Constructing ω-stable Structures: Rank 2 fields

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Abstract

We provide a general framework for studying the expansion of strongly minimal sets by adding additional relations in the style of Hrushovski. We introduce a notion of *separation of quantifiers* which is a condition on the class of expansions of finitely generated models for the expanded theory to have a countable ω-saturated model. We apply these results to construct for each sufficiently fast growing finite-to-one function μ from 'primitive extensions' to the natural numbers a theory T^μ of an expansion of an algebraically closed field which has Morley rank 2. Finally, we show that if μ is not finite-to-one the theory may not be ω-stable.

The stability hierarchy classifies those first order theories which admit a rudimentary dimension theory as ω-stable, superstable or stable; this is a hierarchy with each class strictly containing the preceding one. Any simple algebraic group over an algebraically closed field falls into the first class (and more strongly has finite Morley rank). This paper is part of a general program ([3, 14, 13, 16, 15, 19, 11]) exploring the existence of structures which appear at the lower part of this hierarchy but are not 'classical ' (e.g. not structures studied in algebraic geometry).

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Cherlin and Shelah [10] showed that every superstable (and thus every ω-stable) pure field (language is field language) is algebraically closed. Answering a question of Berline and Lascar [8], and extending a result of Poizat [19] (who constructed a field of rank $\omega \times 2$), we show in this paper that there is an ω-stable bicolored field of Morley rank 2. That is, an algebraically closed field can be expanded by an infinite and coinfinite unary predicate to a structure of Morley rank 2. In the process we construct a general theory of δ-ranked structures which unifies and generalizes the constructions, originated by Hrushovski, of relatively flat structures, either *ab initio* or as fusions. The construction of homogeneous universal structures with respect to a notion of 'strong submodel' is key to this method. We call the structure constructed in this way 'generic'. Our development builds on and extends Holland's setting in ([12],[13]). We work in this generality for several reasons. One gain is the unification of prior work. More important, by clarifying the components of the argument we reduce even more important conjectures to the verification of certain properties stated here. The Cherlin-Zilber conjecture from the mid 70's asserts that every simple ω-stable group of finite Morley rank is an algebraic group (over an algebraically closed field). Borovik, Cherlin, Nesin, Tuna [9] and others are attempting to prove the conjecture by providing a classification theory of groups of finite Morley rank in analogy with the one for finite simple groups. A crucial obstacle to this program is the possible existence of a 'bad field': an ω-stable field of finite Morley rank with a definable infinite subgroup of its multiplicative group. To construct a bad field, we need only modify our construction to require the subset to be a multiplicative subgroup. The 'only' involves significant algebraic difficulties but this paper isolates them within the model theoretic framework. Further refinement of such a construction could provide an ω-stable finite rank simple group which is not algebraic, thereby directly refuting the Cherlin-Zil'ber conjecture. However, unpublished work of Hrushovski and the analysis in [18] indicates that all groups definable in fields obtained by the exact kind of construction discussed here are 'algebraic by commutative by finite'. Thus, the refinement will probably have to proceed along more elaborate lines such as those suggested in [7].

Our general framework allows the construction of expansions of strongly minimal sets to ω-stable structures with Morley rank $\omega \times k$. (When the strongly minimal set is an algebraically closed field and $k = 2$, this was done in [19]). The extension to an arbitrary strongly minimal set is routine; the extension to $k > 2$ requires some additional analysis [4]. More difficult is our construction of structures of finite rank. The key to our argument is characterizing when the generic model obtained from a δ-rank is ω-saturated. We show that certain natural theories do not satisfy this condition and in fact do not have an ω-saturated model; thus the theory is not even ω-stable. We conjecture that our methods will show that these theories are superstable. This would answer a question of [2].

The first section describes a very general situation. We describe constraints on a 'weight' function δ which permits the construction of generic structures with a natural 'stability-like' rank. We do not call this dimension since the exchange property will usually fail. In Section 1 we provide a condition, *separation of quantifiers*, which characterizes those cases where the generic model is ω-saturated. In Section 2, we specialize to expansions of strongly minimal sets by unary predicates. Thus, in particular, we consider the 'bicolored fields' of [19]. We develop the general properties of codes [15, 12] in this context. Our main goal is the construction of bicolored fields of finite rank. As in [16, 15, 3, 12], these are obtained by uniformly bounding the number of 'primitive' extensions of δ-rank 0 by a function μ. In Section 3, we show that if μ is finite-to-one, the theory of the associated generic has separation of quantifiers. We deduce that the generic is ω-stable with Morley rank 2. In Section 4, we show that without this finite-to-one assumption, the theory T^{μ} is not even small.

1 Generalities

Fix a countable first order language L which may have function symbols. We begin with a theory T_0 whose class $\overline{\boldsymbol{K}}_0$ of models is closed under substructure. We let $\boldsymbol{K}_0$ denote the class of *finitely generated* (as structures) elements of $\overline{\boldsymbol{K}}_0$. We describe the properties of a 'weight' function δ on finite sequences from members of $\boldsymbol{K}_0$ which permit the construction of generic structures. The chief novelty of this framework is to clarify the definability conditions on weight which unify the fusion and *ab initio* cases. Then we provide a necessary and sufficient condition for the generic structure to be ω-saturated.

1.1 Notation. We will write $B \subseteq_\omega N$ to indicate B is a finite subset of N. If $A, B \subseteq N$ we write AB for $A \cup B$. For X a subset of N, $\langle X \rangle_N$ denotes the substructure of N generated by X. We will generally omit the subscript. For $A \in \boldsymbol{K}_0$, we write Diag(A) for the *quantifier-free* diagram of A. Note that even when $A = \overline{a}$ is a finite sequence, Diag$(\overline{a})$ is in general a type not a formula.

Throughout this paper, δ is a function from quantifier free diagrams of finite subsets of elements of $\overline{\boldsymbol{K}}_0$ into the natural numbers. In 1.5 and 1.7 we list the properties which are used in our proofs; they hold in the following examples.

1.2 Examples. ab initio For a finite relational language L, $\boldsymbol{K}_0$ is a class of finite L-structures; $\delta(A) = k|A| - e(A)$ (where $e(A)$ counts the number of times the relations of L are realized in A and k is a natural number). We require that $A \in \boldsymbol{K}_0$ implies that for any $B \subseteq A$, $\delta(B) \geq 0$. In this situation T_0 is the universal class determined by $\boldsymbol{K}_0$.

fusion Let T_1 and T_2 be complete, model complete, strongly minimal theories with the definable multiplicity property and elimination of imaginaries in countable languages L_1 and L_2 with $L_1 \cap L_2 = \{=\}$ and respective dimension functions d_1 and d_2. For $N \models T_1 \cup T_2$ and $X \subset_\omega N$, $\delta(X) = d_1(X) + d_2(X) - |X|$. Here, T_0 is $T_1 \cup T_2$ along with the requirement that for each finite X contained in a model of $T_1 \cup T_2$, $\delta(X) \geq 0$.

bicolored T_f is a strongly minimal theory with the definable multiplicity property, elimination of quantifiers and of imaginaries in a countable language L_f with dimension function d_f. Form L by adjoining a unary predicate P (for 'black' points). Let $T_f^{\forall}$ be the L-theory axiomatized by the universal sentences of T_f. For $T_f^{\forall}$ and $X \subset_\omega N$, $\delta(X) = k \cdot d_f(X) - |X \cap P(N)|$. Let T_0 be $T_f^{\forall}$ along with the requirement that for each finite X contained in a model of T_f, $\delta(X) \geq 0$.

The *ab initio* case was introduced in [16] (with $k = 1$) to refute Zil'ber's conjectured classification of strongly minimal sets. This was extended to $k = 2$ in the construction of an $\aleph_1$-categorical non-Desarguesian projective plane in [3]. The fusion case [15] proved the existence of a common expansion of two strongly minimal sets with the same universe. Poizat [19] suggested the study of the third case to prove the existence of fields of finite rank (and constructed fields of rank $\omega \cdot 2$).

Note that $\langle a \rangle = \langle b \rangle$ need not imply that $\delta(a) = \delta(b)$. For example, in bicolored fields with $k = 2$, if a is white and $2a$ is black, $\langle a \rangle = \langle 2a \rangle$ while $\delta(a) = 2$ and $\delta(2a) = 1$.

1.3 Definition. For $N \in \overline{\boldsymbol{K}}_0$ and X and Y finite subsets of N, we write $\delta(X/Y) = \delta(XY) - \delta(Y)$. For U and V subsets of N with $U \subseteq V$, we say that U is a *strong* subset of V, and write $U \leq V$, if for every finite subset X of V, $\delta(X/X \cap U) \geq 0$. If $A \leq A\overline{a}$ we write $\delta(\overline{a}/A)$ for $\min\{\delta(\overline{a}/B) : \overline{a} \cap A \subseteq B \subseteq_\omega A\}$.

If $Y \leq X$ and for every Y' with $Y \subset Y' \subseteq X$, $Y' \not\leq X$, we say that X is a *minimal strong extension* of Y. If, in addition, $\delta(X/Y) = 0$, we say that X is a *primitive* extension of Y.

It is easily verified that an arbitrary intersection of strong subsets of a set V is again strong in V. It follows that for any X and V with $X \subseteq V$, there is a unique smallest subset U of V with $X \subseteq U \leq V$. We may therefore make the following definition.

1.4 Definition. If $X \subseteq B$, the *intrinsic closure* of X in B, denoted $\mathrm{icl_B(X)}$, is the unique smallest set C with $X \subseteq C \leq B$. If $A \leq B$ and $X \subseteq B$, then $\mathrm{icl_B(X/A)} = \mathrm{icl_B(AX) - A}$.

If $Y \subseteq X$, $\delta(X/Y) < 0$ and for any proper subset X' of X, $\delta(X'/Y) \geq 0$, we say that X is a *minimal intrinsic extension* of Y.

Note that, since δ takes on values in the natural numbers, for $A \leq B$ and $X \subseteq B$, $\mathrm{icl_B(X/A)}$ must be finite if $X - A$ is. In particular, $\mathrm{icl_B(X)}$ is finite if X is. It is easily seen that if $Y \subseteq X \subseteq B$ and X is a minimal intrinsic extension of Y, then $X \subseteq \mathrm{icl_B(Y)}$.

We restrict to the case where $\boldsymbol{K}_0$ has the $\leq$- amalgamation property. As usual [6], this produces a unique countable model which is homogeneous with respect to strong extensions of finitely generated substructures and is a union of finitely generated strong substructures. We call this the *generic* model of T_0 and let T denote the theory of this generic model. We will actually establish the amalgamation property for the relevant theories in Section 2.

1.5 Definition. A quantifier free formula $\phi(\overline{x};\overline{y})$ is a *δ-formula* and δ_ϕ is the integer k if the following hold.

1. A sequence $\overline{a}\overline{b}$ satisfying $\phi(\overline{x};\overline{y})$ consists of distinct elements.
2. For some $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{b} \subseteq N$ with $\delta(\overline{a}/\overline{b}) = k$, $N \models \phi(\overline{a};\overline{b})$.
3. For any $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{b} \subseteq N$, if $N \models \phi(\overline{a};\overline{b})$, then $\delta(\overline{a}/\overline{b}) \leq k$.
4. For any $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{a'}, \overline{b} \subseteq N$, if $N \models \phi(\overline{a};\overline{b}) \wedge \phi(\overline{a'};\overline{b})$ and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a'}/\overline{b}) = k$ and $\overline{ba} \leq \langle\overline{ba}\rangle$, $\overline{ba'} \leq \langle\overline{ba'}\rangle$, then $Diag(\overline{ba}) = Diag(\overline{ba'})$. In particular, under these conditions, $\langle\overline{ba}\rangle \simeq \langle\overline{ba'}\rangle$ via $\overline{ba} \mapsto \overline{ba'}$.

We call $\phi(\overline{x};\overline{y})$ a *complete δ-formula* if in addition it satisfies the following condition.

5. For any disjoint subtuples $\overline{x}^1, \overline{x}^2$ from $\overline{x}$, there is a δ-formula $\phi'(\overline{x}^1;\overline{x}^2\overline{y})$ such that $T_0 \models \phi(\overline{x};\overline{y}) \rightarrow \phi'(\overline{x}^1;\overline{x}^2\overline{y})$ and for any $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{b} \subseteq N$, $N \models \phi(\overline{a};\overline{b})$ and $\delta_\phi = \delta(\overline{a}/\overline{b})$ implies $\delta_{\phi'} = \delta(\overline{a}^1/\overline{a}^2\overline{b})$.

Condition 5) hold trivially in the *ab initio* case since each finite sequence in a relational language can be completely described by a first order formula. For our more general situation, the condition is more complicated since we are asserting that a finite description plus 'strong' determines the structure of an infinite set.

1.6 Definition. If $\overline{a}\overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$ and $\phi(\overline{x};\overline{y}) \in Diag(\overline{a};\overline{b})$ is a (complete) δ-formula with $\delta_\phi = \delta(\overline{a}/\overline{b})$, we say that ϕ is a (complete) δ-formula for $\overline{a}$ *over* $\overline{b}$. If also $\overline{b} \leq B \subseteq N$, $\overline{a} \cap B = \emptyset$ and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a}/B)$, we say that ϕ is (complete) δ-formula for $\overline{a}$ over B *based on* $\overline{b}$.

By saying simply that a formula $\phi(\overline{x};\overline{y})$ is a "complete δ-formula for a minimal strong extension," we mean that there is a pair $\overline{a}\overline{b}$ in some model of T_0 such that $\overline{a}$ is a minimal strong extension of $\overline{b}$ and $\phi(\overline{x};\overline{y})$ is a complete δ-formula for $\overline{a}$ over $\overline{b}$. It is easily shown that if such $\overline{a}$ and $\overline{b}$ exist, then, in fact, for *any* $\overline{a'}$ and $\overline{b'}$ satisfying $\phi(\overline{x};\overline{y})$ for which $\delta(\overline{a'}/\overline{b'}) = \delta_\phi$, $\overline{a'}$ must be a minimal strong extension of $\overline{b'}$ as well. The same terminology and remarks apply when "minimal strong extension" is replaced by "primitive extension" or "minimal intrinsic extension." Thus, complete δ-formulas describe the internal structure on the principal kinds of extensions that we discuss.

1.7 Constraints on δ. We require that δ satisfy the following conditions:

1. If $\overline{b} \cap \overline{c} \subseteq \overline{a}$, $\delta(\overline{b}/\overline{ac}) \leq \delta(\overline{b}/\overline{a})$.
2. For any $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$, there is a δ-formula for $\overline{a}$ over $\overline{b}$.

3. If ϕ is quantifier free, $T_0 \vdash \forall \overline{y} \exists^{\leq k} \overline{x} \phi(\overline{x}; \overline{y})$ and $\models \phi(\overline{a}; \overline{b})$, then $\delta(\overline{a}/\overline{b}) \leq 0$. In particular, for any $\overline{a} \subseteq \langle \overline{b} \rangle$, $\delta(\overline{a}/\overline{b}) \leq 0$.

Note that by iterating Condition 2 of 1.7, we easily get (for $N \in \boldsymbol{K}_0$, $\overline{a}, \overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$) the *a priori* stronger (than Constraint 2) condition of existence of a *complete* δ-formula for $\overline{a}$ over $\overline{b}$. Moreover, if $\overline{a} \cap B = \emptyset$ and $B \leq B\overline{a}$, then there is some $\overline{b} \leq B$ with $\delta(\overline{a}/\overline{b}) = \delta(\overline{a}/B)$ and for any such $\overline{b}$, a (complete) δ-formula for $\overline{a}$ over $\overline{b}$ constitutes a (complete) δ-formula for $\overline{a}$ over B based on $\overline{b}$.

Constraint 3 (together with some arithmetic of δ) implies that if $\overline{a} \leq A \in \boldsymbol{K}_0$ then $\langle \overline{a} \rangle \leq A$.

These constraints hold of δ in the cases we have discussed. For the *ab initio* and fusion cases this is established in [16, 15]. Note that in the *ab initio* case the formula ϕ for $\overline{a}\overline{b}$ is just the diagram of $\overline{a}\overline{b}$ and any realization of ϕ is isomorphic to $\overline{a}\overline{b}$.

The isomorphism type of an element N of $\boldsymbol{K}_0$ is determined by $\mathrm{Diag}(\overline{a})$ for any $\overline{a} \leq N$ with $\langle \overline{a} \rangle = N$. Any $A \in \boldsymbol{K}_0$ has the form $\langle \overline{a} \rangle = A$ for some finite sequence $\overline{a}$ with $\overline{a} \leq A$. If $\delta(\overline{a}) = k$, there is a δ-formula $\phi(\overline{x})$ such that $\phi(\overline{a})$ with $\delta_\phi = k$. For any $\overline{a}'$: if $\delta(\overline{a}') = k$, $\overline{a}' \leq \langle \overline{a}' \rangle$ and $\phi(\overline{a}')$ holds, clause 4 of Definition 1.5 implies $\langle \overline{a}' \rangle \approx \langle \overline{a} \rangle$. Thus, since there are only countably many δ-formulas, there are only countably many isomorphism types in $\boldsymbol{K}_0$. Moreover, since for any finite $\overline{a}$ there are only countably many possibilities for $\mathrm{icl}_{\langle \overline{a} \rangle}(\overline{a})$, there are only countably many quantifier free types of finite sets realized in members of $\overline{\boldsymbol{K}_0}$.

1.8 Definition. We denote by $I(\overline{y})$ the collection of all formulas $\forall \overline{x} \neg \phi(\overline{x}; \overline{y})$, where for some $\overline{a}$ and $\overline{b}$, $\overline{a}$ is a minimal intrinsic extension of $\overline{b}$ and $\phi(\overline{x}, \overline{y})$ is a complete δ-formula for $\overline{a}$ over $\overline{b}$.

Let $\overline{b} \in N \in \overline{\boldsymbol{K}_0}$ with $\overline{b} \leq \langle \overline{b} \rangle \leq N$ and $\mathrm{lg}(\overline{y}) = \mathrm{lg}(\overline{b})$: $I^*(\overline{b})$ is $I(\overline{y}) \cup \mathrm{Diag}(\overline{b})$.

Note that if $\overline{b} \subseteq B$, then $B \models I(\overline{b})$ if and only if $\overline{b} \leq B$. This yields immediately

1.9 Lemma. *If $A, B \in \boldsymbol{K}_0$ and A is an elementary submodel of B, then $A \leq B$.*

Thus, $I^*(\overline{c})$ is realized by $\overline{b}$ in a model N just if $\langle \overline{c} \rangle \simeq \langle \overline{b} \rangle$ and $\overline{b} \leq N$. Now suppose $\overline{b} \leq \overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle$ and $\overline{b} \leq M$, the generic. Then there is a strong embedding of $\langle \overline{b}\overline{a} \rangle$ into M over $\overline{b}$. Thus, M satisfies $(\exists \overline{x}) \tau(\overline{x}, \overline{b})$ for each $\tau(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$. If M is saturated then, for $T = \mathrm{Th(M)}$, for each $\tau(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$, $T \cup I^*(\overline{b}) \vdash (\exists \overline{x}) \tau(\overline{x}, \overline{b})$. Thus, if the generic is saturated, every strong embedding of $\overline{b}$ into a model of the theory T extends locally (formula by formula) to strongly embed approximations to a strong extension $\overline{a}$ of $\overline{b}$. We show below that saturation of the generic is necessary for this to hold; otherwise there may be instances of $\overline{b}$ in other models which cannot be so extended. See [1] and Section 4. Then we convert this assertion to a checkable condition on $(\boldsymbol{K}_0, \delta)$ which ensures that the generic structure is ω-saturated. While searching for a good name for this concept we read Hrushovski's [17], where he introduced a similar notion in defining a Robinson theory. We want to maintain this notion as an adjective rather than a part of the definition of the context (as in [17]) because of the interesting examples of generic structures which are not ω-saturated (e.g. [5] and the last section of this paper).

1.10 Definition. We say that T admits *separation of quantifiers* if for any $\overline{b} \leq \overline{b}\overline{a} \leq \langle \overline{b}\overline{a} \rangle \in \boldsymbol{K}_0$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$, for each $\tau(\overline{x}, \overline{y}) \in I^*(\overline{a}, \overline{b})$, $T \cup I^*(\overline{b}) \vdash (\exists \overline{x})\tau(\overline{x}, \overline{b})$.

The significance of this definition is given in the following theorem.

1.11 Theorem. *If T_0 has the $\leq$-amalgamation property and the theory of the generic model of T_0 admits separation of quantifiers then that generic model is ω-saturated.*

Proof. Fix an ω-saturated model N of $Th(M)$. We show M is $L_{\infty,\omega}$-equivalent to N by constructing a back and forth; thus M is saturated. It suffices to show that whenever $\overline{b} \leq M$ and $\overline{b'} \leq N$ with $\mathrm{Diag}(\overline{b}) = \mathrm{Diag}(\overline{b'})$,

1. if $\overline{b}\overline{a} \leq M$ with $\overline{a}$ minimal strong over $\overline{b}$ then there is $\overline{a'}$ with $\overline{b'}\overline{a'} \leq N$ and $\mathrm{Diag}(\overline{b}\overline{a}) = \mathrm{Diag}(\overline{b'}\overline{a'})$;
2. if $\overline{b'}\overline{a'} \leq N$ with $\overline{a'}$ minimal strong over $\overline{b'}$ then there is $\overline{a}$ with $\overline{b}\overline{a} \leq M$ and $\mathrm{Diag}(\overline{b'}\overline{a'}) = \mathrm{Diag}(\overline{b}\overline{a})$.

The second condition is immediate from genericity of M, so we prove only the first. Suppose that $\overline{a}\overline{b} \leq M$. Since T admits separation of quantifiers and $I^*(\overline{b}) \subseteq tp_N(\overline{b'})$, we have that $N \models (\exists \overline{x})\tau(\overline{x}, \overline{b}'))$ for every $\tau(\overline{x}, \overline{y})$ from $I^*(\overline{a}, \overline{b})$. In the ω-saturated N there is an $\overline{a}'$ such that $\overline{b}', \overline{a}'$ realize $I^*(\overline{a}, \overline{b})$ as required. $\square_{1.11}$

We can immediately reformulate separation of quantifiers as follows. For any formula $\tau(\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$ there are formulas $\sigma(\overline{y}) \in I(\overline{y})$ and $\alpha(\overline{y}) \in \mathrm{Diag}(\overline{b})$ such that

$$M \models \forall \overline{y}((\sigma \wedge \alpha)(\overline{y}) \rightarrow \exists \overline{x} \tau(\overline{x}; \overline{y})).$$

Now we provide an equivalent property of $(\boldsymbol{K}_0, \delta)$.

1.12 Definition. We say $(\boldsymbol{K}_0, \delta)$ *admits separation of quantifiers* if for any $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \boldsymbol{K}_0$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$, the following holds: For any formula $\tau(\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$ there are formulas $\sigma(\overline{y}) \in I(\overline{y})$ and $\alpha(\overline{y}) \in \mathrm{Diag}(\overline{b})$ such that whenever $\overline{b'} \subseteq C \in \boldsymbol{K}_0$ and $C \models (\sigma \wedge \alpha)(\overline{b'})$, there is $D \in \boldsymbol{K}_0$ with $C \leq D$ and $\overline{a'} \in D$ such that $D \models \tau(\overline{a'}; \overline{b'})$.

1.13 Lemma. *The following are equivalent:*

1. *$(\boldsymbol{K}_0, \delta)$ admits separation of quantifiers.*

2. T admits separation of quantifiers.

Proof. Let M be the countable generic for $(\boldsymbol{K}_0, \delta)$. Suppose 1 holds. Fix $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \boldsymbol{K}_0$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$ and suppose $\tau(\overline{x}; \overline{y})$ in $I^*(\overline{a}, \overline{b})$. Choose $\sigma(\overline{y})$ and $\alpha(\overline{y})$ by separation of quantifiers for $(\boldsymbol{K}_0, \delta)$. Suppose $\overline{b'} \in M$ and $M \models (\sigma \wedge \alpha)(\overline{b'})$. Let C be the substructure of M generated by some finite, strong subset of M containing $\overline{b'}$; then $C \in \boldsymbol{K}_0$ and $C \leq M$. Since $\sigma \wedge \alpha$ is universal, $C \models (\sigma \wedge \alpha)(\overline{b'})$. Since $(\boldsymbol{K}_0, \delta)$ has separation of quantifiers, there is $D \in \boldsymbol{K}_0$ with $C \leq D$ and $\overline{a'} \in D$ with $D \models \tau(\overline{a'}; \overline{b'})$. By the definition of generic, D can be embedded strongly into M over C, so that $M \models \tau(\overline{a'}; \overline{b'})$ as well, since τ is preserved by strong embeddings. Thus, $M \models \forall \overline{y}((\sigma \wedge \alpha)(\overline{y}) \rightarrow \exists \overline{x} \tau(\overline{x}; \overline{y}))$, as desired.

Now assume that 2 holds. Fix $\overline{a}, \overline{b}$ and τ as in the definition of separation of quantifiers for $(\boldsymbol{K}_0, \delta)$. Using the reformulation of separation of quantifiers for T choose σ and α. Let $\overline{b'} \subseteq C \in \boldsymbol{K}_0$ with $C \models (\sigma \wedge \alpha)(\overline{b'})$. We must find $D \in \boldsymbol{K}_0$ with $C \leq D$ and $\overline{a'} \in D$ such that $D \models \tau(\overline{a'}; \overline{b'})$. Without loss of generality, $C \leq M$. Since α is quantifier free, $M \models \alpha(\overline{b}')$. Suppose $M \models \neg\sigma(\overline{b'})$. Now $\neg\sigma(\overline{b}')$ has the form $(\exists \overline{x})\sigma'(\overline{x}, \overline{c}')$ where $\overline{c}' \in \langle \overline{b}' \rangle$ and $\sigma'(\overline{x}, \overline{y})$ is a complete δ-formula for some minimal intrinsic extension $\overline{e}$ of $\overline{c}$ (where $\overline{c} \in \langle \overline{b} \rangle$ and the isomorphism between $\overline{b}$ and $\overline{b}'$ extends to map $\overline{c}$ to $\overline{c}'$). Thus, there is an $\overline{e}' \in M$ such that for every partition $\overline{e}'_1\overline{e}'_2$ of $\overline{e}'$ with $\overline{e}'_2$ nonempty, $\delta(\overline{e}'_2/\overline{e}'_1\overline{c}') < 0$. In particular, if $\overline{e}' \cap C = \overline{e}'_1$ is a proper subset of $\overline{e}'$ we contradict $C \leq M$. But $\overline{e}' \in C$ contradicts $C \models \sigma(\overline{b'})$. So $M \models (\sigma \wedge \alpha)(\overline{b'})$. Then, by hypothesis, $M \models \tau(\overline{a'}; \overline{b'})$ for some $\overline{a'}$. Take D to be any strong substructure of M containing $C\overline{a'}$. $\square_{1.13}$

1.14 Corollary. *If $(\boldsymbol{K}_0, \delta)$ admits separation of quantifiers and has the $\leq$-amalgamation property then the generic M is saturated.*

2 Bicolored strongly minimal structures

Let T_f be a complete, strongly minimal theory in a countable language L_f in which it has function symbols for all 0-definable functions, which has definable multiplicity and admits both elimination of imaginaries and elimination of quantifiers. The notion of definable multiplicity was introduced in [15]. We do not use the notion directly in this paper but definable multiplicity, elimination of quantifiers and elimination of imaginaries are all used by Hrushovski in the proof of the existence of standard codes, Lemma 2.2. In particular, these hypotheses allow the introduction of the function F considered in Definition 2.1. In this section we consider several expansions of T_f by adding a unary predicate. The general procedure for this expansion stems from [15] and in our case the exposition in [12]. Poizat [19] began the investigation of bicolored fields and proved the existence of infinite rank extensions. Much of this section is similar to [19] and we refer some of the proofs to that paper. But our exposition is a little more formal, in the style of [15] and [12], and provides the basis for the proof in Section 3 that some of these theories have Morley rank 2.

Let $T_f^\forall$ denote the universal part of T_f, so that $T_f^\forall$ is the theory of substructures of models of T_f. Note that by quantifier elimination, for any model N of $T^\forall$, $T_f \cup \mathrm{Diag}_N$ is complete. For $X \subseteq_\omega N \models T_f^\forall$, we let $d_f(X)$ denote the transcendence degree of X in a model of $T_f \cup \mathrm{Diag}_N$, and $acl_f(X) = \{y \in N : \ d_f(y/X) = 0\}$.

We say that a solution $\overline{g}$ of a formula $\phi(\overline{x}; \overline{a})$ is *generic over* X if $R_M(\mathrm{tp}(\overline{\mathrm{g}}/\overline{\mathrm{a}}\mathrm{X})) = \mathrm{R_M}(\phi(\overline{\mathrm{x}}, \overline{\mathrm{a}}))$. In case $X = \emptyset$, we say, simply, that $\overline{g}$ is a generic solution.

We modify slightly the notion of code from [15].

2.1 Definition. An *f-code* $\boldsymbol{c}$ is a seven-tuple $\langle \phi, \theta, \psi, F, m, k, n\rangle$ consisting of L_f-formulas $\phi(x_1, \ldots, x_n; \overline{y})$, $\theta(\overline{y})$, $\psi(\overline{x}_1, \ldots, \overline{x}_m,)$, a function symbol $F(\overline{x}_1, \ldots, \overline{x}_m)$ and integers m, k, and $n = l(\overline{x})$ of which the following are true in any model of $T_f^\forall$.

1. A sequence $\overline{a}\overline{b}$ satisfying $\phi(\overline{x}; \overline{y})$ consists of distinct elements.

2. $\phi(\overline{a}; \overline{b})$ implies $\theta(\overline{b})$.

3. Whenever $\theta(\overline{b})$ holds, $\phi(\overline{x}; \overline{b})$ has Morley degree 1, and $d_f(\overline{g}/\overline{b}) = k$ for a generic solution $\overline{g}$ of $\phi(\overline{x}; \overline{b})$.

4. For each partition $\overline{x} = \overline{x}^1\overline{x}^2$ of $\overline{x}$ there is a k' such that whenever $\theta(\overline{b})$ holds and $\overline{a} = \overline{a}^1\overline{a}^2$ is the corresponding partition of a solution of $\phi(\overline{x}; \overline{b})$, $d_f(\overline{a}^1/\overline{a}^2\overline{b}) \leq k'$. Moreover, equality holds if $\overline{a}$ is a generic solution of $\phi(\overline{x}; \overline{b})$.

5. ψ is symmetric in its arguments, and whenever $\theta(\overline{b})$ holds,
   (a) ψ is satisfied by any independent m-tuple of solutions of $\phi(\overline{x}; \overline{b})$,
   (b) $\models \forall \overline{x}_1, \ldots, \overline{x}_m[\psi(\overline{x}_1, \ldots, \overline{x}_m) \wedge \bigwedge \phi(\overline{x}_i; \overline{b})] \rightarrow F(\overline{x}_1, \ldots, \overline{x}_m) = \overline{b}$
   (c) if $\overline{a}_1, \ldots, \overline{a}_{m-1}$ are any solutions of $\phi(\overline{x}; \overline{b})$ and $\overline{g}$ is a solution generic over $\overline{b}, \overline{a}_1, \ldots, \overline{a}_{m-1}$ then $\psi(\overline{g}, \overline{a}_1, \ldots, \overline{a}_{m-1})$ holds.

While we cannot define the property that $\overline{a}^1, \ldots, \overline{a}^m$ is an independent sequence of realizations of $\phi(\overline{x}; \overline{b})$, the formula ψ provides an approximation which implies crucial properties of an independent sequence. In particular, $F(\overline{a}^1, \ldots, \overline{a}^m) = \overline{b}$ holds for any independent m-tuple of solutions of $\phi(\overline{x}; \overline{b})$),

The following lemma is proved in [15]; it uses the definability of Morley rank and degree and elimination of imaginaries.

2.2 Lemma. *There is a set S_f of f-codes for T_f satisfying the following conditions:*

1. *If $\boldsymbol{c}_1 = \langle \phi_1, \theta_1, \psi_1, F_1, m_1, k_1, n_1\rangle$ and $\boldsymbol{c}_2 = \langle \phi_2, \theta_2, \psi_2, F_2, m_2, k_2, n_2\rangle$ are elements of S_f, $\models \theta_1(\overline{b}^1)$, $\models \theta_2(\overline{b}^2)$ and $\phi_1(\overline{x}; \overline{b}^1)$ and $\phi_2(\overline{x}; \overline{b}^2)$ have a common generic solution, then $\boldsymbol{c}_1 = \boldsymbol{c}_2$ and $acl_f(\overline{b}^1) = acl_f(\overline{b}^2)$.*

2. *For any formula $\rho(\overline{x};\overline{b})$ of Morley degree 1 whose solutions consist of distinct elements, there are $\boldsymbol{c} = \langle \phi, \theta, \psi, F, m, k, n\rangle$ in S_f and $\overline{b'} \in \langle\overline{b}\rangle$ satisfying $\theta(\overline{b'})$ such that $\rho(\overline{x};\overline{b})$ and $\phi(\overline{x};\overline{b'})$ have the same generic solutions.*

3. *If $\boldsymbol{c}$ is obtained from $\boldsymbol{c}'$ by a permutation of the variables $\overline{x}$ and $\boldsymbol{c} \in S_f$, then $\boldsymbol{c}' \in S_f$.*

We fix a set S_f of the sort guaranteed by Lemma 2.2, and refer to its elements as *standard* f-codes.

The last definition and lemma concerned a strongly minimal theory in the base language and so the proof was taken from [15] since that part of the argument is common to the study of fusions and the study here. We now turn to bicolored strongly minimal sets and the properties of the dimension function and appropriate codes. Our exposition follows that of [15] and [12], but similar ideas were expounded by Poizat in [19]. In one technical difference, Poizat requires his 'bicolored fields' to be algebraically closed; we use the word for a wider class in the next paragraph.

Extend L_f by a unary predicate P to $L = L_f \cup \{P\}$. For N an L structure, we shall refer to elements of $P(N)$ as the *black* elements of N and of $N - P(N)$ as the *white* elements. If $X \subseteq_\omega N$, we set

$$\delta(X) = 2d_f(X) - |P(N) \cap X|.$$

We let T_0 be $T_f^\forall$ together with a collection of (universal) sentences ensuring that δ is non-negative in models of T_0. We shall refer to a model of T_0 as a *bicolored* model of $T_f^\forall$. As in the previous section, for $A \subseteq N \vdash T_0$, we let $\langle A\rangle$ denote the L substructure generated by A (which is also the L_f substructure generated by A), and note that $\langle A\rangle$ is again a model of T_0.

In order to apply the results of Section 1, we verify that bicolored strongly minimal sets fit into our general framework.

2.3 Theorem. *The δ in the bicolored case satisfies Constraints 1.7.*

Proof.. It is clear that δ satisfies Constraints 1.7. 1. (Monotonicity in particular is verified in [19].) To construct a δ-formula ϕ, consider $\overline{a}\overline{b}$ with $\delta(\overline{a}/\overline{b}) = k$. Then twice the dimension of $\overline{a}$ over $\overline{b}$ (in the sense of the strongly minimal theory) minus the number of black points in $\overline{a}$ is k. Choose a basis for $\overline{a}$ over $\overline{b}$. There is a particular way the rest of $\overline{a}$ is algebraic over this basis. So, one can write down a complete δ-formula for $\overline{a}$ over $\overline{b}$ of the form $\phi(\overline{x};\overline{y}) = \phi^f(\overline{x};\overline{y}) \wedge \bigwedge_{\models P(a_i)} P(x_i)$, where $\phi^f(\overline{x};\overline{y})$ is an L_f-formula. By hypothesis, ϕ is quantifier free. It is routine to check that the definition of complete δ-formula is fulfilled. For Constraints 1.7. 3, consider an L-formula $\theta(\overline{x};\overline{y})$. If $\theta(\overline{a};\overline{b})$ holds and for some $\overline{a}' \subseteq \overline{a}$, $\delta(\overline{a}'/\overline{b}) \geq 0$ then T_0 does not imply that $\theta(\overline{x},\overline{b})$ is algebraic unless some L_f-formula satisfied by $\overline{a}'$ over $\overline{b}$ is algebraic. Thus, if $\theta(\overline{x},\overline{b})$ is algebraic in T_0, some algebraic formula in L_f

implies θ. So, for any $\overline{c}$ satisfying $\theta(\overline{c}, \overline{b})$, $\delta(\overline{c}/\overline{b}) \leq 0$ as required. All elements of $\langle \overline{b}\overline{a}' \rangle$ have L_f-dimension zero over $\overline{b}\overline{a}'$ so $\overline{b}\overline{a}' \leq \langle \overline{b}\overline{a}' \rangle$ implies the rest of the points are white and we have the second clause of Constraints 1.7. 3. $\square_{2.3}$

These conditions in the bicolored case were first observed in [19].

2.4 Definition. Let $k \subseteq K, L$ be models of T_0. The *free amalgam* k^* of K and L over k is obtained by taking forking independent copies of K and L over k and letting k^* be the structure generated by $L \cup K$ in some L_f-algebraic closure of KL. The predicate P holds in k^* exactly of those elements of which it held in K and L.

The following lemma follows from the definition of free amalgam; the details are in [19].

2.5 Lemma. *If k^* is a free amalgam of K and L over k, then*

1. *for all $A \subseteq k^*$, $\delta(A) \geq \delta(A \cap LK) = \delta(A \cap K) + \delta(A \cap L) - \delta(A \cap k)$.*
2. *if $X \subseteq K - L$, then X is primitive over k if and only if X is primitive over L.*

In particular, if $k \leq K$ in the lemma above, then $\delta(A) \geq 0$ for any $A \subseteq k^*$, so that $k^* \models T_0$. *A fortiori*, we have

2.6 Lemma. *T_0 has the $\leq$-amalgamation property.*

Poizat [19] verifies that the theory of the generic model for T_0 is ω-stable with rank $\omega \times 2$. We will provide a more general analysis of infinite rank expansions of strongly minimal sets in [4]. Our aim now is to construct a theory of Morley rank 2.

There are three kinds of minimal strong extension: $\delta(\overline{a}/\overline{B})$ may be 0, 1 or 2. Most of our analysis will concern those for which $\delta(\overline{a}/\overline{B}) = 0$, i.e. the primitive extensions. Invoking the definition of δ over an infinite set, we see that if $\overline{a}$ is primitive over B there must be a finite $\overline{b}$ contained in B such that $\overline{a}$ is primitive over $\overline{b}$. If $\overline{a}$ is primitive over B, then it is easily seen that either $\overline{a}$ is a white singleton in $acl_f(B)$ or $l(\overline{a}) \geq 2$ and the elements of $\overline{a}$ are all black. The former (white) sort of primitive is easily dealt with in the subsequent analysis. We now specialize our notion of code to describe in L the latter (black) sort.

Suppose that $\overline{a}, B \subseteq N \in \overline{\boldsymbol{K}}_0$ and $\overline{a}$ is a black primitive over B. By our choice of S_f, there are a standard f-code $\langle \phi, \theta, \psi, F, m, k, n \rangle$ and $\overline{b} \subseteq \langle B \rangle$ such that $\overline{a}$ is a generic solution of $\phi(\overline{x}; \overline{b})$ over B. It is easily checked that then $\phi(\overline{x}; \overline{y}) \wedge \bigwedge_{i \leq n} P(x_i)$ is a complete δ-formula for $\overline{a}$ over $\langle B \rangle$ based on $\overline{b}$. Moreover, recalling the remarks following Definition 1.6, if $\overline{a'b'}$ is any other solution of $\phi(\overline{x}; \overline{y}) \wedge \bigwedge_{i \leq n} P(x_i)$ with $\overline{b'} \leq \overline{b'a'}$, then $\overline{a'}$ is primitive over $\overline{b'}$. It is thus reasonable to make the following definition.

2.7 Definition. A *primitive code* $\boldsymbol{c}$ is a seven-tuple $\langle \phi, \theta, \psi, F, m, k, n \rangle$, where $\phi(\overline{x}) = \phi^f \wedge \bigwedge_{i \leq n} P(x_i)$, $\langle \phi^f, \theta, \psi, F, m, k, n \rangle$ is a standard f-code and $\phi^f \wedge \bigwedge_{i \leq n} P(x_i)$ is a complete δ-formula for a primitive. (In particular, $n = 2k$.) We write $\boldsymbol{c}^f = \langle \phi^f, \theta, \psi, F, m, k, n \rangle$ and when we speak of several codes we will write $\phi_{\boldsymbol{c}}, \theta_{\boldsymbol{c}}, \psi_{\boldsymbol{c}}, F_{\boldsymbol{c}}, m(\boldsymbol{c}), k(\boldsymbol{c}), n(\boldsymbol{c})$ for $\phi, \theta, \psi, F, m, k, n$.

Just as the formula ψ in L_f approximated the notion of an independent sequence of m realizations of an f-code, we use ψ to construct an L-formula which approximates (for $k \geq m(\boldsymbol{c})$ the assertion that k-elements are independent realizations of a primitive code.

2.8 Definition. Fix a primitive code c. If $k \geq m(\boldsymbol{c})$, we write $E_{\boldsymbol{c}}(\overline{x}_1, \ldots, \overline{x}_k)$ for the conjunction of the formula $\bigwedge_{i,j} P(x_{i,j})$ with an L_f-formula stating the existence of a $\overline{b}$ satisfying the following conditions:

- $\overline{x}_1, \ldots, \overline{x}_k$ are pairwise disjoint,
- $\phi_{\boldsymbol{c}}(\overline{x}_j; \overline{b})$ for $j = 1, \ldots k$,
- $\psi_{\boldsymbol{c}}(\overline{x}_{i_1}, \ldots, \overline{x}_{i_m})$ for each choice of distinct $i_1, \ldots i_m$ from $\{1, \ldots, k\}$,

Strictly speaking, we should superscript $E_{\boldsymbol{c}}$ with a k, but since k is evident from the arguments, we will not do so. Note that if $\overline{a}_1, \ldots \overline{a}_k \in M \models T_0$, then $M \models E_{\boldsymbol{c}}(\overline{a}_1, \ldots, \overline{a}_k)$ if and only if $\langle \overline{a}_1, \ldots, \overline{a}_k \rangle \models E_{\boldsymbol{c}}(\overline{a}_1, \ldots, \overline{a}_k)$.

Consider a code $\boldsymbol{c}$. The sum of the number of elements in a sequence of $m(\boldsymbol{c})$ realizations of $\boldsymbol{c}$ ($n(\boldsymbol{c}) \cdot m(\boldsymbol{c})$) plus the number of realizations required to determine $\boldsymbol{c}$ ($m(\boldsymbol{c})$) plays an important role a several places in the argument so we formalize the following notation. In particular, this will be the lower bound on $\mu(\boldsymbol{c})$ for the theory T_0^μ to have the amalgamation property.

2.9 Definition.

1. For any code $\boldsymbol{c}$, $q(\boldsymbol{c}) = n(\boldsymbol{c}) \cdot m(\boldsymbol{c}) + m(\boldsymbol{c})$.
2. Henceforth, μ is an arbitrary function from the set of primitive codes into the natural numbers that is invariant under permutation of variables and such that for each primitive code $\boldsymbol{c}$, $\mu(\boldsymbol{c}) \geq q(\boldsymbol{c}) + m(\boldsymbol{c})$.

   If a model M of T_0 satisfies $\neg\exists\ \overline{x}_1, \ldots, \overline{x}_{\mu(c)+1}\ E_{\boldsymbol{c}}(\overline{x}_1, \ldots, \overline{x}_{\mu(c)+1})$, we say that M *respects* $\mu(\boldsymbol{c})$. Otherwise, we say M *violates* $\mu(\boldsymbol{c})$.
3. Now let T_0^μ be T_0, plus, for each primitive code $\boldsymbol{c}$ the sentence

$$\neg\exists\ \overline{x}_1, \ldots, \overline{x}_{\mu(c)+1}\ E_{\boldsymbol{c}}(\overline{x}_1, \ldots, \overline{x}_{\mu(c)+1}).$$

That is, a model of T_0 is a model of T_0^μ if and only if it respects each $\mu(\boldsymbol{c})$. We write $\overline{\boldsymbol{K}}_0^\mu$ for the class of models of T_0^μ, and note that $\overline{\boldsymbol{K}}_0^\mu$ is closed under substructure. We now want to show that $\overline{\boldsymbol{K}}_0^\mu$ has the $\leq$-amalgamation property and investigate the theory of its generic model. We will have to further restrict μ for the finite rank case.

The following argument for amalgamation, which separates the proof of amalgamation for T_0^μ from the more difficult task of showing separation of quantifiers is patterned on that in [19]. We separate them to show the consistency of the theory in Section 4 where the generic is not saturated. There is a redundancy between the main calculation here and the one in Lemma 3.2 below; the later argument could be modified to give directly both amalgamation and separation of quantifiers; indeed it appears in that form in [12]. The argument actually yields a slightly stronger bound: $q(\boldsymbol{c}) + m(\boldsymbol{c}) - 2$.

2.10 Lemma. *T_0^μ has the $\leq$-amalgamation property.*

Proof. Given K, L which are both strong over their intersection k_1 we can identify a maximal extension k of k_1 in K which has a copy in L with that copy and then form the free amalgam k^* of K and L over k. We claim $k^* \models T_0^\mu$.

Suppose, for a contradiction, that for some primitive code $\boldsymbol{c}$, k^* violates $\mu(\boldsymbol{c})$. Let $\overline{a}_1, \ldots, \overline{a}_{\mu(\boldsymbol{c})+1}$ witness the fact. Let $\overline{b} \in k^*$ such that $\phi_{\boldsymbol{c}}(\overline{a}_i, \overline{b})$ for $i = 1, \ldots, \mu(\boldsymbol{c})+1$, so that $\overline{b} = F(\overline{a}_1, \ldots, \overline{a}_m)$. Let A_i be the subset of k^* enumerated by $\overline{a}_i$. We write U_i for $A_i \cap (K - k)$, V_i for $A_i \cap (L - k)$, and W_i for $A_i \cap k$. Write m for $m(\boldsymbol{c})$ and n for $n(\boldsymbol{c})$.

First suppose $\overline{b}$ is not contained in either K or L. Since if m of the A_i are in K or in L then F applied to the m is also in the same structure, we conclude at most $m-1$ of the A_i are in either K or L. Since there are more than $q(\boldsymbol{c}) + m(\boldsymbol{c})$ elements in the sequence we can renumber so that if $A_i \subset K$, then $i \leq m-1$ (although some A_i with $i < m-1$ may not be contained in K), that $V_m \neq \emptyset$, and that for $m+1 \leq i \leq (n+1)m$, U_i and V_i are both nonempty. Now $\overline{b} \in k(A_1, \ldots A_m)$. For $m+1 \leq i \leq (n+1)m$, the dimension of U_i over $\overline{b}W_iV_i$ is at least the dimension of U_i over $A_1 \ldots A_m W_i V_i$ which is at least the dimension of U_i over $k(U_1 \ldots U_m, V_m, V_i)$ and this equals the dimension of U_i over $k(U_1 \ldots U_m)$ since K and L are independent over k. The dimension of U_i over $\overline{b}W_iV_i$ is less than $|U_i|/2$ (since $\overline{a}_i$ is a primitive), so we have shown each $\delta(U_i/k(U_1 \ldots U_m) < 0$ for each i with $m < i \leq (n+1)m$. Since all members of the U_i are black, this implies $\delta(U_1, \ldots U_m/k) < nm$ and then that $\delta(U_1, \ldots U_{(n+1)m}/k) = \delta(U_{m+1}, \ldots U_{(n+1)m}/k(U_1, \ldots U_m)) + \delta(U_1, \ldots U_{(n+1)m}/k) < 0$. This contradicts $k \leq K$.

So we have $\overline{b}$ is contained in one of K, L, say L; we will show $\overline{b} \subset k$. Since L respcets $\mu(\boldsymbol{c})$, one of the A_i is not contained in L whence $A_i \subset (K - k)$. (For, if $U_i \neq A_i$, $|U_i|/2$ is greater than the dimension of U_i over L by primitivity, which equals the dimension of U_i over k by independence contradicting the assumption that $k \leq K$.) Since K and L are independent over k, $\overline{a}_i$ is primitive over k. Thus by Lemma 2.2 for some standard f-code and canonical parameter $\overline{d} \in k$, $\phi^f_{\boldsymbol{d}}(\overline{a}_i, \overline{d})$ holds. Since K and L are independent and $\phi^f_{\boldsymbol{c}}$ has multiplicity one, $\phi^f_{\boldsymbol{c}}(\overline{x}, \overline{b})$ and $\phi^f_{\boldsymbol{d}}(\overline{x}, \overline{d})$ have a common generic solution. By Lemma 2.2, $\overline{d}$ and $\overline{b}$ have the same L_f-algebraic closure. Thus $\overline{b} \in K \cap L = k$.

But if $\overline{b} \subset k$, there must be $A_i \subset (K - k)$ and $A_j \subset (L - k)$. Then they can be identified contradicting the assumption that k is maximal. $\square_{2.10}$

We now have a complete theory, T^μ, the theory of the generic. We will show that under certain conditions on μ this theory is ω-stable with finite rank; in others, it is not even small.

3 Separation of quantifiers for μ finite-to-one

In this section we prove first that if μ is finite-to-one then the class $\boldsymbol{K}_0^\mu$ of the previous section admits separation of quantifiers in the sense of Definition 1.12. By Theorem 1.14 this implies the theory of the generic is ω-saturated and finiteness of Morley rank follows. We begin with some technical results that underly the proof of separation of quantifiers.

3.1 Definition. We say that a tuple $\overline{g}$ *splits over* a set X if $\overline{g}$ lies neither entirely inside, nor entirely outside X.

Note that if $\rho(\overline{x};\overline{y})$ is a complete δ-formula for a minimal strong extension and $\overline{b} \subseteq X \leq M$, $\overline{a} \subseteq M$ with $\rho(\overline{a};\overline{b})$, then $\overline{a}$ cannot split over M. The next lemma depends and expands upon this basic fact.

3.2 Lemma. *Let $\boldsymbol{c}$ be a primitive code and suppose that $M \leq N$ and $\overline{e}_1, \ldots, \overline{e}_r \in N$ satisfy $E_{\boldsymbol{c}}(\overline{e}_1, \ldots, \overline{e}_r)$ with each $\overline{e}_i$ lying in or splitting over M. Then either all of the $\overline{e}_i$ lie in M, or at most $m(\boldsymbol{c}) - 1$ of them lie in M and $r \leq q(\boldsymbol{c})$.*

Proof. Write $m = m(\boldsymbol{c})$ and $\overline{b} = F_{\boldsymbol{c}}(\overline{e}_1, \ldots, \overline{e}_m)$. If at least m of the $\overline{e}_i$ lie in M, then $\overline{b} \subseteq M$. Since $M \leq N$, no $\overline{e}_i$ can split over M, and we're done. So suppose that fewer that m lie in M.

If $r \leq m$, we are done. Otherwise, without loss of generality, suppose that the $\overline{e}_i$ lying in M are among $\overline{e}_1, \ldots, \overline{e}_m$. Let $\overline{g}_1, \ldots, \overline{g}_{r-m}$ list the remainder of the $\overline{e}_i$. Let G be the union of the ranges of $\overline{g}_1, \ldots, \overline{g}_{r-m}$, $G_0 = G \cap M$ and $G_1 = G - M$. Let E be the union of the ranges of $\overline{e}_1, \ldots, \overline{e}_m$, and $E_0 = E \cap M$ and $E_1 = E - M$.

For each $i \leq r-m$, since $\overline{g}_i$ satisfies $\phi_{\boldsymbol{c}}(\overline{x};\overline{b})$ and splits over M, we have $\delta(\overline{g}_i/(\overline{g}_i \cap M)\overline{b}) \leq -1$, whence, since $d_f(\overline{b}/E) = 0$ and $E \cap \overline{g}_i = \emptyset$, $\delta(\overline{g}_i/E(G \cap M)) \leq \delta(\overline{g}_i/(\overline{g}_i \cap M)E) \leq -1$. Thus, since $M \leq N$, basic properies of the dimension δ yield:

$$\begin{array}{rcl} 0 & \leq & \delta(GE/(GE) \cap M) = \delta(G/E(G \cap M)) + \delta(E/(GE) \cap M) \\ & \leq & \sum_i \delta(\overline{g}_i/E(G \cap M)) + \delta(E/(GE) \cap M) \leq -(r-m) + m \cdot n(\boldsymbol{c}), \end{array}$$

giving $r \leq m \cdot n(c) + m$, as required. $\square_{3.2}$

We describe now an extension $C[\overline{g}]$ of a structure C.

