COMPLETE $\mathcal{L}_{\omega_1,\omega}$-SENTENCES WITH MAXIMAL MODELS IN MULTIPLE CARDINALITIES

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ABSTRACT. In [BKS14] examples of incomplete sentences are given with maximal models in more than one cardinality. The question was raised whether one can find similar examples of complete sentences. In this paper we give examples of complete $\mathcal{L}_{\omega_1,\omega}$-sentences with maximal models in more than one cardinality; indeed, consistently in countably many cardinalities. The key new construction is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence with arbitrarily large models but with (κ^+, κ) models for every κ.

We unite ideas from [BFKL13, BKL14, Hjo02, Kni77] to find complete sentences with maximal models in two cardinals. There have been a number of papers finding complete sentences of $\mathcal{L}_{\omega_1,\omega}$ characterizing cardinals beginning with Baumgartner, Malitz and Knight in the 70's, refined by Laskowski and Shelah in the 90's and crowned by Hjorth's characterization of all cardinals below $\aleph_{\omega_1}$ in the 2002. These results have been refined since. But this is the first paper finding complete sentences with maximal models in two or more cardinals.

Our arguments combine and extend the techniques of building atomic models by Fraissé constructions using disjoint amalgamation, pioneered by Laskowski-Shelah and Hjorth, with the notion of homogeneous characterization and tools from Baldwin-Koerwien-Laskowski. This paper combines the ideas of Hjorth and Knight with specific techniques from [BFKL13, BKL14, Sou14, Sou13] and many proofs are adapted from these sources. We thank the referee for a perceptive and helpful report.

Structure of the paper:

In Section 1, we explain the merger techniques for combining sentences that homogeneously characterize one cardinal (possibly in terms of another) to get a single complete sentence with maximal models in prescribed cardinalities.

Section 2 contains the main technical construction of the paper: the existence of a complete sentence ϕ with a unary predicate that has (κ^+, κ) models for every κ. From this construction and the tools of Section 1, we complete the proof of Theorem 1.5 and thus of Corollary 1.7:

for each homogenously characterizable κ,

examples of $\mathcal{L}_{\omega_1,\omega}$-sentences ϕ_κ with maximal models in κ and κ^+ and no larger models.

In Section 3 we present examples,

for each homogenously characterizable κ,

of $\mathcal{L}_{\omega_1,\omega}$-sentences with maximal models in κ and κ^ω and no larger models. The argument can be generalized to maximal models in κ and $\kappa^{\aleph_\alpha}$, for all countable α.

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Finally in Section 4, if κ is a homogeneously characterizable cardinal, we present an $\mathcal{L}_{\omega_1,\omega}$-sentence with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\kappa}$ and no models larger than 2^{κ}.

template

1. The general construction

In this section, for a cardinal κ that admits a homogeneous characterization (Definition 1.1),

we state the theorem that

a complete sentence ϕ_κ of $\mathcal{L}_{\omega_1,\omega}$ that has maximal models in κ and κ^+ and no larger models. The proof applies the notion of a receptive model from [BFKL13] and merges a sentence homogeneously characterizing κ with a complete sentence encoding the old idea of characterizing κ^+ by a κ-like order. We outline here the general plan of the proof, which is carried out in Section 2. This template is extended to functions other than successor in later sections.

We require a few preliminary definitions.

Def:HomChar **Definition 1.1.** *Assume $\lambda \leq \kappa$ are infinite cardinals, ϕ is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence in a vocabulary that contains a unary predicate P, and $\mathcal{M}$ is the (unique) countable model of ϕ. We say*

(1) a model $\mathcal{N}$ of ϕ is of type (κ, λ), if $|\mathcal{N}| = \kappa$ and $|P^{\mathcal{N}}| = \lambda$;
(2) For a countable structure $\mathcal{M}$, $P^{\mathcal{M}}$ is a set of absolute indiscernibles *for $\mathcal{M}$, if $P^{\mathcal{M}}$ is infinite and every permutation of $P^{\mathcal{M}}$ extends to an automorphism of $\mathcal{M}$.*
(3) ϕ homogeneously characterizes *κ, if*
(a) ϕ has no model of size κ^+;
(b) $P^{\mathcal{M}}$ is a set of absolute indiscernibles for $\mathcal{M}$, and
(c) there is a maximal model of ϕ of type (κ, κ).

The next notation is useful for defining mergers. We slightly broaden the notion of 'receptive' from [BFKL13] by requiring some sorts of the 'guest sentence' to restrict to U while others are new sorts in the final vocabulary.

fixvoc1 **Notation 1.2.** *Fix a vocabulary τ containing unary predicates V, U and a binary relation symbol P. The sentence θ_0 says V and U partition the universe and P is a projection of V onto U.*

Let τ_1 extend τ and let θ be a complete τ_1-sentence of $\mathcal{L}_{\omega_1,\omega}$ that implies θ_0. Fix a vocabulary τ' disjoint from τ_1 that contains a unary predicate Q, and let ψ an arbitrary (possibly incomplete) τ'-sentence of $\mathcal{L}_{\omega_1,\omega}$. Let τ_2 contain the symbols of $\tau_1 \cup \tau'$ except for Q.

- *If U defines an infinite* absolutely indiscernible set *in the countable model of θ, we call the pair (θ, U)* receptive. *We call θ* receptive *if there is a U such that (θ, U) is receptive and in that case we also call the countable model of θ a receptive model.*
- *The* merger *$\chi_{\theta,U,\psi,Q}$ of the pair (θ, U) is the conjunction of θ and $\psi^{U,Q}$, where $\psi^{U,Q}$ is the result of substituting U for Q in ψ. Thus $\chi_{\theta,U,\psi,Q}$ is a τ_2-sentence.*
- *If in all models $\mathcal{N}$ of ψ, $Q^{\mathcal{N}}$ is the domain of $\mathcal{N}$, then we will drop Q and write $\chi_{\theta,U,\psi}$.*
- *If $\mathcal{M} \models \theta$ and $\mathcal{N} \models \psi$, the* merger model *$(\mathcal{M}, \mathcal{N})$ denotes a model of $\chi_{\theta,U,\psi,Q}$ where the elements of $Q^{\mathcal{N}}$ have been identified with the elements of $U^{\mathcal{M}}$, which is the intersection of M and N.*
  $\mathcal{M}$ will be called the host *model and $\mathcal{N}$ the* guest *model.*

Note that if ϕ and P homogeneously characterize some κ, then the countable model of ϕ is receptive. Fact 1.3 extends the argument in [BFKL13] to reflect our more general notion of merger.

mergeprop

Fact 1.3. *Let (θ, U) be receptive and ψ a sentence of $\mathcal{L}_{\omega_1,\omega}$.*

1. *The merger $\chi_{\theta,U,\psi,Q}$ is a complete sentence if and only if ψ is complete.*
2. *There is a* 1-1 *isomorphism preserving function between isomorphism types of the countable models of ψ and the isomorphism types of countable models of the merger $\chi_{\theta,U,\psi}$.*
3. *If there is a model M_0 of θ with $|M_0| = \lambda_0$ and $|U^{M_0}| = \rho$ and also a model M_1 of ψ with $|M_1| = \lambda_1$ and $|Q^{M_1}| = \rho$, then there is a model of $\chi_{\theta,U,\psi,Q}$ with cardinality* $\max(\lambda_0, \lambda_1)$.

Remark 1.4. The proof of 1) in [BFKL13] is a bit quick. The completeness also depends on absolute indiscernability. Let $\mathcal{N}$ and $\mathcal{N}'$ be countable models of ψ. Then $Q(\mathcal{N}) \cong_{\tau'} Q(\mathcal{N}')$. By absolute indiscernability that automorphism extends to a τ_1 automorphism of any $\mathcal{M}$ being merged with $\mathcal{N}$ or $\mathcal{N}'$.

