Philosophical implications of the paradigm shift in model theory

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& LOGIC SEMINAR

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Book

Model Theory and the Philosophy of Mathematical Practice:

Formalization without Foundationalism

The book is both a case study of one area of mathematics, model theory, and an argument that developments in that area have more general philosophical interest.

Two parts to talk

1. Describe the paradigm shift in model theory
2. discussion of the four theses of the book

This talk

What is a paradigm shift?

Response today

I describe in some detail a specific ‘paradigm shift in mathematics’ vaguely – a major change in the fundamental questions and techniques of a mathematical area

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Issues for later

What is a general definition of paradigm shift which encompasses this example and others?

Reference: Gillies, Revolutions in Mathematics

What paradigm shift?

Before

The paradigm around 1950 concerned the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

What paradigm shift?

After

After the paradigm shift there is a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

In this framework one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

Section I. Early Model Theory

What hath Tarski (Robinson?) wrought?

Apparently the first modern statement of the ‘extended completeness theorem’ is in Robinson 1951 *Introduction to Model Theory and to the Metamathematics of Algebra*.

Completeness theorem (modern statement)

For every vocabulary τ and every sentence $\phi \in \mathcal{L}(\tau)$

$$(*) \qquad \Sigma \vdash \phi \text{ if and only if } \Sigma \models \phi.$$

Such a statement assumes Tarski’s definition of truth.

Henkin versus Gödel: proof of the completeness theorem

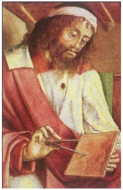
1. Gödel's definition of 'satisfiability in a structure' depends on the ambient deductive system. Specifically, the deductive system must support the existence of a π_2-prenex normal form for each non-refutable sentence.
   And he doesn't give a formal definition of satisfiability for even π_2 sentences.
2. Gödel does not assert the modern form of the extended completeness theorem.
   Rather he says (Theorem IX),

   'Every denumerably infinite set of formulas of the restricted predicate calculus either is satisfiable (that is, all formulas of the system are simultaneously satisfied) or some finite subset is refutable.'
3. He extends the vocabulary of the given theory by new relation symbols, Henkin adds only constants.

Section II.

Axiomatization vrs Formalization

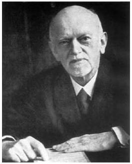
Euclid-Hilbert formalization 1900:

Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen der Geometrie) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language. For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski formalization 1917-1956:

Hilbert  Gödel  Tarski 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

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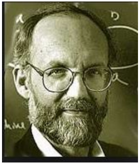
We reverse Bourbaki's aphorism to argue.

Full formalization is an important tool for modern mathematics.

Thesis 3

The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.

Formalization

Feferman  Barwise 

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   - a Specify a class of well formed formulas.
   - b Specify truth of a formula from this class in a structure.
   - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

Structures and Definability

A vocabulary τ is collection of constant, relation, and function symbols.

A τ-structure is a set in which each τ-symbol is interpreted.

A subset A of a τ-structure M is definable in M if there is $\mathbf{n} \in M$ and a τ-formula $\phi(x, \mathbf{y})$ such that

$$A = \{m \in M : M \models \phi(m, \mathbf{n})\}.$$

Note that if property is defined without parameters in M, then it is uniformly defined in all models of $\mathrm{Th}(M)$.

Two Theses

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Theories

Contemporary model theory focuses on theories not logics.

Theories may be given by axioms (first order Peano) or as $\mathrm{Th}(M)$ (true arithmetic).

Examples

algebraically closed fields, dense linear order, the random graph, differentially closed fields, free groups, ZFC,

$\mathrm{Th}(Z, S)$
$\mathrm{Th}(Z, +)$
$\mathrm{Th}(Z, +, 1)$
$\mathrm{Th}(Z, +, 1, \times)$

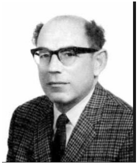
Complete Theories Kahzdan 

Complete theories are the main object of study.

Kazhdan:

> *On the other hand, the Model theory is concentrated on [the] gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should*
> *not distinguish between different models of T, since all the results which are true in one model of T are true in any other model.*
> *One of the main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.*
> *Different models of complete theories are of different flavors and support different intuitions.*

The Significance of Classes of Theories : Definability

Tarski  Robinson 

Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

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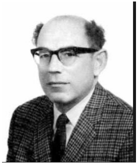
Tarski  Robinson 

Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

Tarski proved quantifier elimination of the reals in 1931.

Such a condition provides a general format for Nullstellensatz-like theorems.

Robinson provides a unified treatment of Hilbert's Nullstellensatz and the Artin-Schreier theorem which led to the notion of differentially closed fields.

Quantifier-elimination provides the epistemelogical virtue of accessiblity.

