First order justification of $C = 2\pi r$

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago*

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Abstract

We provide first order theories EG_π (for basic Euclidean geometry with π) and $\mathcal{E}^2_\pi$ (for geometry over RCF with π) that justify the formulas $C = 2\pi r$ and $A = \pi r^2$ for the circumference and area of a circle. In addition we observe that the second theory is finitistically consistent.

Archimedes enunciated his famous axiom in developing an algorithm for computing π and determining that circumference of the circle is proportional to the diameter and the area to the square of the diameter. Hilbert uses both the Archimedean postulate and an equivalent to Dedekind's postulate in giving a geometric foundation for analytic geometry of the reals. In this note we provide an extension by definitions of each of the first order theory for Euclidean geometry and of Tarski's geometry $\mathcal{E}^2$, which justify the formulas $C = 2\pi r$ and $A = \pi r^2$. The philosophical and historical motivations for the results here are spelled out in [Bal14a, Bal14b]. A crucial point is a definition on the basis of synthetic geometry of arc length for circular arcs.

In the following, multiplication is the segment multiplication as defined by Hilbert. The axiom sets $HP5, EG_{\pi,C,A}$, etc. are defined in Notation 1.3. Our main results are:

Theorem 0.1. *In the first order theory $EG_{\pi,C,A}$, the circumference of a circle is $C = 2\pi r$ and the area of a circle is $A(r) = \pi r^2$.*

Theorem 0.2. *In the first order theory $\mathcal{E}^2_{\pi,C,A}$, the circumference of a circle is $C = 2\pi r$ and the area of a circle is $A(r) = \pi r^2$.*

Theorem 0.3. *The theory $\mathcal{E}^2_{\pi,C,A}$ is a complete consistent first order theory. Indeed the consistency can be proved in primitive recursive arithmetic (PRA).*

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Note that these theories may have non-Archimedean models. Since π is a constant symbol there is a uniform choice in each model of area and circumference for all circles (among the infinitesimally different choices in the monad of the chosen one). The proof of the first theorem is a simple consequence of compactness; the second relies on o-minimality.

1 Background and Definitions

The motivation and significance of this paper depend on some historical facts about the foundations of geometry. Full references and discussion of the following assertions appear in [Bal14a, Bal14b]

Remark 1.1 (Background)**.** Euclid founds his theory of area (of circles and polygons) on Eudoxus' theory of proportion and thus (implicitly) on the axiom of Archimedes.

Hilbert shows any 'Hilbert plane' interprets a field and recovers Euclid's polygonal theory in a first order theory.

Tarski provides a different formalism and axiomatizes the geometries over real-closed fields.

Neither Hilbert nor Tarski directly address circles in their axioms.

The Greeks and Descartes dealt only with geometric objects. The Greeks regarded multiplication as an operation from line segments to plane figures. Descartes interpreted it as an operation from line segments to line segments. Only in the late 19th century, is multiplication regarded as an operation on points (that is 'numbers' in the coordinatizing field).

We formulate our system in a two-sorted vocabulary τ chosen to make the Euclidean axioms (either as in Euclid or Hilbert) easily translatable into first order logic. This vocabulary includes unary predicates for points and lines, a binary incidence relation, a ternary betweenness relation, a quaternary relation for line congruence and a 6-ary relation for angle congruence.

We will work from Hilbert's first order axioms extended by circle-circle intersection to guarantee the existence of circles.

Postulate 1.2. Circle Intersection Postulate *If from points A and B, circles with radius AC and BD are drawn such that one circle contains points both in the interior and in the exterior of the other, then they intersect in two points, on opposite sides of AB.*

We follow Hartshorne[Har00] in the following nomenclature.

Notation 1.3. A *Hilbert plane* is any model of Hilbert's incidence, betweenness[1], and congruence axioms[2]. We abbreviate these axioms by HP. We will write HP5 for these axioms plus the parallel postulate.

By the *axioms for Euclidean geometry* we mean HP5 and in addition the circle-circle intersection postulate 1.2. We will abbreviate this axiom set[3] as EG.