In Section 2 we prove:

knightgen

Theorem 1.5. *There is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence*[1] *$\phi_M$ with unary predicates X, Y and a binary predicate $<$ such that:*

1. *ϕ_M has arbitrarily large models.*
2. *If $\mathcal{M} \models \phi_M$, $|\mathcal{M}| \leq |X^{\mathcal{M}}|^+$.*
3. *For each κ, there exists a model of ϕ_K of type (κ^+, κ).*

Idea of the Proof: We will construct (via a generalized Fraïssé construction) a sentence ϕ_M whose models behave as follows. The sort Y is linearly ordered by $<$. Each element y of Y determines a function $g(\_, y) : X \to Y$ so that for $y \in Y$, $g(\_, y)$ maps X onto the initial segment below y. The mapping is finite-to-one and so bounds the size of any initial segment by $|X|$. The full proof is in Section 2.

Using this result we show if κ is homogeneously characterizable, we can construct a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ that has maximal models in κ and κ^+ and no larger models. Before we proceed

with the proof we introduce the tool by which we turn homogeneously characterizable cardinals into pairs of maximal models.

Thm:ManyMaximal

Theorem 1.6. *Let κ be a homogeneously characterizable cardinal . Then there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence χ in a vocabulary with a new unary predicate symbol B, such that (χ, B) is receptive, χ homogeneously characterizes κ and χ has maximal models with $(|\mathcal{M}|, |B^{\mathcal{M}}|) = (\kappa, \lambda)$, for all $\lambda \leq \kappa$.*

Proof. Fix a receptive pair (θ, U) such that θ homogeneously characterizes κ. Define a new vocabulary $\tau = \{A, B, p\}$ where A, B are unary predicates and p is a binary predicate. Let ϕ be the conjunction of: (a) A, B partition the universe and (b) p is a total function from A onto B such that each $p^{-1}(x)$ is infinite. In the countable model of ϕ, B is a set of absolute indiscernibles.

Now merge θ and ϕ by identifying U and A. The merger $\chi = \chi_{\theta,U,\phi,A}$ is a complete sentence which does not have any models of size κ^+. Let $\mathcal{M}$ be a maximal model of θ with $U^{\mathcal{M}}$ of size κ, and $\mathcal{N}$ a model of ϕ of type (κ, λ), for some $\lambda \leq \kappa$. Then the merger model $(\mathcal{M}, \mathcal{N})$ is a maximal model of χ with $|(\mathcal{M}, \mathcal{N})| = \kappa$ and $|B^{(\mathcal{M},\mathcal{N})}| = \lambda$, which proves the result $\square$

A word of caution: In the countable model of θ, the predicate U defines a set of absolute indiscernibles, and the same is true for the countable model of ϕ and B. So, we started with two models and two sets of absolute indiscernibles. In the merger $\chi_{\theta,U,\phi,A}$, the absolute

[1]The subscript M on ϕ_M is residue of the proof; we have only one such sentence.

indiscernibles of the host model (model of θ) are used to bound the size of A from the guest model (model of ϕ). Moreover, the predicate B from the guest model defines a set of absolute indiscernibles in the merger model too, and it may be small.

Now, combining Theorem 1.6 and Theorem 1.5 we conclude

Corollary 1.7. *Let κ be a cardinal that is a homogeneously characterized by a formula ϕ_κ with set of absolute indiscernibles U. Then there is a complete sentence ϕ^*_κ with additional predicates X and Y that characterizes κ and has maximal models in κ and κ^+.*

Further, if $N \models \phi^*_\kappa$, $|U^N| \le |X^N|$ and $|Y^N| \le |X^N|^+$.

Proof. Let ϕ^*_κ be the merger $\chi_{\phi_\kappa, U, \phi_M, X}$, where the pair (ϕ_κ, U) witnesses the homogeneous characterizability of κ, and the pair (ϕ_M, X) is given by Theorem 1.5. By Fact 1.3, ϕ^*_κ is complete. Suppose $H = (\mathcal{M}, \mathcal{N}) \models \phi^*_\kappa(U, X, Y)$, i.e. $(\mathcal{M}, U^H) \models (\phi_\kappa, U)$ and $(\mathcal{N}, X^H, Y^H) \models \phi_M)$ and p maps U^H onto X^H. This guarantees the cardinality restrictions in the furthermore.

Suppose further than $\mathcal{M}$ is a maximal model of ϕ_κ of size κ. If $|U^{\mathcal{M}}| < \kappa$, then $|X^{\mathcal{N}}| < \kappa$, and by Theorem 1.5 (2) $|\mathcal{N}| \le \kappa$. So, the merger model is maximal and has size κ.

On the other hand, if $|U^{\mathcal{M}}| = \kappa$, then choose a model $\mathcal{N}$ of ϕ_M of type (κ^+, κ), which is maximal among models of ϕ_M with $X = U^{\mathcal{M}}$. By Theorem 1.5, there is no (κ^{++}, κ)-model of ϕ_M and therefore, there must be such a maximal (κ^+, κ) model. The merger model $(\mathcal{M}, \mathcal{N})$ is again a maximal model of ϕ^*_κ, but this time it has size κ^+. $\square$

Particular examples of homogeneously characterizable cardinals are given by [Bau74], [Hjo02], [Sou12], [Sou13], [Sou14], [BKL14].

Theorem 1.8. *If κ is homogeneously characterizable, then the same holds true*

(1) for[2] 2^κ;
(2) for[3] κ^ω;
(3) for[4] $\kappa^{\aleph_\alpha}$, for all countable ordinals α.

Theorem 1.9 (Theorem 4.29, [Sou13])**.** *If $\aleph_\alpha$ is a characterizable cardinal, then $2^{\aleph_{\alpha+\beta}}$ is homogeneously characterizable, for all $0 < \beta < \omega_1$.*

Finally a result of slightly different character; we note a direct proof of a sentence ϕ_n that homogeneously characterize $\aleph_n$ $(n > 0)$ and has $(\aleph_n, \aleph_k)$ models for $k \le n$.

Theorem 1.10 ([BKL14])**.** *For each $n \in \omega$, there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_n such that*

- *ϕ_n homogeneously characterizes $\aleph_n$ with absolute indiscernibles in a predicate P; and*
- *for each $k \le n$, there is a maximal model N_k of ϕ_n of type $(\aleph_n, \aleph_k)$.*

Since in this last example, the complete sentence[5] has maximal models of type $(\aleph_n, \aleph_k)$, for all $k \le n$ there is no need to appeal to Theorem 1.6 for the proof of Corollary 1.7.

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[2] Baumgartner; see also Theorem 3.4 of [Sou13]

[3] Theorem 3.6, [Sou14]

[4] Corollary 5.6, [Sou12]

[5] The proof that these $(\aleph_n, \aleph_k)$ models exist requires the use of both frugal amalgamation and an amalgamation which allows identification. We say a class has frugal amalgamation if for every amalgamation triple A, B, C there is an amalgam on the union of the domains with no identifications. See [BKS09].

2. A complete sentence with (κ^+, κ)-models for all κ

Knight [Kni77] constructed the first example of a complete sentence characterizing $\aleph_1$. We will vary that idea to get the result announced.

Lemma 2.1. *There is a structure $A = (Q, <, g_n)$ in the vocabulary $\langle <, g_n \rangle$, where $<$ is the usual dense order on Q and the g_n are unary, such that for each n and x, $g_n(x) < x$ and for each x the set of $g_n(x)$ is the set of predecessors of x. Moreover this structure has a proper elementary extension isomorphic to itself. Thus A is extendible. If ϕ_A is the Scott sentence of A, ϕ_A characterizes $\aleph_1$.*

The proof of this lemma involves an intricate construction of functions f_a for $a \in \mathcal{P}_\omega(A)$; we now produce a similar structure by a Fraissé style construction, proving Theorem 1.5. We follow the idea of Hjorth in replacing functions f_n by a uniform $f(x, y)$ indexed in the model. This sentence will have arbitrarily large models with two sorts X, Y such that the sort of $|Y| \leq |X|^+$. This construction will allow us in Theorem ?? to construct (κ^+, κ) models of ϕ_M. The predicates G_n are used to enforce local finiteness.

We now describe the sentence ϕ_M; the remainder of the section completes the proof of Theorem 1.5.

Construction 2.2. Let τ contain binary $<$, unary X, Y and ternary $g(x, y, z)$ and $(n+2)$-ary relation symbols $G_n(x_0, \ldots, x_{n-1}, y, z)$ on $X^n \times Y^2$.

$\boldsymbol{K}_0$ is the collection of finite structures such X and Y are disjoint, $<$ linearly orders Y, $g(x, y, z)$ is a *total* function from $X \times Y$ into Y such that $g(x, y, z)$ implies $z \leq y$ and for each $y \in Y$, letting $W_y = \{z \in Y | z \leq y\}$, the function $g_y = g(\_, y) : X \to W_y$ is onto. We often write $g(x, y) = z$ for $g(x, y, z)$.

We want to guarantee that in each model for each $z < y \in Y$ there is an n, $0 < n < \omega$, such that the set $\{x \in X | g(x, y) = z\}$ has size n. In other words, the function $g(\_, y)$ is finite-to-1 when restricted to the set $\{x \in X | g(x, y) < y\}$. Notice that while $g(x, y)$ is allowed to take the value y, the finite-to-1 restriction does not apply to the set $\{x \in X | g(x, y) = y\}$.

For each $z < y \in Y$, we use the $(n+2)$-ary relation symbols $G_n(x_0, \ldots, x_{n-1}, y, z)$ on $X^n \times Y^2$ to indicate that the set $\{x_0, \ldots, x_{n-1}\}$ equals the set $\{x \in X | g(x, y) = z\}$, which is of size n. The relations G_n satisfy the following:

The universal closure of each formula of the following form(s):

$$G_n(x_0, \ldots, x_{n-1}, y, z) \to \bigwedge_{i<j<n} x_i \neq x_j;$$

$$G_n(x_0, \ldots, x_{n-1}, y, z) \to (z < y) \wedge \bigwedge_{i<n} g(x_i, y, z);$$

$$G_n(x_0, \ldots, x_{n-1}, y, z) \wedge g(w, y, z) \to \bigvee_{i<n} w = x_i;$$

and for every permutation $\pi : n \to n$,

$$G_n(x_0, \ldots, x_{n-1}, y, z) \leftrightarrow G_n(x_{\pi(0)}, \ldots, x_{\pi(n-1)}, y, z)$$

and each sentence of the form:

$$\forall y \, \forall z < y \bigvee_{0<n<\omega} \exists x_0 \exists x_1 \, \exists x_2 \ldots \exists x_{n-1}, \; G_n(x_0, x_1, \ldots x_{n-1}, y, z);$$

Note that since in every member of $\boldsymbol{K}_0$ and for every y there is an x with $g(x, y) = y$, we will never have to check this condition in an amalgamation. The following easy lemma will be applied several times.

Lem:bigX **Lemma 2.3.** *If $A \in \boldsymbol{K}_0$ and $m \in \omega$, there is a structure $B \in \boldsymbol{K}_0$ with A a substructure of B and $|X^B \setminus X^A| \geq m$.*

Proof. Adjoin m elements to X^A and let $g^B(x, y) = y$, for all new x and each $y \in Y^A$. For each n exactly the same elements satisfy G_n in A and B. □

gp **Definition 2.4.** *We call (A, B) a* good pair, *if $A \subset B$, $A, B \in \boldsymbol{K}_0$ and $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$.*

Lem:DAPForK0 **Lemma 2.5.** *$\boldsymbol{K}_0$ has the disjoint amalgamation property under substructure.*

Proof. Suppose A is a substructure of B and C. Applying Lemma 2.3 to B with $m = |Y^B \setminus Y^A|$ and to C with $m = |Y^C \setminus Y^A|$ we may assume $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$ and $|X^C \setminus X^A| \geq |Y^C \setminus Y^A|$ (and they intersect in A). Extend the partial order on $Y^B \cup Y^C$ by linearly ordering the a_i, b_j that are in an interval (c_i, c_j) in any way that preserves the ordering of the a's and of the b's.

Let $X^D = X^B \cup X^C$ and $Y^D = Y^B \cup Y^C$. The goal is to define $g^D(x, y)$ on the pairs $(x, y) \in (X^B \setminus X^A) \times (Y^C \setminus Y^A) \cup (X^C \setminus X^A) \times (Y^B \setminus Y^A)$ to obtain an amalgam $D \in \boldsymbol{K}_0$.

For each $y \in Y^C \setminus Y^A$, let $W^B_y = \{z \in Y^B \setminus Y^A | z \leq y\}$. Define $g^D(x, y)$ so that for each $y \in Y^C \setminus Y^A$ and each $x \in X^B \setminus X^A$, $g^D(x, y) \in W^B_y$ and each element in W^B_y is mapped to by at least one $x \in X^B \setminus X^A$. This is possible by the application above of Lemma 2.3. If W^B_y is empty, define $g^D(x, y) = y$.

Symmetrically, for each $x \in X^C \setminus X^A$ and each $y \in Y^B \setminus Y^A$, define $g^D(x, y)$ to be an element in $W^C_y = \{z \in Y^C \setminus Y^A | z \leq y\}$, so that each element in $W^C_y \subset Y^C \setminus Y^A$ is mapped to at least once. Again, if the set is empty, define $g^D(x, y) = y$.

Since no newly defined triple x, y_1, y_2 that satisfies $g(x, y_1) = y_2$ has both y_1 and y_2 in C, or both y_1 and y_2 in B, we can extend the definition of the G^D_n's consistent with the new assignments of g while extending G^B_n and G^C_n and so making $D \in \boldsymbol{K}_0$. □

We summarize the results of this construction. We call a structure $\boldsymbol{K}_0$-generic if it is homogeneous and universal for $\boldsymbol{K}_0$-structures and is a union of finite structures.

CountableGeneric **Theorem 2.6.** *There is a countable $\boldsymbol{K}_0$-generic (and atomic) model M (with Scott sentence ϕ_M), which is an increasing union of members of $\boldsymbol{K}_0$ which are each closed under g. Thus, M is locally (but not uniformly locally) finite. Moreover $M \models (\forall y_1, y_2)[y_1 < y_2 \rightarrow (\exists x) g(x, y_2) = y_1]$ since each model in the union does.*

Thus, if $N \models \phi_M$, Y^N is a dense linear order and for every $y \in Y^N$, $|X^N| \geq |\{y' \in Y^N : y' < y\}|$. In particular, if $|X^N| \leq \kappa$, then the size of Y^N is bounded by κ^+.

Proof. Since the G_n are only defined on models of cardinality at least n, there are only countably many models in $\boldsymbol{K}_0$, so the amalgamation property (and trivially joint embedding) guarantee the existence of $\mathcal{M}$. □

a1ok **Corollary 2.7.** *There is an $(\aleph_1, \aleph_0)$-model of ϕ_M.*

Proof. We borrow the technique of Lemma 1.10 of [BFKL13]. Expand the vocabulary τ to τ' by adding a new unary predicate Q. Consider the class $\boldsymbol{K}'_0$ of finite τ'-structures N such that:

(1) $N \restriction \tau \in \boldsymbol{K}_0$;
(2) $X^N \subseteq Q^N$; and
(3) $Y^N \cap Q^N$ is an initial segment of Y^N.

By virtue of $\boldsymbol{K}_0$ satisfying disjoint amalgamation, the same is also true for $\boldsymbol{K}'$. We claim that the generic model M' for $\boldsymbol{K}'$ has a proper τ-substructure M_0 with universe $Q^{M'}$, $M_0 \restriction \tau \cong M' \restriction \tau$, and $X^{M_0} = X^{M'}$.

Since M' is $\boldsymbol{K}'$-generic, both $M' \restriction \tau$ and $M_0 \restriction \tau$ must be $\boldsymbol{K}_0$-generic. By Theorem 2.6, they are isomorphic. The fact that $X^{M_0} = X^{M'}$ follows from condition (2).

Now, consider the question whether M_0 is a proper submodel of M'. It is immediate that M_0, M' agree on $<$, and $X^{M_0} = X^{M'}$ implies that they agree on the G_n's too. The only case that needs to be considered is if there exist $x \in X^{M_0}$ and $y_1 \in Y^{M_0}$, and $g^{M_0}(x, y_1)$ does not equal $g^{M'}(x, y_1)$. But this can happen only if $g^{M'}(x, y_1) \notin Y^{M_0}$, i.e. $g^{M'}(x, y_1) \notin Q^{M'}$. In this case, $g^{M'}(x, y_1) < y_1$, $y_1 \in Q^{M'}$ and $g^{M'}(x, y_1) \notin Q^{M'}$, contradicting condition (3).

So, we proved all three conditions on $M_0 \restriction \tau$ and $M' \restriction \tau$. Taking an uncountable increasing chain of atomic models, we get an $(\aleph_1, \aleph_0)$ model $M^* \models \phi_M$. □

hat **Definition 2.8.** *$C \in \hat{\boldsymbol{K}}_0$ if*

(1) for every finite $A \subset C$ there exists some finite substructure A' of C with $A \subset A'$ and $A' \in \boldsymbol{K}_0$.
(2) $(Y^C, <)$ is a linear order without endpoints.

Condition 1) is often expressed by saying C is locally finite. The following lemma is a crucial observation.

onto **Lemma 2.9.** *If $C \in \hat{\boldsymbol{K}}_0$, then g^C_y maps onto $W^C_y = \{z \in C : z \leq y\}$.*

Proof. Let $z \in W^C_y$. By local finiteness there is a finite A such that g^A_y maps onto W^A_y; in particular there is an $x \in A$ with $g^A(x, y) = g^C(x, y) = z$. □

The structure N is called $\boldsymbol{K}_0$-*rich*, or just *rich*, if for every $A \subset N$ which is in $\boldsymbol{K}_0$ and every extension $B \in \boldsymbol{K}_0$ of A, there is an embedding of B into N over A. If $N \in \hat{\boldsymbol{K}}_0$ and N is rich then standard arguments show $N \models \phi_M$.

Our first goal is to prove that ϕ_M has model in all cardinalities. Then we will merge this sentence with a complete sentence ϕ that homogeneously characterizes some cardinal κ by identifying X with the set of absolute indiscernibles of ϕ. This will result in a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that has maximal models in two cardinalities: κ and κ^+.

In the following argument we write $g^C(x, y)$ to emphasize when we are using the existing model C to define the extended function $g^D(x, \_)$.

Thm:AP **Theorem 2.10.** *Let $C \in \hat{\boldsymbol{K}}_0$ be infinite. Let $A, B \in \boldsymbol{K}_0$, $A \subset B, C$. Then the triple (A, B, C) can be disjointly amalgamated to a structure $D \in \hat{\boldsymbol{K}}_0$ with $|X^D| = |X^C|$ and $|Y^D| = |Y^C|$.*

Proof. Extend the partial order on $Y^B \cup Y^C$ to a *total* order without endpoints. Without loss of generality by Lemma 2.3 assume that $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$. The goal is to extend g^C and g^B to a function g^D on all pairs $(x, y) \in (X^B \setminus X^A) \times (Y^C \setminus Y^A) \cup (X^C \setminus X^A) \times (Y^B \setminus Y^A)$ so that $D \in \hat{\boldsymbol{K}}$. In stage 1, we almost achieve this goal. But we may need a further finite extension of D to D' and this is stage 2 of the construction.

In making this extension no new values are assigned to $g(a,b)=c$ where both b,c are in Y^C, or both b,c are in Y^B. So there is no danger of injury to any predicate $G_n(\mathbf{x},b,c)$ with $b,c \in B$, or $b,c \in C$.

Stage 1: We organize the case structure by the possible locations of an (x,y) on which g^D must be defined. The first case is $(x,y) \in X^B \setminus X^A \times Y^C \setminus Y^A$. For fixed y and each x define $g^D(x,y)$ to be an element of the set $W_y^B = \{z \in Y^B \setminus Y^A | z \le y\}$, ensuring that each element in W_y^B is mapped to at least once. This is possible given the assumption $|X^B \setminus X^A| \ge |Y^B \setminus Y^A|$. If W_y^B is empty, let $g^D(x,y) = y$. Since only finitely many extensions $g^D(x,\_)$'s have been defined and all of them take values outside of C, it is trivial to define the G_n^D. Note that since each $z \in Y^C$, $z \le y$, is already in the range of g_y^C, g_y^D maps onto $\{z \in Y^D | z \le y\}$.

The second case of stage 1 is $(x,y) \in X^C \setminus X^A \times Y^B \setminus Y^A$. It requires more work, since we must define $g^D(x,y)$ for infinitely many x and finitely many y without violating the constraints on the G_n. We choose a surrogate $y_0 > Y^B$ with $y_0 \in Y^C \setminus Y^A$ that we use to define $g^D(x,y)$ for all (x,y) satisfying this case. For every $x \in X^C \setminus X^A$ and $y \in Y^B \setminus Y^A$, we define $g^D(x,y)$ by three subcases.

(1) if $g^C(x,y_0) \ge y$, let $g^D(x,y) = y$.
(2) If $g^C(x,y_0) < y$ and $g^C(x,y_0) \notin Y^A$, let $g^D(x,y) = g^C(x,y_0)$.
(3) If $g^C(x,y_0) < y$ and $g^C(x,y_0) \in Y^A$, let[6] $g^D(x,y) = y$.

Each $g^D(x,\_)$ maps into $W_y^D = \{z \in D : z \le y\}$. Further the range of $g^D(\_,y)$ contains all of W_y^D, except for the elements (if any) in the set

$$P_y = \{z \in Y^C \setminus Y^A | z < y \text{ and } g^C(x,y_0) = z \to x \in X^A\}.$$

Indeed, let $z < y$. If $z \in Y^B$, then there is an x in X^B with $g^B(x,y) = z$ since $B \in \boldsymbol{K}_0$. If $z \in Y^C \setminus Y^A$, there exist $x_1 \in X^C$ with $g^C(x_1,y_0) = z$ since $C \in \hat{\boldsymbol{K}}_0$. If in addition $z \notin P_y$, then x_1 can be assumed to belong to $X^C \setminus X^A$. Then $g^C(x_1,y_0) = g^D(x_1,y) = z$, by subcase (2). If z belongs to P_y, then x_1 is necessarily an element of X^A, and none of subcases (1)-(3) applies since $x_1, y \in B$ and $g^D(x_1,y)$ equals $g^B(x_1,y)$. Thus, there is no x with $g^D(x,y) = z \in P_y$.

Stage 2: We now ensure that all elements in P_y are also included in the range of $g^D(\_,y)$. For this we add a finite number of new x's, X_{new}, call the extended domain D', and extend g^D to $g^{D'}$. Since X^A is finite, the same is true for P_y, and $|P_y| \le |X^A|$. We map each $y \in Y^B \setminus Y^A$ to P_y using these new elements so that for each $y \in Y^B \setminus Y^A$ and each $z \in P_y$, there exists at least one $x \in X_{\text{new}}$ such that $g^{D'}(x,y) = z$. For every $y \in Y^C$ and $x \in X_{\text{new}}$ set $g^{D'}(x,y) = y$.

We now define the $G_n^{D'}$. In stage 1 we did this for all $y \in Y^C$ and $z \in Y^B$ with $z < y$. By property (1) of Definition 2.8, for each $z \in Y^C \setminus Y^A$, there are only finitely many $x \in X^C$ that agree that the value of $g^C(x,y_0)$ is z. This implies that for each $y \in Y^B \setminus Y^A$ and each $z \in (Y^C \setminus Y^A) \setminus P_y$, $z < y$, there are only finitely many $x \in X^C \setminus X^A$ such that $g^{D'}(x,y) = z$. If $z \in P_y$, then there are only finitely many $x \in X_{\text{new}}$ such that $g^{D'}(x,y) = z$. Extend the G_n's from B and C to D' as the new assignments of $g^{D'}$ require.

This completes the construction of the amalgam D'; next we show that each of B,C is a substructure of D'. That is, the G_n's are preserved when we pass from B,C to D'. To check from B to D, we must show that for each $y,z \in Y^B$ there is no $x \in X_{\text{new}} \cup (X^C \setminus X^A)$, with $y > z$ and $g^{D'}(x,y) = z$ as this would violate $G_n^B(\mathbf{x},y,z)$. If $x \in X_{\text{new}}$ and $y \in Y^B \setminus Y^A$, then $g^{D'}(x,y) \ne z$ since $g^{D'}(x,y)$ belongs to $P_y \subset Y^C \setminus Y^A$. If $x \in X_{\text{new}}$ and $y \in Y^A$,

[6] Since each $g^B(\_,y)$ is onto, there must be an $x_1 \in X^B$, $g^B(x_1,y) = g^C(x,y_0)$ holds; following the recipe in 2) would make $G_n^D(\mathbf{x},y,z)$ differ from $G_n^B(\mathbf{x},y,z)$. This forces the introduction of the X_{new} in stage 2.

then $g^{D'}(x,y) = y \neq z$. If $x \in X^C \setminus X^A$, there do not exist $y, z \in Y^B$ with $y > z$ and $g^{D'}(x,y) = z$. For such a $g^{D'}(x,y)$ could not be defined using subcase (2), as in subcase (2) $z = g^{D'}(x,y) = g^C(x,y_0) \notin Y^A$ (and so certainly not in Y^B since $z \in Y^C$). So, for any $y, z \in Y^B$ with $y > z$, $|\{x : g^{D'}(x,y) = z\}| = |\{x : g^B(x,y) = z\}|$, which makes B a substructure of D'.

To check that the G_n's are preserved from C to D', let $y, z \in Y^C$ with $y > z$. We must show there is no $x \in X_{\text{new}} \cup (X^B \setminus X^A)$ such that $g^{D'}(x,y) = z$. If $x \in X_{\text{new}}$, then $g^{D'}(x,y) = y$. If $x \in X^B \setminus X^A$, then $g^{D'}(x,y) \in W^B_y \subset Y^B \setminus Y^A$ by the first case of Stage 1. Thus, both B and C are substructures of D'.

It remains to prove that $D' \in \boldsymbol{K}_0$. Property (2) is immediate. Property (1) is the one that needs work.

Let D_0 be a finite substructure of D'. Without loss of generality assume that $X_{\text{new}} \cup B \cup \{y_0\} \subset D_0$. Applying property (1) on C, there exists a substructure D_1 of C with $D_1 \in \boldsymbol{K}_0$ and $D_0 \cap C \subset D_1$. We claim that $D_2 = D_0 \cup D_1$ is a substructure of D' and is in $\boldsymbol{K}_0$, which proves the result.

We must show D_2 is closed under g. We follow the same case structure as in the construction.

In the first case of Stage 1, where $(x,y) \in X^B \setminus X^A \times Y^{D_1} \setminus Y^A$, $g^D(x,y)$ was defined to be either an element $w \in W^B_y \subset Y^B \setminus Y^A$ or y. In either event it is in D_2.

Consider now the second case of Stage 1: $(x,y) \in X^{D_1} \setminus X^A \times Y^B \setminus Y^A$. In either subcase (1) or (3) $g^{D'}(x,y) = y$ and the result is immediate. In subcase (2) suppose $g^{D'}(x,y) = g^C(x,y_0) = z'$. Then $z' \in D_1$ since x, y_0 are in the substructure D_1. Any other (x,y) with $x \notin X_{\text{new}}$ are either both in B or both in D_1 so closure under $g^{D'}$ follows since B and D_1 are in $\boldsymbol{K}_0$.

And finally we consider the elements added in stage 2. If $(x,y) \in X_{\text{new}} \times Y^{D_1}$, then $g^{D'}(x,y) = y \in D_2$. If $(x,y) \in X_{\text{new}} \times Y^B$, then $g^{D'}(x,y) \in P_y \subset g^C(X^A, y_0)$. Since $A \cup \{y_0\} \subset D_1$ and D_1 is closed under g^C, $g^C(X^A, y_0)$ is a subset of D_1.

We have shown D_2 is closed under $g^{D'}$. It remains to prove that $G^{D'}_n \restriction D_2 = G^{D_2}_n$ and $D_2 \in \boldsymbol{K}_0$.

Suppose $z < y \in Y^{D_2}$; we must show $G^{D'}(\_, y, z) \restriction D_2 = G^{D_2}(\_, y, z)$. If both y, z are in B or both are in C, we showed this in showing B, C substructures of D' (since D_1 is given to be a substructure of C).

Again we follow the case structure of the construction.

If $y \in Y^{D_1} \setminus Y^A$ and $z \in Y^B \setminus Y^A$, $y > z$, then $g^{D'}(x,y) = z$ was defined in Stage 1 of the construction for some $x \in X^B \setminus X^A$. So $\{x : x \in D' \ \&\ g^{D'}(x,y) = z\} = \{x : x \in B \ \&\ g^{D'}(x,y) = z\}$ as required.

If $y \in Y^B \setminus Y^A$, $z \in Y^{D_1} \setminus Y^A$, $y > z$ and $g^{D'}(x,y) = z$ for some x, then the value was defined either using subcase (2) in Stage 1 or using the new elements X_{new} in Stage 2. If under subcase (2), there are only finitely many $x \in X^C \setminus X^A$ such that $g^C(x, y_0) = z$. Since $z, y_0 \in D_1$ and $D_1 \in \boldsymbol{K}_0$, it follows that all these x's belong to D_1 and thus to D_2. If $z \in P_y$, the Stage 2 construction showed $\{x : x \in D' \ \&\ g^{D'}(x,y) = z\} \subset X_{\text{new}} \subset D_2$.

Overall, $\{x : g^{D_2}(x,y) = z\} = \{x : g^D(x,y) = z\}$ and D_2 is a substructure of D.

Finally we must argue that $D_2 \in \boldsymbol{K}_0$. But we constructed it so that g^D is total and each $g^{D'}_y = g^{D_2}_y$ maps onto $W^{D_2}_y$, so $D_2 \in \boldsymbol{K}_0$ and hence $D' \in \hat{\boldsymbol{K}}$. $\square$

allcard **Corollary 2.11.** *There are models of ϕ_M in all infinite cardinalities.*

Proof. Proceed by induction. The countable case has been established. Let N_0 be a structure of size κ that satisfies ϕ_M. Construct inductively a sequence of models $N_\alpha \in \hat{\boldsymbol{K}}$ for $\alpha < \kappa^+$, so that every $\alpha < \kappa^+$, $N_{\alpha+1}$ is the amalgam given by Theorem 2.10 of N_α and some $B \in \boldsymbol{K}_0$ over a finite substructure $A \in \boldsymbol{K}_0$ of N_α and take unions at limits. Organize the induction so that for every $\alpha < \kappa^+$, every finite $A \subset N_\alpha$ with $A \in \boldsymbol{K}_0$ and every finite extension $B \in \boldsymbol{K}_0$ of A, at some stage $\beta > \alpha$, B is amalgamated with N_β over A. N_{κ^+} is rich and so is model of ϕ_M. □

Theorem 2.6 and Corollary 2.11 complete the first two parts of the proof of Theorem 1.5. The models of ϕ_M of size κ produced by Corollary 2.11 are of type (κ, κ), i.e. both $|X|$, $|Y|$ have the same cardinality κ. We want to prove the analogue of Corollary 2.7 for cardinals greater than $\aleph_1$. The argument does not directly generalize because, unlike in $\aleph_0$, the ϕ_M is not κ categorical. We need more work to overcome this difficulty.

More precisely, the difficulty is that in Theorem 2.10, we need to add the elements X_{new}. Here, we construct from $C \models \phi_M$, a proper extension D with $X^D = X^C$.

filt **Definition 2.12.** *Let C be a model of ϕ_M of size κ. Then C* admits a filtering *if there exists a strictly increasing sequence $(C_\alpha)_{\alpha<\kappa}$ with $\bigcup_{\alpha<\kappa} C_\alpha = C$ that satisfies*

(1) for all α, $C_\alpha \in \hat{\boldsymbol{K}}_0$;
(2) for all successors $\alpha + 1$, $C_{\alpha+1}$ is the disjoint amalgam of C_α and some $B_\alpha \in \boldsymbol{K}_0$ over a substructure $A_\alpha \subset C_\alpha$ with $A_\alpha \in \boldsymbol{K}_0$;
(3) for all α, the pair (B_α, A_α) is a good pair (cf. Definition 2.4); and
(4) for each limit α, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$.

If in addition the sequence $(C_\alpha)_{\alpha<\kappa}$ satisfies the following property, then C admits an ample filtering*:*

(5) For each good pair (A, B) and each $\hat{A} \subset C_\alpha$ with $\hat{A} \cong A$ and each $n \in \omega$, there are κ-many stages α such that $A_\alpha = \hat{A}$ and $B_\alpha \cong B$, and $|X^{C_{\alpha+1}}| \geq |X^{C_\alpha} \cup X^{B_\alpha}| + n$.

All models of ϕ_M produced by Lemma 2.11 admit a filtering and it is not hard to modify the proof of the same lemma to produce models of ϕ_M of type (κ, κ) that admit ample filtering. In fact a stronger statement is true.

AmpleFiltering **Lemma 2.13.** *Let C be a model of ϕ_M of size κ. If C admits a filtering, then there is some model C' of ϕ_M such that $C \subset C'$ and C' admits an ample filtering.*

Proof. Let $(C_\alpha)_{\alpha<\kappa}$ be a filtering of C. If it is an ample filtering, then take $C' = C$. If not, then proceed for another κ more stages to construct an extension of $(C_\alpha)_{\alpha<\kappa}$ to $(C_\alpha)_{\alpha<\kappa\cdot 2}$ and take $C' = \bigcup_{\alpha<\kappa\cdot 2}$. In order to guarantee clause 5) in the definition of ample filtering we must modify the argument for 2.10, to increase the size of the X_{new} at each stage. Specifically, require now that for every $A \subset C_\alpha$, $A \in \boldsymbol{K}_0$, $\alpha < \kappa \cdot 2$, every finite extension $B \in \boldsymbol{K}_0$ of A and every $n \in \omega$, there are κ-many stages $\beta_i^{(\alpha)} > \alpha$, $i < \kappa$, and κ-many disjoint isomorphic copies of B, B_i, such that B_i is amalgamated with $C_{\beta_i^{(\alpha)}}$ over A at stage $\beta_i^{(\alpha)}$ and in the amalgam the size of $|X^{C_{\beta_i^{(\alpha)}+1}}|$ is larger than $|X^{C_{\beta_i^{(\alpha)}}} \cup X^{B_i}| + n$. The construction can be carried through in κ stages. It follows that C' is rich, and therefore, a model of ϕ_M. □

Def:suitext **Definition 2.14.** *Call (A', B') a* suitable extension *of (A, B), with $A \subset B, A'$ and $A', B \subset B'$, if*

(1) all four structures are in $\boldsymbol{K}_0$;
(2) $X^{A'} \subset A$ and $X^{B'} \subset B$;

(3) $|X^{B'} \setminus X^{A'}| =$[7] $|X^B \setminus X^A| \geq |Y^{B'} \setminus Y^{A'}|$.

The following fact will be applied several times.

endext **Lemma 2.15.** *If $A, A' \in \boldsymbol{K}_0$, $A \subset A'$ and $X^{A'} = X^A$, then $Y^{A'}$ is an end-extension of Y^A.*

Proof. Assume otherwise, that is there exist points $y_0 \in Y^A$ and $y_1 \in Y^{A'} \setminus Y^A$ such that $y_0 > y_1$. By Theorem 2.6, there must be some $x \in X^{A'}$ such that $g^{A'}(x, y_0) = y_1$. Since we assumed that $X^{A'} = X^A$, x must also belong to X^A. But then $g^A(x, y_0)$ is defined and belongs to Y^A, which means that it can not equal y_1. Because A is a substructure of A', $g^{A'}(x, y_0) = g^A(x, y_0)$ and we get a contradiction. Thus, under the assumption $X^{A'} = X^A$, it is necessary for $Y^{A'}$ to be an end-extension of Y^A. $\square$

suitext **Corollary 2.16.** *If (A', B') is a suitable extension of (A, B), then $Y^{A'}$ is an end-extension of Y^A and the same is true for $Y^{B'}$ and Y^B.*

We need the following variation of Theorem 2.10.

Thm:AP2 **Lemma 2.17.** *Let A, B, C be as in the assumptions of theorem 2.10 and let C' be a disjoint amalgam of the triple (A, B, C). Assume that $D \supset C$ is in $\hat{\boldsymbol{K}}_0$ with $X^D = X^C$ and let A', B' be structures in $\boldsymbol{K}_0$ so that $A' \subset B', D$ and let (A', B') be a suitable extension of (A, B). If $|X^{C'}| \geq |X^C \cup X^B| + |X^{A'}|$, then the triple (A', B', D) can be disjointly amalgamated to D', so that $D' \supset D$ and $X^{D'} = X^{C'}$.*

Proof Sketch. We have the disjoint amalgam C' of C and B over A. By Lemma 2.15 and Corollary 2.16, $Y^{A'}$, $Y^{B'}$ and Y^D are end-extensions of Y^A, Y^B and Y^C respectively.

To construct D' we proceed as in the proof of Theorem 2.10. In the first case of stage 1 we need that $|X^{B'} \setminus X^{A'}| \geq |Y^{B'} \setminus Y^{A'}|$. This is given by the assumption that (A', B') is a suitable extension of (A, B). In the second case of stage 1, we needed a surrogate element $y_0 \in Y^D \setminus Y^{A'}$ with $y_0 > Y^{B'}$. This is possible because $(Y^D, <)$ is a linear order without endpoints and $Y^{B'}$ is finite. In stage 2, we needed a set of new elements X_{new}. The size of X_{new} is bounded by the size of P_y, which in return is bounded by the size of $X^{A'}$. Instead of adding elements X_{new}, we now use elements that belong to $X^{C'} \setminus (X^C \cup X^B)$ to map $y \in Y^B - Y^A$ to P^y; for all other $x \in X^{C'}$ and any $y \in Y^B \setminus Y^A$, $g^{D'}(x, y) = y$. We assumed $|X^{C'} \setminus (X^C \cup X^B)| \geq |X^{A'}|$ so this is easy.

Since in the proof of Theorem 2.10, no new elements are added to Y, in the current construction of the amalgam D' of B' and D over A', we get that $Y^{D'}$ is an end extension of $Y^{C'}$. $\square$

The next theorem proves that under the assumption of ample filtering, a model C of ϕ_M of type (κ, κ) can be properly extended to a model D of ϕ_M, while $X^C = X^D$. This means that we extend only the Y-sort. To guarantee that D is rich, if (A, B) is a good pair and $A \subset D_\alpha$, we embed B into D over A with the image of $X^B \subset C$. By Lemma 2.15, it is necessary that Y^D is an end-extension of Y^C. Furthermore, we also prove that the resulting model D admits ample filtering, which enables us to apply the theorem inductively.

In this construction, the C of Lemma 2.17 is C_α, C' is $C_{\alpha+1}$, D is D_α, and D' is $D_{\alpha+1}$.

EndExtension **Lemma 2.18.** *Assume $\kappa > \aleph_0$ and C is a model of ϕ_M of type (κ, κ) that admits an ample filtering. Then there exists a model $D \supset C$ of ϕ_M such that $X^D = X^C$. Furthermore, D admits an ample filtering and Y^D is an end-extension of Y^C.*

[7] by (2)

Proof. Fix a sequence $\langle C_\alpha, A_\alpha, B_\alpha : \alpha < \kappa\rangle$ witnessing that C admits an ample filtering. In particular, at each stage $\alpha + 1$, $C_{\alpha+1}$ is the disjoint amalgam of C_α and B_α, over $A_\alpha \subset C_\alpha$ with $A_\alpha, B_\alpha \in \boldsymbol{K}_0$.

We now create a new sequence $\langle D_\alpha, A'_\alpha, B'_\alpha : \alpha < \kappa\rangle$ of structures in $\boldsymbol{K}_0$, so that

(1) for every α, (A'_α, B'_α) is a suitable extension of (A_α, B_α) and
(2) if $D = \cup_\alpha D_\alpha$, then $(D_\alpha)_{\alpha<\kappa}$ witnesses that D admits ample filtering.

Let $A \subset C$, $A \in \boldsymbol{K}_0$, and $B \in \boldsymbol{K}_0$ is an extension of A. Consider all possible structures A', B' such that (A', B') is a suitable extension of (A, B). By assumption 5) on the sequence (C_α), there are κ-many stages such that $A_\alpha = A$, $B_\alpha \cong B$, and $|X^{C_{\alpha+1}}| \geq |X^{C_\alpha} \cup X^{B_\alpha}| + n$, where $n = |X^{A'}|$. Use Lemma 2.17 to disjointly amalgamate D_α and B'_α over A'_α to $D_{\alpha+1} \in \boldsymbol{K}_0$. Theorem 2.17 guarantees that (a) $D_{\alpha+1} \supset C_{\alpha+1}$ and (b) $X^{D_{\alpha+1}} = X^{C_{\alpha+1}}$. At limit stages take unions.

Organize the induction so that there are κ-many stages such that $A'_\alpha = A'$ and $B'_\alpha \cong B'$. Let $D = \bigcup_{\alpha<\kappa} D_\alpha$. It follows by the definitions that $D \supset C$, $X^D = X^C$, and that the sequence $(D_\alpha)_{\alpha<\kappa}$ is an ample filtering of D. We have $D \in \hat{\boldsymbol{K}}_0$. To complete the proof, we need the following claim.

rich **Claim 2.19.** *Let $A', B' \in \boldsymbol{K}_0$ and $A' = D \cap B'$. Then there is an embedding of B' into D over A'. Thus, D is also rich, which proves that D is a model of $\phi_{\mathcal{M}}$.*

Proof. First assume that $|X^{B'} \setminus X^{A'}| \geq |Y^{B'} \setminus Y^{A'}|$. If this is not the case, extend B using Lemma 2.3. Let $A = A' \cap C$ and let $B = A \cup X^{B'}$. Since $X^D = X^C$, the pair (A', B') is a suitable extension of (A, B).

Using Lemma 2.18, find some $\alpha < \kappa$ such that $A' = A'_\alpha$, B is isomorphic to B_α, and B' is isomorphic to B'_α. It follows, again by Lemma 2.18, that $A = A' \cap C = A'_\alpha \cap C = A_\alpha$ and $D_{\alpha+1}$ is the disjoint amalgam of D_α and B'_α over A'. The embedding that sends B' into $D_{\alpha+1}$ over A' proves the claim. □

The proof of the claim finishes the proof. □

Note that while $X^{D_{\alpha+1}}$ may well extend X^{D_α}, $X^{D_\alpha} \subset X^C$.

To complete the proof of Theorem 1.5 we must show for each κ, there exist a model of ϕ_M of type (κ^+, κ).

Proof. If $\kappa = \aleph_0$, the result holds true by Corollary 2.7. Assume κ is uncountable and let C_0 be a model of type (κ, κ) given by Corollary 2.11. By Lemma 2.13, assume that C_0 admits ample filtering.

Construct a sequence $(C_\alpha)_{\alpha<\kappa^+}$ such that each C_α is a model of ϕ_M that admits ample filtering, $X^{C_\alpha} = X^{C_0}$ and $Y^{C_{\alpha+1}}$ is an end-extension of Y^{C_α}. Use Theorem 2.18 with C as C_α and D as $C_{\alpha+1}$ for the successor stages. At limit stages take unions. The construction continues past the limit stages, because the union of models of the same cardinality that admit ample filtering, also admits ample filtering. □

This completes the proof of Theorem 1.5 and thus Corollary 1.7.

Lastly, notice that if $\mathcal{N}' \models \phi_{M'}$, $|X^{\mathcal{N}'}| = \kappa$ and $|Y^{\mathcal{N}'}| = \kappa^+$, the size of the predicate $U(\mathcal{N}')$ can not be of size κ^+. This makes it impossible to use this technique to homogeneously characterize κ^+.

sec:ltoomega

3. Maximal models in κ and κ^ω

Working similarly to Section 1 we construct a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that admits maximal models in κ and κ^ω, and has no larger models. But we must define a sentence that transfers from κ to κ^ω rather than κ^+.

ltoomega

Theorem 3.1. *Let ϕ be a complete $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence (in vocabulary τ) with a set of absolute indiscernibles U. Then there is a complete $\mathcal{L}_{\omega_1,\omega}(\tau')$-sentence ϕ^* (in vocabulary $\tau' \supset \tau$) with the property:*

If ϕ homogeneously characterizes κ, then ϕ^* homogeneously characterizes κ^ω.

Proof sketch: Here is the basic idea of the construction.

Fix an infinite set X and consider the structure N with universe the disjoint union of ω, $X^{<\omega}$ and a subset of X^ω. Fix a vocabulary τ_1 with unary predicates K, V, F denoting these sets, binary predicates H, R and a ternary predicate E. Interpret the 'height' predicate $H(\cdot,\cdot)$ on $K \times V$ so that $H(n,v)$ holds if and only if $v \in X^n$, and the predicate $R(u,v)$ on V^2 which holds if v is an immediate successor of u, and the restriction function E on $K \times F \times V$ such that $E(k,f,v)$ holds if and only if $f \restriction k = v$. Of course, by a sentence in $\mathcal{L}_{\omega_1,\omega}(\tau_1)$ we can require that K is the standard ω and each of V and F are sets of functions.

If X is countable and F is also countable, the resulting structure has a Scott sentence μ. Furthermore, if for every $v \in X^n$ there exist infinitely many $f \in F$ with $f \restriction n = v$, by Theorem 3.4 of [Sou14] this structure is back and forth equivalent with the model where X is countable and F is the set of eventually constant sequences.

Now using the τ-sentence ϕ we are able (in an expanded model N_*) to bound $|X| = |K^{N_*}|$ by κ and $|V^{N_*}|$ by κ^ω. Form τ' by adding a binary symbol $M(\cdot,\cdot)$ to $\tau_1 \cup \tau_2$ and predicates $S(x,\cdot)$ for each τ-relation $S(\cdot)$.

Assert that the sets $M(u,\cdot)$ for $u \in V$ are disjoint and require that for each $u \in V$, the set $M(u,\cdot)$ (under the relations $S(u,\cdot)$) is a model of ϕ. Require further that the set $R(u,\cdot)$ of the immediate successors of u is also the set $U(u,\cdot)$ of absolute indiscernibles of the model $M(u,\cdot)$ of ϕ. Since ϕ homogeneously characterizes κ, if $N_* \models \phi^*$, $R^{N_*}(u,\cdot)$ cannot be larger than κ so the number of immediate successors of u can not be more than κ. The resulting tree has height ω and is $\leq \kappa$-splitting. To prevent the first level $V(0,\cdot)$ from growing arbitrarily we require that $V(0,\cdot)$ has only one element (the root).