Classification of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories

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1. The significance of (complete) first order theories
2. The significance of classes of (complete) first order theories: Quantifier reduction

   'Applied' Quantifier reduction in a natural language is essential for mathematical application.

   'Pure' Quantifier elimination by fiat exposes the fundamental model theoretic structure.

3. stability hierarchy

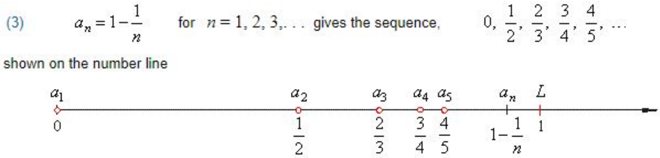
Model Theory in the 60's: Morley-Vaught

The collection of formulas p is a complete type over A if it satisfies one of the following equivalent conditions.

1. p is a maximal consistent set of formulas $\phi(x, \mathbf{a})$ with parameters from A.
2. p is a member of the Stone Space of the Lindenbaum algebra of A.
3. The solutions of p are an orbit of the group of automorphisms of the monster model which fix A.

Understanding Types

The theory of an dense linear order without end points with an infinite increasing sequence named.



Question to audience

Let $\mathcal{A} = (\mathbb{Q}, <, a_n)_{n<\omega}$ be the rational numbers and let a_n denote $1 - 1/n$.

Do 1 and 1.1 realize different types over $\{a_n : n < \omega\}$?

Different flavors of models

Are $\mathcal{A}_{sat} = (\mathbb{Q}, <, a_n)_{n<\omega}$ and $\mathcal{A}_{ord} = (\mathbb{Q} - \{1\}, <, a_n)_{n<\omega}$ isomorphic?

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Note that $\mathcal{A}_{prime} = ((-\infty, 1), <, a_n)$ is the third countable model of T.

Types as descriptions: stability

A complete n-type over the empty set is a description of an n-tuple (over the empty set).

Replace T by $\mathrm{Th}(M, A)$ where $M \models T$ and $A \subset M$.
A complete n-type in $S_n(\mathrm{Th}(M, A))$ is a description of an n-tuple over A.

Definition

Write $S_n(M, A)$ for $S_n(\mathrm{Th}(M, A))$.

The complete theory T is λ-stable if for every $M \models T$ and every $A \subset M$,

$$|A| \leq \lambda \Rightarrow S_n(M, A) \leq \lambda.$$

Semantic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T is stable in no λ.
2. (strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$
3. (superstable) T is stable in those $\lambda \geq 2^{\aleph_0}$.
4. (ω-stable) T is stable in all infinite λ.

Syntactic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T has the order property; some formula $\phi(\mathbf{x}, \mathbf{y})$ defines a linear order on an infinite subset of M^n.
2. (stable) For every formula ϕ, there is an integer n and a formula ϕ_n asserting 'there is no sequence of n-elements with the ϕ-order property'.
3. (superstable) There is a global rank R_C (with respect to n-inconsistency) such that $R_C(\psi) < \infty$ for all ψ.
4. (ω-stable) There is a global rank R_M (with respect to inconsistency) such that $R_M(\psi) < \infty$ for all ψ

From all theories towards classification

Theorem

1. the (strict) hierarchies on the last two slides are the same.
2. The defining conditions are either arithmetic or Π^1_1, so absolute in ZFC.

Historical Consequence

After the paradigm shift first order model theory is no longer so tightly entangled with axiomatic set theory.

What constitutes syntax?

A theory T is unstable if there is a formula with the order property. This formula may change from theory to theory.

1. In a dense linear order one ϕ is $x < y$;
2. In a real closed field one is $(\exists z)(x + z^2 = y)$,
3. In the theory of $(\mathbb{Z}, +, 0, \times)$ one is $(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. In the theory of complex exponentiation $(\mathcal{C}, +, \times, \exp)$, one first notices that $\exp(u) = 0$ defines a substructure which is isomorphic to $(\mathbb{Z}, +, 0, \times)$ and uses the formula from arithmetic.
5. In infinite boolean algebras an unstable formula is $x \neq y$ & $(x \wedge y) = x$; here the domain of the linear order is *not* definable.

It is this flexibility, grounded in the formal language, which underlies the wide applicability of stability theory.

Shelah classification strategy 

A property P is a dividing line if
both P and $\neg P$ are virtuous.

Stable and superstable are dividing lines

ω-stable and $\aleph_1$-categorical are virtuous but not dividing lines.

Thesis 4

The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself (through the medium of geometric stability theory) plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics.

Three kinds of geometry

1. First order Euclidean geometry
2. first order formalizations of real and complex algebraic geometry
3. combinatorial geometry

What hath Hilbert wrought (elementary geometry)

Hilbert shows that First order axioms for euclidean geometry suffice for polygonal geometry including area and proportion and the basics of circle including right angle trigonometry.