3.3 Notation. Let $\overline{b} \in C \in \overline{\boldsymbol{K}}_0^\mu$. Suppose $\overline{g}$ is an L_f-generic realization of $\rho^f(\overline{x},\overline{b})$ over C, where ρ^f is the L_f-part of a complete δ-formula for a minimal strong extension of $\overline{b}$ and D is the L_f model of $T_f^\forall$ generated by $C\overline{g}$ expanded by making all elements of $D - C\overline{g}$ white. We write $C[\overline{g}]$ for D.

We list some elementary properties of $C[\overline{g}]$.

3.4 Lemma. *Let C, $\overline{b}$ and $\overline{g}$ be as above.*

1. $C \leq C[\overline{g}]$.
2. *If* $\overline{e} \subseteq C[\overline{g}]$, $\overline{e} \not\subseteq C$ *with* $\overline{e}$ *and* $\overline{g}$ *primitive over* C, *then* $\overline{e} = \overline{g}$, *up to reordering.*
3. *If* $\overline{b} \subseteq X \subseteq C$ *and* $\overline{e} \subseteq C$, *then (since* $\delta(\overline{g}/X) = \delta(\overline{g}/X\overline{e})$*)* $\delta(\overline{e}/X) = \delta(\overline{e}/X\overline{g})$. *In particular, a complete* δ*-formula for* $\overline{e}$ *over* $\langle X\rangle$ *serves as a complete* δ*-formula for* $\overline{e}$ *over* $\langle X\overline{g}\rangle$.

If $\overline{g}$ is, in fact, primitive over $\overline{b}$, then by Lemma 2.2 and Definition 2.7 there are $\overline{c} \subseteq \langle\overline{b}\rangle$ and a primitive code $\boldsymbol{c}$ with $\phi_{\boldsymbol{c}}(\overline{g};\overline{c})$. As the example in Section 4 of that paper shows, even if $C[\overline{g}]$ respects $\mu(\boldsymbol{c})$, it may yet violate $\mu(\boldsymbol{d})$ for some other $\boldsymbol{d}$. Key to the proof of separation of quantifiers is that there be only finitely many such $\boldsymbol{d}$. The following lemma, together with μ being finite to one, will guarantee this.

3.5 Lemma. *Suppose $\boldsymbol{d}$ is a primitive code, and that* $\overline{e}_1,\ldots,\overline{e}_H \subseteq C[\overline{g}]$ *with* $E_{\boldsymbol{d}}(\overline{e}_1,\ldots,\overline{e}_H)$, *and* $H > \mu(\boldsymbol{d}) \geq q(\boldsymbol{d}) + m(\boldsymbol{d})$. *Then, either* $\overline{e}_i = \overline{g}$ *for some i, up to a possible reordering of variables, or* $H \leq 3l(\overline{g})$.

Proof. First, suppose that at least $m(d)$ of the $\overline{e}_i$ lie in C. Then, $F_{\boldsymbol{d}}(\overline{e}_1,\ldots\overline{e}_m)$ lies in C. Then, noting that $C \leq C[\overline{g}]$, none of the $\overline{e}_i$ splits over C. Since $H > \mu(\boldsymbol{d})$, C does not contain all of the $\overline{e}_i$, and every element of $C[\overline{g}] - C\overline{g}$ is white, so at least one, call it $\overline{e}_l$, lies inside $\overline{g}$. But $\overline{e}_l$ is then primitive over C and is contained in the primitive $\overline{g}$ over C, so that (setwise) $\overline{g} = \overline{e}_i$.

We can thus assume that fewer than $m(\boldsymbol{d})$ of the $\overline{e}_i$ lie in C. By Lemma 3.2, at most $q(\boldsymbol{d})$ of them lie in or split over C. By assumption, $H - q(\boldsymbol{d}) \geq m(\boldsymbol{d})$, so at least $m(\boldsymbol{d})$ of the (pairwise disjoint) $\overline{e}_i$ lie inside $\overline{g}$, making $m(\boldsymbol{d}) \leq m(\boldsymbol{d})n(\boldsymbol{d}) \leq l(\overline{g})$, whence $q(\boldsymbol{d}) \leq 2l(\overline{g})$. Thus, since $\overline{e}_1,\ldots,\overline{e}_H$ all either lie in or split over C (at most $q(\boldsymbol{d})$ of them), or lie in $\overline{g}$ (at most $l(\overline{g})$ of them), $H \leq q(bd) + l(\overline{g}) \leq 3l(\overline{g})$, as desired. □3.5

3.6 Theorem. *If μ is finite to one then K^μ admits separation of quantifiers.*

Proof. We verify Definition 1.12. Fix $\overline{b} \leq \overline{a}\overline{b} \leq \langle\overline{a}\overline{b}\rangle \in \overline{\boldsymbol{K}}_0^\mu$ with $\overline{a}$ minimal strong over $\langle\overline{b}\rangle$ and $\tau(\overline{x};\overline{y}) \in I^*(\overline{a}\overline{b})$. To simplify notation we rename $I^*(\overline{b})$ as the set of all finite conjunctions of formulas in the original $I^*(\overline{b})$. (We suppress the splitting of β into an α and σ which we used in definition.) We must show that there is $\beta(\overline{y})$ in the new $I^*(\overline{b})$ such that whenever $\overline{b}' \subseteq C \in \overline{\boldsymbol{K}}_0^\mu$, and $C \models \beta(\overline{b}')$, there is $D \in \overline{\boldsymbol{K}}_0^\mu$ with $C \leq D$ and $D \models \exists\overline{x}\tau(\overline{x};\overline{b}')$.

Let $\rho(\overline{x};\overline{y})$ be a complete δ-formula for $\overline{a}$ over $\overline{b}$. Choose $\overline{g}$ so that $\rho(\overline{x};\overline{y})$ is a complete δ-formula for $\overline{g}$ over $\overline{b}'$ with $\delta(\overline{g}/C) = \delta(\overline{g}/\overline{b}')$. We will find β as above so that if $C \models \beta(\overline{b}')$, then either $C[\overline{g}] \models \tau(\overline{g};\overline{b}')$ and $C[\overline{g}] \in \overline{\boldsymbol{K}}_0^\mu$ (so that we may take $D = C[\overline{g}]$ for some such $\overline{g}$), or C itself models $\exists\overline{x}\tau(\overline{x};\overline{b}')$ (so that we may take $D = C$).

We may assume, without loss of generality, that C is algebraically closed, since if we set C' to be the field algebraic closure of C with all new points colored white, then $C \models \psi(\overline{b}')$ if and only if $C' \models \psi(\overline{b}')$ for all $\psi \in I^*(\overline{b})$, $C \models \exists \overline{x} \tau(\overline{x}; \overline{b}')$ if and only if $C' \models \exists \overline{x} \tau(\overline{x}; \overline{b}')$, and $C[\overline{g}] \simeq C'[\overline{g}]$.

We must find a $\beta(\overline{y})$ such that if $C \models \beta(\overline{b}')$, then either $C[\overline{g}] \in \overline{\boldsymbol{K}}_0^\mu$ (and we may take $D = C[\overline{g}]$) or $C \models \exists \overline{x} \tau(\overline{x}; \overline{b}')$.

The minimal strong extension $\overline{a}$ of $\langle \overline{b} \rangle$ is of one of three types: It may be a single white point; it may be a single black point d_f-independent from $\overline{b}$; and it may be a black primitive.

We consider the first two cases together. Set $a = \overline{a}$. Let $\boldsymbol{d}$ be any primitive code and suppose that $C[g] \models E_{\boldsymbol{d}}(\overline{e}_1, \ldots, \overline{e}_r)$ for some $r > m(\boldsymbol{d})$. As $C \in \overline{\boldsymbol{K}}_0^\mu$, to show $C[\overline{g}] \in \overline{\boldsymbol{K}}_0^\mu$, it is enough to show that $\overline{e}_, \ldots \overline{e}_r$ must all lie in C. Since $C[g] - C$ contains at most one black point and $l(\overline{e}_i) \geq 2$, for each i, at most one of the $\overline{e}_i$ does not lie entirely inside C and none can lie completely outside C. In particular, at least $m(\boldsymbol{d})$ of the $\overline{e}_i$ do lie entirely inside C, so that $F_{\boldsymbol{d}}(\overline{e}_1, \ldots, \overline{e}_m) \subseteq C$. Thus, since $C \leq C[g]$, none of the $\overline{e}_i$ can split over C, giving $\overline{e}_, \ldots, \overline{e}_r \subseteq C$, as desired.

Now we must guarantee $C[g] \models \tau(g; \overline{b}')$. If g is white and field algebraic over C then $g \in C$ and there is nothing to show. Otherwise, g must be independent from C. Suppose that $\tau(x, \overline{y})$ is a formula from $I^*(a, \overline{b})$ that we must satisfy. (If we guarantee each such formula is satisfied by $g, \overline{b}'$, then the finite conjunction is as well.) If τ is in $\mathrm{Diag}(\overline{a}, \overline{b})$, this is easy. So we concentrate on the case that τ is $(\forall \overline{z}) \neg \tau''(x, \overline{y}, \overline{z})$ where τ'' has the form

$$\tau^f(x, \overline{y}, \overline{z}) \wedge \bigwedge_{i < \lg(\overline{z})} P(z_i) \wedge \overline{z} \cap \overline{xy} = \emptyset$$

and is satisfied by $\overline{a}\overline{b}$ in (the white algebraic closure of) $\langle \overline{a}\overline{b} \rangle$. Moreover, for any $a', \overline{b}', \overline{c}'$, if $\tau^f(a', \overline{b}', \overline{c}')$, $d_f(\overline{c}'/a'\overline{b}') \leq h = \frac{\lg(\overline{z})}{2} - 1$; $\delta(\overline{c}/a'\overline{b}') < 0$. Let $\tau'(\overline{z}, \overline{y})$ be the L_f-formula which holds of $\overline{c}'\overline{b}'$ (in any M containing $\overline{c}'\overline{b}'$) if for generic g, $\tau^f(g, \overline{b}', \overline{c}')$ holds. Then,

$$T_0 \cup I^*(\overline{b}) \vdash \neg(\exists \overline{z}) \tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i < \lg(\overline{z})} P(z_i).$$

To see this note that if there is black $\overline{c}'$ and $g', \overline{b}'$ with g' independent from $\overline{c}'$ over $\overline{b}'$ and $\tau^f(g', \overline{b}', \overline{c}')$, then in fact $\delta(\overline{c}'/\overline{b}) < 0$. Choose $\beta(\overline{y})$ from $I^*(\overline{y})$ so that

$$T_0 \vdash \beta(\overline{y}) \rightarrow \neg(\exists \overline{z}) \tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i < \lg(\overline{z})} P(z_i).$$

Suppose $C \models \beta(\overline{b}')$ and some $\overline{c}' \in C[g]$ satisfies $\tau''(g, \overline{b}', \overline{c}')$ where τ'' was described above. Then the members of $\overline{c}'$ are black (and so in C). So $C \models \tau'(\overline{b}', \overline{c}') \wedge \bigwedge_{i < \lg(\overline{z})} P(c_i')$ which contradicts that $C \models \neg(\exists \overline{z}) \tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i < \lg(\overline{z})} P(z_i)$.

Now, we consider the less trivial case in which $\overline{a}$ is a black primitive. By part 4 of Definition 2.1, (and since $\delta(\overline{a}/\overline{b})=0$)

$$T_0 \cup I^*(\overline{b}) \cup \{\rho(\overline{x};\overline{y})\} \vdash I^*(\overline{a};\overline{b}).$$

Thus, for any consequence $\omega(\overline{x};\overline{y})$ of $T_0 \cup I^*(\overline{a};\overline{b})$, there is $\beta(\overline{y}) \in \bigwedge I^*(\overline{b})$ such that

$$T_0 \vdash \beta(\overline{y}) \wedge \rho(\overline{x};\overline{y}) \rightarrow \omega(\overline{x};\overline{y}).$$

We use this fact to construct, by stages, a formula $\beta(\overline{y}) \in I^*(\overline{b})$ such that if $C \models \beta(\overline{b}')$ then either $C[\overline{g}] \in \overline{\boldsymbol{K}}_0^{\mu}$ or $C \models \exists \overline{x} \tau(\overline{x};\overline{b}')$.

Find a primitive code $\boldsymbol{d}_0$ and $\overline{c} \subseteq \mathrm{acl}_f(\overline{b})$ such that $\overline{a}$ is a generic solution of $\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c})$. Let $\gamma(\overline{u};\overline{y})$ isolate the L_f-type of $\overline{c}$ over $\overline{b}$. Let $\chi(\overline{u},\overline{y})$ be the formulas such that $\chi(\overline{c}\overline{d})$ holds if and only if the Morley rank of $\rho(\overline{x};\overline{d}) \wedge \phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c})$ equals the Morley rank of $\rho(\overline{x};\overline{d})$. Then $T_0 \cup \mathrm{Diag}(\overline{b})$ proves

$$\exists \overline{u}\ \gamma(\overline{u};\overline{y}) \wedge \chi(\overline{u},\overline{y}),$$

and $T_0 \cup I^*(\overline{b})$ proves

$$\forall \overline{u}[\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{u}) \wedge \gamma(\overline{u};\overline{y}) \rightarrow \tau(\overline{x};\overline{y})].$$

Next, by our choice of μ finite to one, we may list all primitive codes $\boldsymbol{d}_1,\ldots \boldsymbol{d}_l$ with $\mu(\boldsymbol{d}_i) \leq 3l(\overline{g})$. Then $T_0 \cup I^*(\overline{a};\overline{b})$ proves

$$\bigwedge \neg E_{\boldsymbol{d}_i}(\overline{w}_1,\ldots,\overline{w}_{\mu(\boldsymbol{d}_i)+1}),$$

where the conjunction ranges over $i=0,\ldots,l$ and all choices of $\overline{w}_1,\ldots,\overline{w}_{\mu(\boldsymbol{d}_i)+1}$ from among $\overline{xy}$, and

$$\bigwedge \forall \overline{v}_1,\ldots,\overline{v}_{m(\boldsymbol{d})},\overline{z}\ (E_{\boldsymbol{d}_i}(\overline{z},\overline{v}_1,\ldots,\overline{v}_m) \wedge\ \overline{v}_1,\ldots,\overline{v}_m \subseteq \overline{xy} \rightarrow \overline{z} \cap \overline{xy} = \emptyset \vee \overline{z} \subseteq \overline{xy}),$$

where the conjunction ranges over $i=0,\ldots,l$.

We will use the following notation. Fix a code $\boldsymbol{d}$. For each σ a sequence of $m(\boldsymbol{d})$, $n(\boldsymbol{d})$ tuples from $\{1,\ldots p)\}$ and sequence of constants $\overline{g}$ or variables $\overline{x}$ of length p, $\overline{g}^{\sigma}$ or $\overline{x}^{\sigma}$ denotes the subsequence of $\overline{g},\overline{x}$ respectively indexed by σ. This will be used with $p=\mathrm{lg}(\overline{g})$.

For each such $\boldsymbol{d}$ and σ, if $\neg E_{\boldsymbol{d}}(\overline{a}^{\sigma})$ holds in the white algebraic closure of $\langle \overline{a}\overline{b} \rangle$ then $T_0 \cup I^*(\overline{a};\overline{b})$ proves

$$\neg E_{\boldsymbol{d}}(\overline{x}^{\sigma}).$$

By the remark above, we may find $\beta_1(\overline{y}) \in I^*(\overline{y})$ such that $T_0 \cup \{\beta_1(\overline{y}) \wedge \rho(\overline{x};\overline{y})\}$ proves each of the last five displayed formulas. Immediately, if $C \models \beta_1(\overline{b}')$, then also $C[\overline{g}] \models \beta_1(\overline{b}')$ (since $C \leq C[\overline{g}]$). Thus,

1. $\overline{g}$ is a generic solution of $\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c}')$ for some $\overline{c}' \subseteq \mathrm{acl}_f(\overline{b}')$;
2. $C \models \forall \overline{uz}\ [\gamma(\overline{u};\overline{b}') \wedge \phi^f_{\boldsymbol{d}_0}(\overline{z};\overline{u}) \to \tau(\overline{z};\overline{y})]$.
3. $E_{\boldsymbol{d}_i}(\overline{y}_1,\dots\overline{y}_{\mu(\boldsymbol{d}_i)+1})$ is not realized in $\overline{g}\overline{b}'$ for $i=0,\dots,l$;
4. if $\overline{e}'_1,\dots\overline{e}'_{m(\boldsymbol{d}_i)} \subseteq \overline{g}\overline{b}'$ and $\overline{e}' \subseteq C[\overline{g}]$ with $E_{\boldsymbol{d}_i}(\overline{e}',\overline{e}'_1,\dots,\overline{e}'_{m(\overline{d}_i)})$, then $\overline{e}'$ does not split over $\overline{g}\overline{b}'$, for $i=0,\dots,l$;
5. if $\neg E_{\boldsymbol{d}_i}(\overline{a}^\sigma)$ then $\neg E_{\boldsymbol{d}_i}(\overline{g}^\sigma)$.
6. $C[\overline{g}] \models \tau(\overline{g};\overline{b}')$.

For each code $\boldsymbol{d}$ with $m(\boldsymbol{d})\cdot n(\boldsymbol{d}) \le l(\overline{x})$ and sequence σ as described above such that $E^f_{\boldsymbol{d}}(\overline{a}^\sigma)$ holds in $\mathrm{acl}_f(\overline{ab})$, we will define a formula $\beta^2_{\boldsymbol{d},\sigma}(\overline{y})$; $\beta_2(\overline{y})$ is the conjunction of all these formulas. Write $m=m(\boldsymbol{d})$.

<u>Case 1</u>. $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$.

We choose $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{y})$ such that if $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ and $C[\overline{g}] \models E^f_{\boldsymbol{d}}(\overline{g}^\sigma)$, then $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}_f(\overline{b}')$.

Since $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$, there are an L_f-formula $\lambda(\overline{u};\overline{y})$ and an integer l such that $T^f \vdash \forall\overline{y}\exists^{\le l}\overline{u}\lambda(\overline{u};\overline{y})$ and $T_0 \cup I^*(\overline{ab}) \vdash \lambda(F_{\boldsymbol{d}}(\overline{x}^\sigma);\overline{y})$. Thus, there is $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{b})$ such that $T_0 \vdash \beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \wedge \rho(\overline{x};\overline{y}) \to \lambda(F_{\boldsymbol{d}}(\overline{x}^\sigma);\overline{y})$. This choice of $\beta^2_{\boldsymbol{d},\sigma}(\overline{y})$ immediately satisfies our requirements.

<u>Case 2</u>. $F_{\boldsymbol{d}}(\overline{a}^\sigma) \not\subseteq \mathrm{acl}_f(\overline{b})$.

In this case we choose the formula $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{y})$ so that if $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ then all solutions of $E_{\boldsymbol{d}}(\overline{z},\overline{g}^\sigma)$ in C lie in $\overline{b}'$.

Let $\overline{e}$ satisfy the unique L-type $q \in S(\mathrm{acl}_f(\overline{ab}))$ of a realization of $E^f_{\boldsymbol{d}}(\overline{z},\overline{a}^\sigma)$ outside $\mathrm{acl}_f(\overline{ab})$ in a strong extension of $\mathrm{acl}_f(\overline{ab})$. Then $\overline{e}$ depends on $\overline{a}$ over $\overline{b}$, as if not, we could successively realize q by $\overline{f}_1,\dots,\overline{f}_{m(\boldsymbol{d})}$, each independent from its predecessors over $\overline{ab}$, whence $\overline{f}_1,\dots,\overline{f}_{m(\boldsymbol{d})}$ would be independent from $\overline{a}$ over $\overline{b}$, giving $F_{\boldsymbol{d}}(\overline{a}^\sigma) = F_{\boldsymbol{d}}(\overline{f}_1,\dots,\overline{f}_m) \subseteq \mathrm{acl}_f(\overline{f}_1,\dots,\overline{f}_m)$, and $\mathrm{acl}_f(\overline{f}_1,\dots,\overline{f}_m) \cap \mathrm{acl}_f(\overline{a}) \subseteq \mathrm{acl}_f(\overline{b})$, whence $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$.

Since $\overline{e}$ is not independent from $\overline{a}$ over $\overline{b}$, there are l, strictly less than the rank of $\rho(\overline{x};\overline{b})$, and an L_f-formula $\psi(\overline{x};\overline{yz})$, satisfied by $\overline{a}\overline{b}\overline{e}$, such that for any (by the definability of rank) $\overline{c}\overline{d}$ the rank of $\psi(\overline{x};\overline{c}\overline{d})$ is l. Then,

$$T_0 \cup I^*(\overline{ab}) \vdash \forall \overline{z} \subseteq P\ [E^f_{\boldsymbol{d}}(\overline{z};\overline{x}^\sigma) \wedge \overline{z} \not\subseteq \overline{xy} \to \psi(\overline{x};\overline{yz})].$$

To see this, note that if $E^f_{\boldsymbol{d}}(\overline{e};\overline{a}^\sigma)$ holds $\delta(\overline{e}/\overline{a}\overline{b}) < 0$. So $T_0 \cup I^*(\overline{a}\overline{b}$ implies that $\overline{e}$ is not both black and contained in $\mathrm{acl}_f(\overline{a}\overline{b})$. The uniqueness of q allows us to fix ψ and l. Find $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{b})$ such that $T_0 \cup \{\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \wedge \rho(\overline{x};\overline{y})\}$ proves this displayed consequence of $T_0 \cup I^*(\overline{a}\overline{b})$.

Now suppose that $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$, and $\overline{e}' \subseteq C$ with $C[\overline{g}] \models E^f_{\boldsymbol{d}}(\overline{e}',\overline{g}^\sigma)$. Then since $\psi(\overline{g};\overline{b}'\overline{e})$ must fail (as $\overline{g}$ is independent from C over $\overline{b}'$), it follows that $\overline{e}' \subseteq \overline{b}'$.

Finally, set $\beta(\overline{y}) = \beta_1(\overline{y}) \wedge \beta_2(\overline{y})$. (Recall that $\beta_2(\overline{y})$ is a conjunction over the $\beta^2_{\boldsymbol{d},\sigma}$.) Note that $T_0 \cup I^*(\overline{b}) \vdash \beta(\overline{y})$.

Now suppose that $C \models \beta(\overline{b}')$. As $C[\overline{g}]$ satisfies $\exists\overline{x}\tau(\overline{x};\overline{b}')$, if $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_0$, we finish. So, suppose that $C[\overline{g}] \notin \overline{\boldsymbol{K}}^\mu_0$, so that there are a primitive code $\boldsymbol{d}$ and $\overline{e}'_1,\ldots,\overline{e}'_{\mu(\boldsymbol{d})+1} \subseteq C[\overline{g}]$ with $E_{\boldsymbol{d}}(\overline{e}'_1,\ldots\overline{e}'_{\mu(\boldsymbol{d})+1})$. We argue that $C \models \exists\overline{x}\tau(\overline{x};\overline{b}')$. We first show that $F_{\boldsymbol{d}}(\overline{e}'_1,\ldots,\overline{e}'_{m(\boldsymbol{d})}) \subseteq C$. By Lemma 3.5, either $\overline{e}'_{i_0} = \overline{g}$, up to reordering, for some i_0, or $\mu(\boldsymbol{d}) + 1 \leq 3l(\overline{g})$.

In the first case, all of the other $\overline{e}'_i$ must lie in C, so $F_{\boldsymbol{d}}(\overline{e}'_i,\ldots,\overline{e}'_{m(\boldsymbol{d})}) \subseteq C$ is immediate.

In the second case, we have $\boldsymbol{d} = \boldsymbol{d}_i$ for some $i = \{1,\ldots,l\}$. If $F_{\boldsymbol{d}}(\overline{e}'_1,\ldots,\overline{e}'_{m(\boldsymbol{d})}) \not\subseteq C$, then fewer than $m(\boldsymbol{d}_i)$ lie in C and by Lemma 3.2, either none splits over C, or at most $q(\boldsymbol{d})$ lie in or split over C. In the first subcase, all lie in C or lie in $\overline{g}$, so that, since fewer than $m(\boldsymbol{d})$ lie in C and $\mu(\boldsymbol{d}) + 1 - (m(\boldsymbol{d}) - 1) = q(\boldsymbol{d}) + m(\boldsymbol{d}) + 2 > m(\boldsymbol{d})$, at least $m(\boldsymbol{d})$ lie in $\overline{g}$. In the second subcase, since $\mu(\boldsymbol{d}) + 1 - q(\boldsymbol{d}) = m(\boldsymbol{d}) + 1 > m(\boldsymbol{d})$, again, at least $m(\boldsymbol{d})$, say $\overline{h} = \langle \overline{e}_{j_1} \ldots \overline{e}_{j_m} \rangle$ lie in $\overline{g}$. Choose σ so that $\overline{g}^\sigma = \overline{h}$. We have $C[\overline{g}] \models E_{\boldsymbol{d}}(\overline{g}^\sigma)$. Thus, by condition 5), $E_{\boldsymbol{d}}(\overline{a}^\sigma)$ holds in $\langle \overline{a}\overline{b} \rangle$. Since $C \models \beta_1(\overline{b}')$, the $\overline{e}'_i$ do not all lie in $\overline{b}'\overline{g}$, and no $\overline{e}'_i$ splits over $\overline{b}'\overline{g}$, so at least one, call it $\overline{e}'$, lies in $C - \{\overline{b}'\}$. Our choice of $\beta^2_{\boldsymbol{d},\sigma}$ depended on whether $F_{\boldsymbol{d}}(\overline{a}^\sigma)$ were in $\mathrm{acl}_f(\overline{b})$. If it was not, since $C \models \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ we could not have such an $\overline{e}'$. So $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$. Then $C \models \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ immediately gives $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}_f(\overline{b}')$.