By definition, a *Euclidean plane* is a model of EG: Euclidean geometry.

We write $\mathcal{E}^2$ for a geometrical axiomatization of the plane over a real closed field (RCF) (Theorem 2.2).

We transfer Tarski's system to one in the vocabulary above as follows.

Theorem 1.4. *Tarski [Tar59] gives a theory equivalent to the following system of axioms $\mathcal{E}^2$. It is first order complete for the vocabulary τ. The axioms are:*

1. *Euclidean geometry (EG);*
2. *Either of the following two sets of axioms which are equivalent over 1).*
   (a) *An infinite set of axioms declaring that every polynomial of odd-degree has a root.*
   (b) *The axiom schema of continuity described just below.*

The connection with Dedekind's approach is seen by Tarski's actual formulation as in [GT99]; the first order completeness of the theory is imposed by an **Axiom Schema of Continuity** - a definable version of Dedekind cuts:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(axy)] \rightarrow (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \rightarrow B(xby)],$$

where α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x and $B(x, y, z)$ means y is between x and z. This schema allow the solution of odd degree polynomials. By the completeness of real closed fields, this theory is also complete[4].

[1]These include Pasch's axiom (B4 of [Har00]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[2]These axioms are equivalent to the common notions of of Euclid and Postulates I-V augmented by one triangle congruence postulate, usually taken as SAS since that is where Euclid makes illegitimate use of the superposition principle.

[3]In the vocabulary here, there is a natural translation of Euclid's axioms into first order statements. The construction axioms have to be viewed as 'for all– there exist sentences. The axiom of Archimedes as discussed below is of course not first order. We write Euclid's axioms for those in the original [Euc56] vrs (first order) axioms for Euclidean geometry, EG. Note that EG is equivalent to (i.e. has the same models) as the system laid out in Avigad et al [ADM09], namely, planes over fields where every positive element as a square root). The latter system builds the use of diagrams into the proof rules.

[4]Tarski proves the equivalence of geometries over real closed fields with his axiom set in [Tar59].

2 π in Euclidean geometry

Hilbert [Hil71] (bi)interprets a field in any geometry satisfying $HP5$, after naming two arbitrary points as $0, 1$ to fix a line. As the example of geometry over the real algebraic numbers shows, there is no guarantee that there is a straight line segment whose 'length' is π. We remedy this with the following extension of the system.

Note that having named $0, 1$, each element of the *surd field* (the maximal real quadratic extension of the rational field), F_s, is denoted by a term $t(x, 0, 1)$ built from the field operations and $\sqrt{}$. Such terms name the perimeter of regular polygons inscribed (circumscribed) in the unit circle.

Definition 2.1 (Axioms for π). *1. Add to the vocabulary a new constant symbol π. Let i_n (c_n) be the perimeter of a regular $3 * 2^n$-gon inscribed[5] (circumscribed) in a circle of radius 1. Let $\Sigma(\pi)$ be the collection of sentences (i.e. type)*

$$i_n < 2\pi < c_n$$

for $n < \omega$.

2. EG_π denotes deductive closure in the vocabulary τ along with the constant symbols $0, 1, \pi$ of the axioms EG of a Euclidean plane and $\Sigma(\pi)$.

We formulated these axioms as properties of the point π rather than of the segment 0π because it is slightly more compact notation and more congenial to a modern reader. But shortly, we will describe the polygons approximating a circle in term of segments in the geometrical rather than the field language as it is more convenient. The compactness theorem easily yields:

Theorem 2.2. *EG_π is a consistent but incomplete theory. It is not finitely axiomatizable[6].*

Proof. A model of EG_π is given by closing $F_s \cup \{\pi\} \subseteq \Re$ under Euclidean constructions. To see EG_π is not finitely axiomatizable, for any finite subset Σ_0 of Σ choose a real algebraic number p satisfying Σ_0; close $F_s \cup \{p\} \subseteq \Re$ under constructibility to get a model of EG which is not a model of EG_π. $\square_{2.2}$

In EG it is straightforward to define a linear order on (equivalence classes under congruence of) straight line segments; indeed this is what makes Hilbert's field an ordered field. We now extend that order to certain arcs of circles. To avoid complications, we restrict our discussion of 'length' to arc of circles and straight lines with the following notation. Recall that Euclid uses the word 'line' to refer to any curve and

[5]I thank Craig Smorynski for pointing out that is not so obvious that the perimeter of an inscribed n-gon is monotonic in n and reminding me that Archimedes started with a hexagon and doubled the number of sides at each step.