I show that by adding a constant for π one can justify in first order logic the formulas for area and circumference of a circle.

These theories are constructively consistent (in PRA).

Hilbert uses the Archimedean and Dedekind axioms only for:

1. metamathematical investigations
2. asserting the identity of the plane satisfying all the axioms with geometry over the reals.

$\aleph_1$-categorical theories

Morley  Lachlan  Zilber 

Theorem

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.

Strongly minimal sets are the building blocks of structures whose first order theories are categorical in uncountable power.

Dimension: the essence of geometry

Dimension is a natural generalization of the notion of two and three dimensional space.

With coordinatization, the dimension tells us how many coordinates are needed to specify a point.

unidimensionality and categoricity in power

This dimension (for a countable language) and uncountable strongly minimal (more generally $\aleph_1$-categorical) structure is the same as the cardinality of the model.

The role of geometry

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

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Bourbaki's 3 great mother structures

order, groups, topology

ADD geometry

Geometry and Algebra are inevitable

Hrushovski 

Zilber / Hrushovski

Abstract model theoretic conditions imply algebraic consequences.

e.g. A group is definable in any $\aleph_1$-categorical theory that is not almost strongly minimal.

More technical hypothesis imply

1. the group is an abelian or a matrix group over an ACF of rank at most 3 or
2. there is a definable field.

The hypothesis do not mention anything algebraic.

Which theories are well behaved

The Main Gap: T has many models or is 'controlled by the countable'

Let T be a countable complete first order theory.

1. Either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ or
2. T is superstable without the *omitting types order property* or the *dimensional order property* and is shallow whence
   1. each model of cardinality λ is decomposed into countable models indexed by a tree of countable height and width λ.
   2. and thus, for any ordinal $\alpha > 0$, $I(T, \aleph_\alpha) < \beth_\delta(|\alpha|)$ (for a countable ordinal δ depending on T);

Either there is uniform way to assign invariants or there is the maximal number of models in every uncountable power.

It's inevitable: Abstract Model theory to algebra

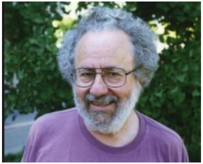
Hart, Hrushovski, Laskowski

Any model of a complete theory, whose uncountable spectrum is

$$I(\aleph_\alpha, T) = \min\left(2^{\aleph_\alpha}, \beth_{d-1}(|\alpha + \omega| + \beth_2)\right)$$

for some finite $d > 1$,

interprets an infinite group.

Can infinity be tamed? Davis 

Martin Davis wrote:

> *Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents.*

No! We systematically make this separation in important cases. What ”Gödel showed us is that the wild infinite could not really be separated from the tame mathematical world if we insist on starting with the wild worlds of arithmetic or set theory.

The crucial contrast is between:
a foundationalist approach – demand global foundations
and a foundational approach – search for mathematically important foundations of different topics.

Formalization as a mathematical tool

First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Differentially closed fields;
   Solves problems dating back to Painlevè 1900

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First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Differentially closed fields;
   Solves problems dating back to Painlevè 1900
2. Definable analysis
   The functions studied are composed from those named in the vocabulary. The domain of the model is the domain of the functions.
   o-minimality
   Applications to Hardy Fields, and asymptotic analysis real exponentiation, number theory

Axiomatic Analysis: Example

Fuchsian differential equation

$$S_{\frac{d}{dt}}(y) + (y')^2 R_\Gamma(y) = 0$$

In his famous 'Leçons de Stockholm', Painlevé conjectured that over any differential field extension K of $\mathcal{C}(t)$,

$$\mathrm{tr.deg} K(y, y', y'') = 0 \text{ or } 3.$$

Nagloo discovered that irreducibility in this setting is equivalent to a general model theoretic property called strong minimality.

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Theorem (Casale-Freitag-Nagloo)

Over any differential field extension K of $\mathcal{C}(t)$,

$$\text{tr. deg} K(y, y', y'') = 0 \text{ or } 3.$$

Previously, only partial results, most notably most notably work of Nishioka from the 1970s and 1980s.

Reliability or Clarity

. . . a long-term look at achievements in mathematics shows that genuine mathematical achievement consists primarily in making clear by using new concepts *. . .*

We look for uses of mathematical logic in bringing out these roles of of concepts in mathematics. Representations and methods from the reliability programs are not always appropriate.

We need to be able to emphasize special *features of a given mathematical area and its relationship to others, rather than how it fits into an absolutely general pattern. (Manders 1987)*

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Logic is the analysis of methods of reasoning

versus

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Logic is a tool for doing mathematics.

More precisely,
Mathematical logic is tool to solve not only its own problems but to organize and do traditional mathematics.

Part II

Two Theses

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Thesis 3: a multiplicity of logics

The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.

Thesis 4: The role of geometry

The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself (through the medium of geometric stability theory) plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics.