldsymbol{K}_{\mathcal{B},\mu}$, we call the associated theory $T_{\mathcal{B},\mu}$.

$(*)$ implies that every two element subset of the generic is strong and so every model is 2-transitive.

# Avoiding Finite Cycles II

## Lemma

If $\mu \in \mathcal{B}$, $\boldsymbol{K}_{\mathcal{B},\mu}$ has the amalgamation property.

If $\mu \in \mathcal{B}$ then for any model, $(M, R)$, of $T_{\mathcal{B},\mu}$ and any $(a, b)$, all $(a, b)$-cycles are infinite and $(M, R)$ is uniform.

Setting finitely many of the $\mu(\gamma_{\boldsymbol{i}}) = m_i$ for finitely many $i$ allows finitely many cycles.

# III.1 Sparse Steiner systems

# plus ça change, plus c'est la même chose

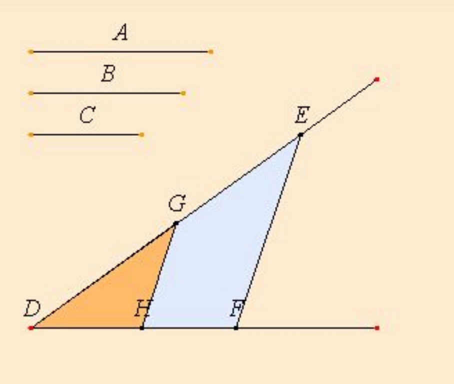


Figure: Constructing the 4th proportional

# plus ça change, plus c’est la même chose

The projective form of Euclid’s construction of the 4th proportional. $X$ is the point at infinity.

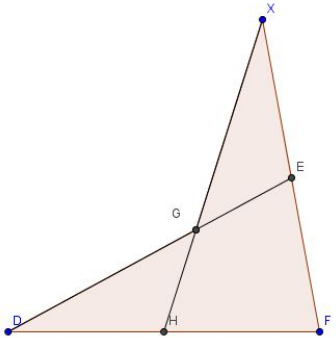


Figure: Pasch configuration/group configuration

# Anti-Pasch Steiner triple system

The Pasch configuration is omitted in the acl-geometry by flatness. We now omit the group configuration aka as a $(4, 6)$-configuration in the object language.

## Amalgamation

The subclass of $\boldsymbol{K}_\mu$ that omit the Pasch configuration has the amalgamation property.

## Corollary

There is a strongly minimal anti-Pasch Steiner triple systems

## Definition

[CGGW10] A Steiner triple system is $\infty$-sparse if for every $n \geq 4$ there is no set $A$ of size $n + 2$ with $\delta(A) = 2$. (no configuration on $n + 2$ points with $n$ lines)..

We have a 4-sparse; [CGGW10] have an $\infty$-sparse example. But using $\delta$ can we explain this better.

# Questions

1. For which $k$, $\mu$ are the models of $T_\mu$ (definably) coordinatized?
2. Which strongly minimal sets whose $\mathrm{acl}$-geometry is non-trivial, admit a quasigroup (not necessarily definable)?
   Or even a definable binary function?
   Can one do anything for those $k$ where there is no associated variety of universal algebras in the finite?
3. Can one develop something like the cycle analysis for other $(2, k)$ Steiner systems?
4. Can these methods help with finite systems?
   1. The function $\delta$ concerns finite structures. Can the notions of primitive and the amalgamation be useful tools?
   2. Problem: Almost none of the elements of $\boldsymbol{K}_{0,\mu}$ where $\mu(\alpha) = k$ are Steiner systems.
      Is there a way to pull back properties of the infinite structure to the finite?

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