[6]Ziegler ([Zie82], shows that EG is undecidable.

restrictively defines 'straight line'. As in Hilbert, here straight line is the basic notion. We will use the capitalized word *'Line' segment* to mean either a straight line segment or an arc (segment of a circle). An approximant is a bent line given by a (connected) piece of a circumscribed polygon.

Definition 2.3. *By a* bent line[7] $b = X_1 \ldots X_n$ *we mean a sequence of straight line segments* X_iX_{i+1} *such that each end point of one is the initial point of the next.*

1. *Note each bent line* $b = X_1 \ldots X_n$ *has a length* $[b]$ *given by the straight line segment composed of the sum of the segments of* b*.*
2. *An* approximant *to the arc* $X_1 \ldots X_n$ *of a circle with center* P*, is a bent line satisfying:*
   (a) $X_1, \ldots X_n, Y_1, \ldots Y_n$ *are points such that all* PX_i *are congruent and each* Y_i *is in the exterior of the circle.*
   (b) *Each of* X_1Y_1*,* Y_iY_{i+1}*,* Y_nX_n *is a straight line segment.*
   (c) X_1Y_1 *is tangent to the circle at* X_1*;* $Y_{n-1}X_n$ *is tangent to the circle at* X_n*.*
   (d) *For* $1 \leq i < n$*,* Y_iY_{i+1} *is tangent to the circle at* X_i*.*

Definition 2.4. *Let* $\mathcal{S}$ *be the set (of equivalence classes of) straight line segments. Let* $\mathcal{C}_r$ *be the set (of equivalence classes under congruence) of arcs on circles of a given radius*[8] r*. Now we extend the linear order on* $\mathcal{S}$ *to a linear order* $<_r$ *on* $\mathcal{S} \cup \mathcal{C}_r$ *as follows. For* $s \in \mathcal{S}$ *and* $c \in \mathcal{C}_r$

1. *The segment* $s <_r c$ *if and only if there is a chord* XY *of a circular arc* $AB \in c$ *such that* $XY \in s$*.*
2. *The segment* $s >_r c$ *if and only if there is an approximant* $b = X_1 \ldots X_n$ *to* c *with length* $[b] = s$ *and with* $[X_1 \ldots X_n] >_r c$*.*

It is easy to see that this order is well-defined since each chord of an arc is shorter than than any approximant to the arc (by repeated use of the triangle inequality, Euclid I.20).

Now we want to argue that π, as implicitly defined by the theory EG_π, serves its geometric purpose. For this, we add a new unary function symbol C mapping our fixed line to itself and satisfying the following scheme.

Definition 2.5. *A unary function* $C(r)$ *mapping* $\mathcal{S}$*, the set of equivalence classes (under congruence) of straight line segments, into itself that satisfies the conditions below is called* a circumference function

[7] This is less general than Archimedes (page 2 of [Arc97]) who allows segments of arbitrary curves 'that are concave in the same direction'.

[8] It at least requires some work to compare the length of arcs on circles of different radius and with chords of different lengths. We work around the issue now; our assignment of angle measure in Lemma 3.8 solves the problem in some models. Is there a more direct/more general solution?

ι_n: $C(r)$ *is greater than the perimeter of a regular inscribed* 3×2^n*-gon.*

γ_n: $C(r)$ *is less than the perimeter of a regular inscribed* 3×2^n*-gon.*

Definition 2.6. *The theory* $EG_{\pi,C}$ *is the extension of the* $\tau \cup \{0, 1, \pi\}$*-theory* EG_π *obtained by the explicit definition* $C(r) = 2\pi r$*.*

By similarity of the polygons $i_n(r) = ri_n$ and $c_n(r) = rc_n$, the ordering specified in Definition 2.5 will be satisfied if $C(r)$ is replaced by 'the circumference of a circle of radius r'. Note that while the approximations are given by standard 3×2^n-gons, defined by a schema, the translation to circles of different radius is done by multiplication within the geometry. So the approximations can be calculated for circles of any radius (including infinite or infinitessimal radius if the field is non-archimedean.)

Thus we have shown that for each r there is an $s \in \mathcal{S}$ whose length, $2\pi r$ is less than the perimeters of all inscribed polygons and greater that those of the inscribed polygons. We can verify that by choosing n large enough we can make i_n and c_n as close together as we like (more precisely, for given m differ by $< 1/m$). Our definition of EG_π then makes the following metatheorem immediate.

Theorem 2.7. *In* $EG^2_{\pi,C}$*,* $C(r) = 2\pi r$ *is a circumference function (i.e. satisfies all the conditions* ι_n *and* γ_n*).*

We have *not* established this claim for each arc in $\mathcal{C}_r$ for even one r. We will accomplish that task in Lemma 3.8.

In an Archimedean field there is a unique interpretation of π and thus a unique choice for a circumference function with respect to the vocabulary without the constant π. Since we added the constant π to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad of $2\pi r$ would equally well fit our condition for being the circumference. There may be automorphisms of a model of EG_π, but they must fix F_s pointwise and just move π in its cut in F_s.

We now note a consequence of EG_π, that allows us to treat the area of a circle. Using the notion of segment multiplication [Hil62] establishes the usual formulas for the area of polygons. We build on that here by considering also the area of circles.

Lemma 2.8 (Encoding a second approximation of π)**.** *Let* I_n *and* C_n *denote the area of the regular* 3×2^n*-gon inscribed or circumscribing the unit circle and for* $n < \omega$ *let* σ_n *denote the sentence:* $I_n < \pi < C_n$

Then EG_π *proves each* σ_n *is satisfied by* π*.*

Proof. The (I_n, C_n) define the cut for π in the surd field F_s reals and the (i_n, c_n) define the cut for 2π and it is a fact about the surd field that these are the same cut. (I.e. for every t, there exists an N_t such that if $k, \ell, m, n \geq N$ the distances between any pair of i_k, c_ℓ, I_m, I_n is less than $1/t$.) $\square_{2.8}$

Now, as in the circumference case, by formalizing a notion of equal area, including a schema for approximation by finite polygons (which for conciseness we omit), we can define a formal area function $A(r)$ which gives the area of circle just if is squeezed between the areas of a family of inscribing and circumscribing polygons.

Definition 2.9. *The theory $EG_{\pi,C,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory EG_{π} obtained by the explicit definition $A(r) = \pi r^2$.*

In the vocabulary with this function named we have, since the $I_n(C_n)$ converge to one half of the limit of the $i_n(c_n)$ and thus describe the same cut:

Theorem 2.10. *In $EG_{\pi,C,A}$, the area of a circle is $A(r) = \pi r^2$.*

3 π in Geometries over Real Closed Fields

A first order theory T for a vocabulary including a binary relation $<$ is *o-minimal* if every 1-ary formula is equivalent in T to a Boolean combination of equalities and inequalities [dD99]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [Tar31].

Theorem 3.1. *Form $\mathcal{E}^2_{\pi}$ by adjoining $\Sigma(\pi)$ to $\mathcal{E}^2$. $\mathcal{E}^2_{\pi}$ is first order complete for the vocabulary τ along with the constant symbols $0, 1, \pi$.*

Proof. By Hilbert, there are well-defined field operations on the line through 01. By Tarski, the theory of this real closed field is complete. The field is bi-interpretable with the plane [Tar51] so the theory of the geometry $\mathcal{E}^2$ is complete as well. Further by Tarski, the field is o-minimal. The type over the empty set of any point on the line is determined by its position in the linear ordering of the subfield F_s. Each i_n, c_n is an element of the field F_s. This position in the linear order of 2π in the linear order on the line through 01 is given by Σ. Thus, by o-minimality, $\mathcal{E}^2_{\pi}$ is complete. $\square_{2.2}$

We now rely on the definitions of bent line, circumference function etc. from Section 2. Using them we extend the theory $\mathcal{E}^2_{\pi}$.

Definition 3.2. *We define two new theories expanding $\mathcal{E}^2_{\pi}$.*

1. *The theory $\mathcal{E}^2_{\pi,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory $\mathcal{E}^2_{\pi}$ obtained by the explicit definition $C(r) = 2\pi r$.*
2. *The theory $\mathcal{E}^2_{\pi,C,A}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory T_{π} obtained by the explicit definition $A(r) = \pi r^2$.*

As an extension by explicit definition, $\mathcal{E}^2_{\pi,C,A}$ is complete and o-minimal. As before, by similarity $i_n(r) = r i_n$ and $c_n(r) = r c_n$, and so the approximations of π

by inscribed and circumscribed polygons will satisfy the conditions of Notation 2.5 if $C(r)$ is replaced by 'the circumference of a circle of radius r. As in Theorem 2.7, our definition of $\mathcal{E}^2_\pi$ then gives.

Theorem 3.3. *The theory $\mathcal{E}^2_{\pi,C,A}$, is a complete, decidable, and o-minimal extension of $EG_{\pi,C,A}$ and $\mathcal{E}^2_\pi$.*

1. *In $\mathcal{E}^2_{\pi,C}$, $C(r) = 2\pi r$ is a circumference function (i.e. satisfies all the ι_n and γ_n).*
2. *In $\mathcal{E}^2_{\pi,C,A}$, the area of a circle is $A(r) = \pi r^2$.*

Proof. We are adding definable functions to $\mathcal{E}^2_\pi$ so o-minimality and completeness are preserved. The theory is recursively axiomatized and complete so decidable. The formulas continue to compute area and circumference correctly as they extend $EG_{\pi,C,A}$. $\square_{3.3}$

We now extend the known fact that the theory of Real closed fields is 'finitistically justified' (in the list of such results on page 378 of [Sim09]) to $\mathcal{E}^2_{0,1,\pi,A,C}$. For convenience, we lay out the proof with reference to results[9] recorded in [Sim09].

Fact 3.4. The theory $\mathcal{E}^2$ is bi-interpretable with the theory of real closed fields. And thus it (as well as $\mathcal{E}^2_{\pi,C,A}$) is finitistically consistent (i.e provably consistent in primitive recursive arithmetic (PRA).).

Proof. By Theorem II.4.2 of [Sim09], RCA_0 proves the system $(Q, +, \times, <)$ is an ordered field and by II.9.7 of [Sim09], it has a unique real closure. Thus the existence of a real closed ordered field and so $Con(RCOF)$ is provable in RCA_0. (Note that the construction will imbed the surd field F_s.)

Lemma IV.3.3 [FSS83] asserts the provability of the completeness theorem (and hence compactness) for countable first order theories from WKL_0. Since every finite subset of $\Sigma(\pi)$ is easily seen to be satisfiable in any RCOF, it follows that the existence of a model of $\mathcal{E}^2_\pi$ is provable in WKL_0. Since WKL_0 is π^0_2-conservative over PRA, we conclude PRA proves the consistency $\mathcal{E}^2_\pi$. As $\mathcal{E}^2_{\pi,C,A}$ is an extension by explicit definitions its consistency is also provable in PRA. $\square_{3.4}$

A crucial feature of modern mathematics is to replace the 'area is proportional to' in Euclid by formulas which specify the proportionality constant. We have so far found the proportionality constant only for specific problems. In the remainder of the section, we consider several ways of systematizing the solution of families of such problems. First, we look for models where every angle determines an arc that corresponds to the length of a straight line segment. Then we consider several model theoretic schemes to organize such problems.

[9] We use RCOF here for what we have called RCF before as the argument here is quite sensitive to adding the order relation to the language. Note that Friedman[Fri99] strengthens the results for PRA to exponential function arithmetic (EFA).

We first look at angle measure, motivated by the strong hypothesis in Birkhoff's geometry. Euclid's 3rd postulate, "describe a circle with given center and radius", implies that a circle is uniquely determined by its radius and center. In contrast Hilbert simply defines the notion of circle and proves the uniqueness (See Lemma 11.1 of [Har00].). In either case we have: two segments of a circle are congruent if they cut the same central angle. So establishing an angle measure is the same as assigning a straight line segment as the length of each arc.

We contrast this with metric geometry (e.g. as axiomatized in [Bir32]).

Remark 3.5. Birkhoff [Bir32] introduced the following axiom in his system[10].

> POSTULATE III. The half-lines ℓ, m, through any point O can be put into $(1, 1)$ correspondence with the real numbers $a(\mathrm{mod} 2\pi)$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m respectively, the difference $a_m - a_\ell(\mathrm{mod} 2\pi)$ is $\angle AOB$.

This is a parallel to Birkhoff's 'ruler postulate' which assigns each segment a real number length. Thus, Birkhoff takes the real numbers as an unexamined background object. At one swoop he has introduced multiplication, and assumed the Archimedean and completeness axioms. So even 'neutral' geometries studied on this basis are actually are greatly restricted. He argues that his axioms define a categorial system isomorphic to $\Re^2$. So it is equivalent to Hilbert's.

The next task is to find a more modest version of Birkhoff's postulate: a first order theory with countable models which assign a measure to each angle in the model between 0 and 2π. Recall that we have a field structure on the line through 01 and the number π on that line. We will make one further explicit definition.

Definition 3.6. *A* measurement of angles *function is a map μ from congruence classes of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side BC, $\mu(\angle ABD) = \mu(\angle ABC) + \mu(\angle CBD)$*

Theorem 3.7. *For every countable model M of $\mathcal{E}^2_\pi$, there is a countable model M' containing M such that a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M'$.*

Proof. We adapt the proof of Theorem 2.7. Fix an angle XPY where X, Y are on the circumference of a unit circle with center P. Replace the inscribed and circumscribed polygons of Definition 2.5 by building polygons inscribed and circumscribing the sector (also using the two radii as two sides, but choosing new points to refine the approximation by bisecting each central angle at each stage). As in the proof of Theorem 2.2, we obtain the arc length as a type over the triple PXY in $\mathcal{E}^2_\pi$. Call an end point of an arc determined in this way an *angle measure cut*.

[10]This is the axiom system used in virtually all U.S. high schools since the 1960's.

Given a model N of $\mathcal{E}^2$, let N' be a countable elementary extension of N realizing all the, countably many, angle measure cuts determined by N. Now proceed inductively, let $M_0 = N$ and $M_{n+1} = M'_n$. Then M_ω is required model where μ is defined on all angles. $\square_{3.7}$

We have constructed a countable model M such that each arc of a circle in M has length measured by a straight line segment in M. There is no Archimedean requirement; adding the Archimdean axiom here would determine a unique number rather than a monad. Since each of the cuts we realized in the previous construction was given by a recursive type over a finite set, a recursively saturated model[11] will realize the relevant type to verify the following theorem.

Corollary 3.8. *If M is a countable recursively saturated model of $\mathcal{E}^2_\pi$ a measure of angles function μ is defined on the (congruence class of) each angle determined by points $P, X, Y \in M$. The unique countable model of the Scott sentence of the countable recursively saturated model then has the property that each angle has a measure.*

Note that in any model satisfying the hypotheses of Corollary 3.8, we can carry out elementary right angle trigonometry (angles less than 180^o). Unit circle trigonometry, where periodicity extends the sin function to all of the line violates o-minimality. (The zeros of the $\sin$ function are an infinite discrete set.)