So in either case, we have $F_{\boldsymbol{d}}(\overline{e}'_1,\ldots,\overline{e}'_{m(\boldsymbol{d})}) \subseteq C$ and some $\overline{e}'_i \in \overline{g}$. Since $C \leq C[\overline{g}]$, this $\overline{e}'_i$ is a primitive over C, contained in the primitive $\overline{g}$ over C, so $\overline{e}'_i = \overline{g}$, up to reordering. Now $C \models \beta_1(\overline{b}')$ gives us that for some $\overline{c}'$ satisfying $\gamma(\overline{u};\overline{b}')$, $\overline{g}$ is a generic solution of $\phi_{\boldsymbol{d}_0}(\overline{x};\overline{c}')$. This forces $\boldsymbol{d} = \boldsymbol{d}_0$, up to reordering, and $F_{\boldsymbol{d}}(\overline{e}'_1,\ldots\overline{e}'_m) \subseteq \mathrm{acl}_f(\overline{c}')$. Then again from $C \models \beta_1(\overline{b}')$ (condition 2), we have $C \models \tau(\overline{e_i}';\overline{b}')$. $\square_{3.6}$

Now we note that the example we constructed does have rank 2. This requires a couple of lemmas to show that types of δ-rank 0 are algebraic. Note that if $\overline{b}$ and $\overline{c}$ are primitive over $\overline{a}$ either they intersect in $\overline{a}$ (and so each is primitive over the other) or they each have negative dimension over the intersection. The number of such intersecting pairs is thus bounded by $\delta(\overline{a})$ which in turn is less than twice $\lg(\overline{a})$. We summarise this argument, which is implicit in many papers, as:

3.7 Lemma. *Let $\rho(\overline{x};\overline{y})$ be a complete δ-formula for a primitive extension. Then there is $r \geq 0$ (depending only on the length of $\overline{y}$) such that for any $N \models T_0$ and $\overline{b} \subseteq N$, for any $A \subseteq N$ and any s, if A is a collection of $r \times s$ distinct solutions of $\rho(\overline{x};\overline{b})$, then there are $\overline{a}^1,\ldots,\overline{a}^s \in A$ with $\overline{a}^i$ primitive over $\overline{b}\overline{a}^1\ldots\overline{a}^{i-1}$ for $0 < i \leq s$.*

To see this let $r = 2lg(a)+1$. We conclude that primitive types are algebraic.

3.8 Lemma. *If $\rho(\overline{x};\overline{y}) = \rho^f(\overline{x};\overline{y}) \wedge \bigwedge_i P(x_i)$ is a complete δ-formula for a primitive extension, then there is q such that $T_0^\mu \vdash \forall\overline{y}\exists^{\leq q}\overline{x}\rho(\overline{x};\overline{y})$.*

Proof. Fix $N \models T_0^\mu$ and let $\overline{b} \subseteq N$. Let A be the set of solutions of $\rho(\overline{x};\overline{b})$ in $\overline{N}$. We find a uniform bound on $|A|$.

Find a $\overline{b'} \in N$ and a primitive code $\boldsymbol{c}$ such that $\overline{b'}$ satisfies $\theta_{\boldsymbol{c}}$ and $\rho^f(\overline{x};\overline{b})$ and $\phi_{\boldsymbol{c}}(\overline{x};\overline{b'})$ have the same generic solutions. Find the r for $\rho(\overline{x};\overline{y})$ guaranteed by Lemma 3.7. Set $q = r + \mu(\boldsymbol{c})$. If $|A| > q$, then A contains a set of $\mu(\boldsymbol{c})+1$ solutions $\overline{a}_1,\ldots,\overline{a}_{\mu(\boldsymbol{c})+1}$ such that $\overline{a}_i$ is primitive over $\overline{b}\overline{a}_1\ldots\overline{a}_{i-1}$ for $0 < i \leq \mu(\boldsymbol{c})+1$. Such a sequence of $\overline{a}_i$ must be L_f-independent. By Definition 2.1 and Definition 2.8 for any such sequence of $\overline{a}_i$, we have $E_{\boldsymbol{c}}(\overline{a}_1,\ldots,\overline{a}_{\mu(\boldsymbol{c})+1})$, so that $N \not\models \mathrm{T}_0^\mu$. $\square_{3.8}$

3.9 Theorem. *If μ is finite-one and for each $\boldsymbol{d}$, $\mu(\boldsymbol{d}) \geq m(\boldsymbol{d}) + q(\boldsymbol{d})$ then the universe of the generic model has rank 2.*

Proof. First we show that the set of black points is strongly minimal. Let M be the generic model. It suffices to show that for any finite set $X \subseteq M$, any two black points which are not in $\mathrm{acl}_L(X)$ are automorphic over X. Without loss of generality X is intrinsically closed. Note that by Lemma 3.8 every element with $d_M(a/X) = 0$ is in $\mathrm{acl}_L(X)$, $\mathrm{acl}_L(X)$ is strong in M. Thus if a, a' are black points which are not in $\mathrm{acl}_L(X)$, they must L_f-transcendental over X and thus realize the same L_f-type. But their quantifier free L-types are also the same. For there can be no black point in $\mathrm{acl}_f(Xa) - \mathrm{acl}_f(X)$ (otherwise $\delta(a/X) = 0$) and similarly for a'. But then $X \leq Xa \leq \langle Xa\rangle \leq M$ and the same for a'. Now by genericity the required automorphism exists. Thus the set of black points in M is minimal and since M is ω-saturated the set of black points is strongly minimal.

We can use the same argument to analyze the white points: $d_M(a/\emptyset) = 1$ if there is a black point in the L_f-algebraic closure of a; otherwise $d_M(a/\emptyset) = 2$. All nonalgebraic a of the first category have rank 1 since they are interalgebraic with a generic of the strongly minimal set of black points. All nonalgebraic a of the second category realize the same type by genericity. Thus T has Morley rank 2. $\square_{3.9}$

This completes the proof of our main positive result: there is a rank two theory of fields. In the next section we show the necessity of restricting to finite-to-one μ.

4 The generic of $\boldsymbol{K}_0^\mu$ need not be ω-saturated

We show that without the condition that μ be finite-one, the generic model, M, of T_0^μ need not be ω-saturated. Indeed, we will show that the theory of the generic model need not be small.

We will define a finite set A ($\overline{b}\overline{c}\overline{dh}$ below) and a primitive B ($\overline{e}\overline{f}\overline{g}$ below) over A. We find also for $i > 0$ sets A_i ($\overline{a}_i\overline{b}\overline{c}\overline{dh}$ below) which are minimal intrinsic extensions of A (in $\boldsymbol{K}_0$). We will then choose a μ_0 so that if $N \models T^{\mu_0}$ and $A \subset A_i \subset N$, then there is no embedding of B into N over A (Proposition 4.2). This yields, for M the generic model of T^{μ_0}, that the type asserting, 'there is a strong embedding of A into M but it cannot be extended to an embedding of B into M' is consistent. Clearly, such a type cannot be realized in a generic model so M is not ω-saturated.

The intuition for finding such pairs A_i and B is that B will be a primitive that, in the presence of A_i, forces for a code $\boldsymbol{d}$, more than $\mu(\boldsymbol{d})$ independent realizations of $\boldsymbol{d}$ over A. To fill in this outline, we must fully describe the appropriate A, B and A_i. We begin by constructing B then a single $\overline{a}$ (determined by a formula α). After Proposition 4.2 we construct mutually incompatible formulas α_i which each have the crucial properties of $\overline{\alpha}$. Then A_i is $A \cup \{\overline{a}_i\}$ where $\overline{a}_i$ satisfies $\overline{\alpha}_i$.

We write MRD (k, i) to indicate a formula has Morley rank k and degree i. For both simplicity and concreteness, we restrict in this section to bicolored fields.

We will prove the following theorem. We say a generic solution $\overline{a}_1 \ldots \overline{a}_n$ of a formula $\theta(\overline{x}_1, \ldots \overline{x}_n)$ is r-wise independent if every r-element subset of $\{\overline{a}_1 \ldots \overline{a}_n\}$ is independent.

4.1 Theorem. *There exists a function μ_0 such that for each code $\boldsymbol{c}$, $\mu_0(\boldsymbol{c}) \geq q(\boldsymbol{c}) + m(\boldsymbol{c})$ but the generic, M, of $\boldsymbol{K}_0^{\mu_0}$ is not saturated.*

Proof. Fix r and let $k = 2r$. Let $\overline{t}$ be a sequence of k variables and $\overline{s}, \overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}$ each be sequences of r variables. Find field formulas β and γ satisfying the following conditions. (The formulas can actually be taken in the language of the field addition. Thus, by a linear variety below we mean a conjunction of equations $\sum a_i x_i = 0$ where the a_i are integers.)

- $\beta(v, w, x, \overline{s})$ has MRD $(r + 1, 1)$ and a generic solution is $r + 1$-wise independent.
- $\gamma(\overline{t}, y, z, v, w, x)$ has MRD $(k + 3, 1)$ and a generic solution is $k + 3$-wise independent.

Now let $\overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s})$ be $\bigwedge_{i \leq r} \beta(v_i, w_i, x_i, \overline{s})$ together with the assertion that a solution consists of distinct elements, those corresponding to $\overline{v}, \overline{w}, \overline{x}$ being black, and those corresponding to $\overline{s}$ being white. Further, let $\overline{\gamma}(\overline{t}, \overline{yzv}, \overline{wx})$ be $\bigwedge_{i \leq r} \gamma(\overline{t}, y_i, z_i, v_i, w_i, x_i)$ together with the assertion that a solution consists of distinct, black elements.

Let
$$\phi(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s}) = \overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s}) \wedge \overline{\gamma}(\overline{t}, \overline{y}, \overline{z}, \overline{v}, \overline{w}, \overline{x})$$

together with the assertion that a solution consists of distinct elements.

Let $\overline{e}\overline{f}\overline{g};\overline{b}\overline{c}\overline{d}\overline{h}$ be a generic solution of ϕ; that is we have the substitutions: $\overline{b}$ for $\overline{v}$, $\overline{c}$ for $\overline{w}$, $\overline{d}$ for $\overline{x}$, $\overline{e}$ for $\overline{y}$, $\overline{f}$ for $\overline{z}$, $\overline{g}$ for $\overline{t}$, and $\overline{h}$ for $\overline{s}$. Note that only $\overline{h}$ is white.

It is easily checked that $\delta(\overline{b}\overline{c}\overline{d}\overline{h}) = r$. For, $d_f(b_i c_i d_i \overline{h}) = d_f(b_i c_i d_i/\overline{h}) + d_f(\overline{h}) = 1 + r$ for each i, so $\delta(\overline{b}\overline{c}\overline{d}\overline{h}) = 2(2r) - 3r = r$. With a bit more difficulty, one shows that $\overline{e}\overline{f}\overline{g}$ is primitive over $\overline{b}\overline{c}\overline{d}\overline{h}$ with complete δ-formula ϕ. We first show $\overline{e}\overline{f}\overline{g}$ is primitive over $\overline{b}\overline{c}\overline{d}$ with complete δ-formula $\overline{\gamma}$. Namely, $d_f(\overline{e}\overline{f}\overline{g}\overline{b}\overline{c}\overline{d})$ is $d_f(\overline{e}\overline{f}\overline{b}\overline{c}\overline{d}/\overline{g}) + d_f(\overline{g})$ which equals $k + r \cdot d_f(e_i f_i b_i c_i d_i/\overline{g}) = k + 3 \cdot r$. Since $(\overline{e}\overline{f}\overline{g})$ has cardinality $2r + k = 2k$ and $d_f(\overline{b}, \overline{c}, \overline{d}) = 3 \cdot r$, $\delta(\overline{e}\overline{f}\overline{g}/\overline{b}\overline{c}\overline{d}) = 0$. Moreover, each subset has strictly positive δ so $\overline{g}, \overline{e}, \overline{f}$ is primitive over $\overline{b}\overline{c}\overline{d}$. But then also $\overline{g}, \overline{e}, \overline{f}$ is primitive over $\overline{b}\overline{c}\overline{d}\overline{h}$ and $\overline{\gamma}$ and, *a fortiori*, ϕ is a complete δ-formula for this extension.

This describes the primitive B $(\overline{e}\overline{f}\overline{g})$ over A $(\overline{b}\overline{c}\overline{d}\overline{h})$ in our outline. We proceed to find an intrinsic closure of A by adding a sequence $\overline{a}$. (So this is an exemplar for one of our $\overline{a}_i$.)

Suppose that $\alpha(u, v, w, x)$ is a field formula with MRD $(3, 1)$ any of whose generic solutions is 3-wise independent. We write $\overline{\alpha}(\overline{u}, \overline{v}, \overline{w}, \overline{x})$ for $\bigwedge_{i \leq r} \alpha(u_i, v_i, w_i, x_i)$ together with the assertion that the elements of a solution are distinct and black.

Let

$$\rho_\alpha(u, v, w, x, y, z; \overline{t}) = \gamma(\overline{t}, y, z; v, w, x) \wedge \alpha(u, v, w, x)$$

together with the assertion that the elements of a solution are distinct and black. As above, ρ_α is, by inspection, a complete δ-formula for a primitive extension. So $abcdef$ is primitive over $\overline{g}$.

By Lemma 2.2.2, for each $\overline{g}$, there is a primitive code $\boldsymbol{d}_\alpha$ and a $\overline{g}' \in \langle \overline{g} \rangle$ such that $\rho_\alpha(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; \overline{g})$ and $\phi_{\boldsymbol{d}_\alpha}(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; \overline{g}')$ have the same generic solutions. Since any two independent sequences of the same length are automorphic, we can choose a primitive code $\boldsymbol{d}_\alpha$ and a (vector valued) function $F_\alpha \in L_f$ such that if $\overline{g}$ is d_f-independent, then $\theta_{\boldsymbol{d}_\alpha}(F_\alpha(\overline{g}))$, and $\rho_\alpha(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; \overline{g})$ and $\phi_{\boldsymbol{d}_\alpha}(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; F_\alpha(\overline{g}))$ have the same generic solutions.

We will now show:

4.2 Proposition. *If $\mu(\boldsymbol{d}_\alpha) < r$, and M is the generic model of T^μ and $\overline{b}\overline{c}\overline{d}\overline{h} \subseteq M$, then*

$$M \models [(\exists \overline{u})\overline{\alpha}(\overline{u}, \overline{b}, \overline{c}, \overline{d}) \longrightarrow \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})].$$

Thus, for any $N \models T^\mu$,

$$N \models [(\exists \overline{u})\overline{\alpha}(\overline{u}, \overline{b}, \overline{c}, \overline{d}) \longrightarrow \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})].$$

Proof. In any model N of T^μ, note that if $\overline{b}\overline{c}\overline{d}\overline{h}$ is a solution of $\overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s})$ and $\overline{a}$ is a solution of $\overline{\alpha}(\overline{u}, \overline{b}, \overline{c}, \overline{d})$ disjoint from $\overline{b}\overline{c}\overline{d}\overline{h}$, then for each i, $\delta(a_i/\overline{b}\overline{c}\overline{d}\overline{h}) = -1$. But, $\delta(\overline{a}\overline{b}\overline{c}\overline{d}\overline{h}) \geq 0$, since $N \models T_0$, and $\delta(\overline{b}\overline{c}\overline{d}\overline{h}) \leq r$. So it must be that for any such solution of $\overline{\beta} \wedge \overline{\alpha}$, we have $\delta(\overline{a}\overline{b}\overline{c}\overline{d}\overline{h}) = 0$, and the set $\overline{a}\overline{b}\overline{c}\overline{d}\overline{h}$ is intrinsically closed. Now if N is the generic model M, the $\leq$-homogeneity guarantees that all such solutions are automorphic; thus $\overline{b}\overline{c}\overline{dh}$ is a generic solution of $\overline{\beta}$. If, moreover, $\overline{b}\overline{c}\overline{dh}$ extends in M to a solution $\overline{a}\overline{b}\overline{c}\overline{d}\overline{e}\overline{f}\overline{g}\overline{h}$ of

$$\omega_\alpha(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{yz}\overline{t}\overline{s}) = \phi(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s}) \wedge \overline{\alpha}(\overline{u}, \overline{vw}, \overline{x})$$

then $\delta(\overline{a}\overline{b}\overline{c}\overline{d}\overline{e}\overline{f}\overline{g}\overline{h}) = \delta(\overline{e}\overline{f}\overline{g}/\overline{a}\overline{b}\overline{c}\overline{dh}) + \delta(\overline{a}\overline{b}\overline{c}\overline{dh}) = 0+0 = 0$ as well, so that all solutions in M of ω_α are automorphic, and their restrictions to the appropriate variables are generic solutions of $\overline{\beta}$, $\overline{\alpha}$ and $\overline{\gamma}$. In particular, $a_1b_1c_1d_1e_1f_1, \ldots, a_rb_rc_rd_re_rf_r$ are r independent solutions of $\rho_\alpha(u, v, w, x, y, z; \overline{g})$.

If $\overline{a}\overline{b}\overline{c}\overline{d}\overline{e}\overline{f}\overline{g}\overline{h}$ is any solution of $\omega_\alpha(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}, \overline{t}, \overline{s})$, in M the discussion in the previous paragraph shows $M \models E_{\boldsymbol{d}_\alpha}(a_1b_1c_1d_1e_1f_1, \ldots, a_rb_rc_rd_re_rf_r)$. We chose $m(\boldsymbol{d}_\alpha) \leq r$, so the expression is defined. Thus if M satisfies both $(\exists \overline{u})\overline{\alpha}(\overline{u}, \overline{b}, \overline{c}, \overline{d})$ and $(\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})$, then it satisifies $(\exists \overline{u})(\exists \overline{y}, \overline{z}, \overline{t})\omega_\alpha(\overline{u}, \overline{b}, \overline{c}, \overline{d}, \overline{y}, \overline{z}, \overline{t}, \overline{h})$ which contradicts $M \models T^\mu$. Thus M satisfies the required implication. But since T^μ is complete the implication holds in every model. $\square_{4.2}$

Now, we make copies of the $\overline{a}$ and define the appropriate μ for our counterexample. Find (generically) distinct formulas α_i, $i = 1, 2, 3, \ldots$ satisfying the conditions imposed above on α, and such that $m(\boldsymbol{d}_\alpha) = 1$. (We only need a uniform bound on $m(\boldsymbol{d}_\alpha)$; we get this by choosing the α_i from among linear varieties, for which $m(\boldsymbol{d}_\alpha)$ is always 1.) Fix a $\mu = \mu_0$ for which $\mu_0(\boldsymbol{d}_{\alpha_i}) < r$ for each i (can do provided $r \geq 8$, and still satisfy $\mu_0(\boldsymbol{d}_{\alpha_i}) \geq m(\boldsymbol{d}_{\alpha_i})n(\boldsymbol{d}_{\alpha_i}) + 2m(\boldsymbol{d}_{\alpha_i}) = 1 \cdot 6 + 2 = 8$). (Note that such a μ_0 is not finite-one.)

Now we construct the type which is omitted in the generic. By the choice of the α_i, for fixed $\overline{b}\overline{c}\overline{dh}$ satisfying $\overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s})$ and $j \neq l$, $\overline{\alpha}_j(\overline{u}, \overline{b}, \overline{c}, \overline{d})$ and $\overline{\alpha}_l(\overline{u}, \overline{b}, \overline{c}, \overline{d})$ cannot both have solutions disjoint from $\overline{b}\overline{c}\overline{dh}$, since if $\overline{a}$ and $\overline{a}'$ were such, we would have $\delta(\overline{aa}'\overline{b}\overline{c}\overline{dh}) < 0$. The construction before Proposition 4.2 showed that for each j, $\overline{\alpha}_j(\overline{u}, \overline{b}, \overline{c}, \overline{d})$ is consistent with T^μ. Our last observation then implies that the set of formulas $\{\overline{\alpha}_j(\overline{u}, \overline{b}, \overline{c}, \overline{d})\} \cup \{\neg\overline{\alpha}_\ell(\overline{u}, \overline{b}, \overline{c}, \overline{d}) : \ell \neq j\}$ is also consistent with T^μ.

Since $\overline{e}\overline{f}\overline{g}$ is primitive over $\overline{b}\overline{c}\overline{dh}$ with complete δ-formula ϕ, if $\overline{b}\overline{c}\overline{dh} \leq M$, the generic, is a solution of $\overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s})$ then $M \models (\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})$. Thus, if M is ω-saturated, then

$$T^{\mu_0} \cup I^*(\overline{b}, \overline{c}, \overline{d}, \overline{h}) \vdash (\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s}).$$

But

$$T^{\mu_0} \cup I^*(\overline{b}, \overline{c}, \overline{d}, \overline{h}) \cup \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s})$$

is consistent. To see this, take any finite subset Σ of $I^*(\overline{b}, \overline{c}, \overline{d}, \overline{h})$. The argument of the previous paragraph shows there is some i such that there is a model C of T^{μ_0} with $\overline{b}, \overline{c}, \overline{d}, \overline{h}$ satisfying $\Sigma(\overline{b}, \overline{c}, \overline{d}, \overline{h}) \cup (\exists \overline{u})\overline{\alpha}_i(\overline{u}, \overline{b}, \overline{c}, \overline{d})$. But then since Proposition 4.2 holds for each $\overline{\alpha}_i$, $C \models \neg\exists \overline{y}, \overline{z}, \overline{t}\phi(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})$ as required. $\square_{4.1}$

In fact, we can show a stronger result. Recall that a theory is small if for each n there are only countably many n-types over the empty set. If T has a countable saturated model, then T is small. To find a T^{μ_0} which is not small, we will have to modify the argument above slightly.

As before we will define a finite set A ($\overline{b}\overline{c}\overline{dh}$ below). But now we will choose an infinite family of primitives B_i ($\overline{e}^i\overline{f}^i\overline{g}^i$ below) over A. We will show we can determine independently whether each of the B_i is realized. This gives $2^{\aleph_0}$ types of $4r$-ary sequences. The sequence $\overline{b}\overline{c}\overline{dh}$ realizes p_X in N if exactly the B_i for $i \in X$ are embedded in N. To prove the consistency of this set of types, we find for $s \in 2^{<\omega}$ a set A_s ($\overline{a}_s\overline{b}\overline{c}\overline{dh}$ below) such that if A_s is the minimal intrinsic extension of A (in $\boldsymbol{K}_0$), the B_i for $i \in s$ are forbidden (Proposition 4.4). We turn to the formal proof.

4.3 Theorem. *There exists a function μ_0 such that for each code $\boldsymbol{c}$, $\mu(\boldsymbol{c}) \geq q(\boldsymbol{c}) + m(\boldsymbol{c})$ but T^{μ_0} is not small.*

Find (generically) distinct formulas γ_i, $i = 1, 2, 3, \ldots$ satisfying the conditions imposed above on γ, and for each $n < \omega$ and $s \subseteq n$, (generically) distinct formulas α_s, satisfying the conditions imposed above on α. Form formulas $\overline{\beta}$, $\overline{\gamma}_i$ etc. as before.

For $i < \omega$, let

$$\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s}) = \overline{\beta}(\overline{v}, \overline{w}, \overline{x}, \overline{s}) \wedge \overline{\gamma}_i(\overline{t}, \overline{y}, \overline{z}, \overline{v}, \overline{w}, \overline{x})$$

together with the assertion that a solution consists of distinct elements.

For $i < n$ and $s \subseteq n$, let

$$\rho_{i,s}(u, v, w, x, y, z; \overline{t}) = \gamma_i(\overline{t}, y, z; v, w, x) \wedge \alpha_s(u, v, w, x)$$

together with the assertion that the elements of a solution are distinct and black.

Now find a primitive code $\boldsymbol{d}_{i,s}$ and, by Definition 2.2.2, a (vector valued) function $F_{i,s} \in L_f$ such that if $\overline{g}$ is d_f-independent, then $\theta_{\boldsymbol{d}_{i,s}}(F_{i,s}(\overline{g}))$, and $\rho_{i,s}(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; \overline{g})$ and $\phi_{\boldsymbol{d}_{i,s}}(\overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}; F_{i,s}(\overline{g}))$ have the same generic solutions. Fix μ_0 with $\mu_0(\boldsymbol{d}_{i,s}) < r$ if and only if $i \in s$.

Just repeating the proof of Proposition 4.2 for each i and s, we have:

4.4 Proposition. *If for each i, s, $\mu_0(\boldsymbol{d}_{i,s}) < r$, M is the generic model of T^{μ_0} and $\overline{b}\overline{c}\overline{dh} \subseteq M$, then*

$$M \models [(\exists \overline{u})(\overline{\alpha}_s(\overline{u}, \overline{b}, \overline{c}, \overline{d}) \longrightarrow \bigwedge_{i \in s} \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d})].$$

Thus, for any $N \models T^{\mu_0}$ and $\overline{b}, \overline{c}\overline{dh} \subseteq N$,

$$N \models [(\exists \overline{u})(\overline{\alpha}_s(\overline{u}, \overline{b}, \overline{c}, \overline{d}) \longrightarrow \bigwedge_{i \in s} \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d})].$$

Proof of Theorem 4.3:

For each $X \subseteq \omega$, we define a type p_X in $4r$-variables, $\overline{v}, \overline{w}, \overline{x}, \overline{s}$. Namely, $i \in X$ if and only if $(\exists \overline{y}, \overline{z}, \overline{t})\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{v}, \overline{w}, \overline{x}, \overline{s})$ is in p_X.

To see the consistency of p_X we show that for any n, there is a tuple $\overline{b}, \overline{c}, \overline{d}$ (really, $\overline{b}_s, \overline{c}_s, \overline{d}_s$ but we drop the s for notational convenience) in the generic model M which satisfy the conditions in p_s for $s = X \cap n$. Fix such an n and s. We first construct a structure in $\boldsymbol{K}_0$. Let $A = \{\overline{b}, \overline{c}, \overline{d}, \overline{h}\}$ be a generic solution of β. If $i \notin s$, adjoin to A, sequences $\overline{e}^i, \overline{f}^i, \overline{g}^i$ which are generic solutions of ϕ_i over A. Now add a sequence $\overline{a}_s$ satisfying $\overline{\alpha}_s(\overline{a}_s, \overline{b}, \overline{c}, \overline{d})$ to form a set A_s. Note $\delta(A_s) \geq 0$ and the field generated by A_s does not violate μ_0. So there is an embedding of A_s into the generic model M of T^{μ_0} and $M \models (\exists \overline{y}, \overline{z}, \overline{t})\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})$ if $i \notin s$. Now by Proposition 4.4, for each $i \in s$, $M \models \neg(\exists \overline{y}, \overline{z}, \overline{t})\phi_i(\overline{y}, \overline{z}, \overline{t}; \overline{b}, \overline{c}, \overline{d}, \overline{h})$ $\square_{4.3}$

It is open whether in this situation T^{μ_0} is stable.

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