Tarski ends [Tar59] by comparing the properties of three first order theories of geometry $\mathcal{E}^2$, EG, and the weak second order theory[12] of $\Re^2$. Tarski concludes:

> The author feels that, among these various conceptions, the one embodied in $\mathcal{E}^2$ distinguishes itself by the simplicity and clarity of underlying intuitions and by the harmony and power of its metamathematical implications.

We hope the ability to develop the formulas for the area and circumference of circles in a very mild extension of $\mathcal{E}^2$ bears witness to his judgement.

References

[ADM09] J. Avigad, Edward Dean, and John Mumma. A formal system for Euclid's elements. *Review of Symbolic Logic*, 2:700–768, 2009.

[Arc97] Archimedes. The measurement of the circle. In *On the sphere and cylinder I*, pages 1–56. Dover Press, 1897. Translation and comments by T.L. Heath (including 1912 supplement).

[11] See for example [Bar75].

[12] Weak second order logic allows quantification over finite sequences of elements.

[Bal14a] John T. Baldwin. Axiomatizing changing conceptions of the geometric continuum I: Euclid and Hilbert. http://homepages.math.uic.edu/~jbaldwin/pub/axconIsub.pdf, 2014.

[Bal14b] John T. Baldwin. Axiomatizing changing conceptions of the geometric continuum II: Archimedes- Descartes-Hilbert-Tarski. http://homepages.math.uic.edu/~jbaldwin/pub/axconIIfin.pdf, 2014.

[Bar75] J. Barwise, editor. *Admissible sets and structures*. Perspectives in Mathematical Logic. Springer-Verlag, 1975.

[Bir32] G.D. Birkhoff. A set of postulates for plane geometry. *Annals of Mathematics*, 33:329–343, 1932.

[dD99] L. Van den Dries. *Tame Topology and O-Minimal Structures*. London Mathematical Society Lecture Note Series, 248, 1999.

[Euc56] Euclid. *Euclid's elements*. Dover, New York, New York, 1956. In 3 volumes, translated by T.L. Heath; first edition 1908; online at http://aleph0.clarku.edu/~djoyce/java/elements/.

[Fri99] Harvey Friedman. A consistency proof for elementary algebra and geometry. preprint: http://www.personal.psu.edu/t20/fom/postings/9908/msg00067.html, 1999.

[FSS83] Harvey Friedman, Steve Simpson, and Rick Smith. Countable algebra and set existence axioms. *Annals Pure and Applied Logic*, pages 141–181, 1983.

[GT99] S. Givant and A. Tarski. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5:175–214, 1999.

[Har00] Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer-Verlag, 2000.

[Hil62] David Hilbert. *Foundations of geometry*. Open Court Publishers, 1962. reprint of Townsend translation 1902: Gutenberg e-book #17384 http://www.gutenberg.org/ebooks/17384.

[Hil71] David Hilbert. *Foundations of geometry*. Open Court Publishers, 1971. original German publication 1899: translation from 10th edition, Bernays 1968.

[Sim09] S.G. Simpson. *Subsystems of Second Order Arithmetic*. Cambridge, 2009.

[Tar31] Alfred Tarski. Sur les ensemble définissable de nombres réels I. *Fundamenta Mathematica*, 17:210–239, 1931.

[Tar51] Alfred Tarski. *A decision method for elementary algebra and geometry*. University of California Press, Berkeley and Los Angeles, Calif., 1951. 2nd edition, <http://www.ams.org/mathscinet/search/publications.html?pg1=IID&s1=170920>.

[Tar59] A. Tarski. What is elementary geometry? In Henkin, Suppes, and Tarski, editors, *Symposium on the Axiomatic method*, pages 16–29. North Holland Publishing Co., Amsterdam, 1959.

[Zie82] M. Ziegler. Einige unentscheidbare körpertheorien. *Enseignement Math.*, 28:269280, 1982. Michael Beeson has an English translation.