The detailed axiomatization of this structure by a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ and the proof that it characterizes κ^ω appear in [Sou14].

kkomega

Theorem 3.2. *Assume $\lambda \leq \lambda^\omega < \kappa < \kappa^\omega$ and ϕ_κ homogeneously characterizes κ. Then there is a complete sentence ϕ^*_κ that has maximal models in κ and κ^ω, and no models larger than κ^ω.*

Proof. By Theorem 1.6, we can assume ϕ_κ has maximal models of type (κ,λ) for all $\lambda \leq \kappa$. Let ϕ^*_κ be the sentence from Theorem 3.1. If for every $u \in V$, the set $M(u,\cdot)$ is a maximal model of ϕ_κ of type (κ,λ), then the resulting tree is λ-splitting and the associate model is a maximal model of ϕ^*_κ of size $\max\{\kappa,\lambda^\omega\} = \kappa$.

Further, if for every $u \in V$, the set $M(u,\cdot)$ is a maximal model of ϕ_κ of type (κ,κ), then the resulting tree is κ-splitting and yields a maximal model of ϕ^*_κ of size $\max\{\kappa,\kappa^\omega\} = \kappa^\omega$.

□

Replacing the construction that characterized κ^ω from [Sou14] with the construction that characterized $\kappa^{\aleph_\alpha}$, $\alpha < \omega_1$, from [Sou12] (cf. Theorem 3) one can prove the following theorem.

kkalpha **Theorem 3.3.** *Assume $\alpha < \omega_1$, $\lambda \leq \lambda^{\aleph_\alpha} < \kappa < \kappa^{\aleph_\alpha}$ and there is a sentence ϕ_κ that homogeneously characterizes κ. Then there is a complete sentence ϕ^*_κ that has maximal models in κ and $\kappa^{\aleph_\alpha}$, and no models larger than $\kappa^{\aleph_\alpha}$.*

The next theorem describes where JEP holds/fails in the examples of this section. Note that the notion of strong embedding $\prec_{\boldsymbol{K}}$ specified below, maybe different than elementary substructure in the fragment generated by ϕ^*_κ.

Let ϕ_κ be a sentence that homogeneously characterizes κ with P a set of absolute indiscernibles and let ϕ^*_κ be as in Theorem 3.1. Let $\boldsymbol{K}$ be the collection of models that satisfy ϕ^*_κ and let $\mathcal{N}_0 \prec_{\boldsymbol{K}} \mathcal{N}_1$ if $\mathcal{N}_0 \subset \mathcal{N}_1$ and for each $u \in V$, $M(u,\cdot)^{\mathcal{N}_0} \cup R(u,\cdot)^{\mathcal{N}_0} \prec M(u,\cdot)^{\mathcal{N}_1} \cup R(u,\cdot)^{\mathcal{N}_1}$, where $\prec$ is understood as elementary substructure in the fragment generated by ϕ_κ.

JEP **Theorem 3.4.** *If the models of ϕ_κ satisfy JEP($< \kappa$), then the same is true for $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$.*

Proof. Let $\mathcal{N}_0, \mathcal{N}_1 \in \boldsymbol{K}$ and $|\mathcal{N}_0|, |\mathcal{N}_1| < \kappa$. It follows that the tree contained in either $\mathcal{N}_0$ or $\mathcal{N}_1$ must satisfy $|V(0,\cdot)| = 1$ and for each $u \in V(n,\cdot)$, $|M(u,\cdot) \cup R(u,\cdot)| < \kappa$. The goal is to embed $\mathcal{N}_0, \mathcal{N}_1$ into a common $\mathcal{N} \in \boldsymbol{K}$.

First embed the root a_0 of the tree in $\mathcal{N}_0$ and the root a_1 of the tree in $\mathcal{N}_1$ into the root of the tree in $\mathcal{N}$, call it a. Since the models of ϕ_κ satisfy JEP($< \kappa$), joint embed $M(a_0,\cdot)^{\mathcal{N}_0} \cup R(a_0,\cdot)^{\mathcal{N}_0}$ and $M(a_1,\cdot)^{\mathcal{N}_1} \cup R(a_1,\cdot)^{\mathcal{N}_1}$ to a common model $M(a,\cdot)^{\mathcal{N}} \cup R(a,\cdot)^{\mathcal{N}}$, say through embeddings f_0, f_1. If $v \in R(a,\cdot)^{\mathcal{N}}$ and $v \notin range(f_0) \cup range(f_1)$, then attach a copy of $(\aleph_0)^\omega$ with root v into $\mathcal{N}$. If $v \in range(f_0) \setminus range(f_1)$, then embed $M(f_0^{-1}(v),\cdot)^{\mathcal{N}_0} \cup R(f_0^{-1}(v),\cdot)^{\mathcal{N}_0}$ into $M(v,\cdot)^{\mathcal{N}} \cup R(v,\cdot)^{\mathcal{N}}$. Similarly work if $v \in range(f_1) \setminus range(f_0)$. If $v \in range(f_0) \cap range(f_1)$, then joint embed the models $M(f_0^{-1}(v),\cdot)^{\mathcal{N}_0} \cup R(f_0^{-1}(v),\cdot)^{\mathcal{N}_0}$ and $M(f_1^{-1}(v),\cdot)^{\mathcal{N}_1} \cup R(f_1^{-1}(v),\cdot)^{\mathcal{N}_1}$ into $M(v,\cdot)^{\mathcal{N}} \cup R(v,\cdot)^{\mathcal{N}}$. Then repeat by induction the same process for each $u \in V(n,\cdot)$, and finally embed each $f \in F^{\mathcal{N}_0}$ and $f \in F^{\mathcal{N}_1}$ into the corresponding branch in $F^{\mathcal{N}}$. The reader can verify that $\mathcal{N}_0, \mathcal{N}_1 \prec_{\boldsymbol{K}} \mathcal{N}$. $\square$

Notice that disjoint-JEP fails for $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$. One reason is that the root of the tree in $\mathcal{N}_0$ and the root of the tree in $\mathcal{N}_1$ must embed into the same element, namely the root of the tree in $\mathcal{N}$.

ManyCardinalities 4. Consistency of Maximal models in countably many cardinalities

In this section we construct a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that consistently admits maximal models in countably many cardinalities.

We adapt ideas from [Hjo02] and [Sou13] to characterize pairs of cardinals $\kappa, 2^\kappa$. Combining this result with our basic technique we are able to construct sentences with maximal models in cardinalities 2^λ, for all $\lambda \leq \kappa$, where κ is a homogeneously characterizable cardinal; this is interesting when $\kappa < 2^\lambda$. The following version of Theorem 4.29 of [Sou13] establishes the necessary transfer result.

Thm:CharPowerset **Theorem 4.1.** *There is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi(X,Y)$ such that if $M \models \phi(X,Y)$ and $|X| = \kappa$ then $|Y| \leq 2^\kappa$; the maximum is attained.*

Proof Sketch: Fix a set X and a binary relation on X which defines a dense linear order. As in Section 4 of [Sou13], the key idea is the following condition crystallizes the property of the meet function on 2^X. Construct (via a generalized Fraïssé construction) a map f from Y^2 to X such that for distinct[8] $a_0, a_1, a_2 \in Y$, if $f(a_0,a_1) \neq f(a_0,a_2)$ then $f(a_1,a_2) = \min\{f(a_0,a_1), f(a_0,a_2)\}$. But if $f(a_0,a_1) = f(a_0,a_2)$ then $f(a_1,a_2) > f(a_0,a_1) = f(a_0,a_2)$.

[8]For further restrictions see Definition 4.6 of [Sou13].

The resulting structure has a Scott sentence ϕ' in $\mathcal{L}_{\omega_1,\omega}$ such that if $\mathcal{M} \models \phi'$, $|Y^{\mathcal{M}}| \leq 2^{|X^{\mathcal{M}}|}$ and $2^{|X^{\mathcal{M}}|}$ is possible. Namely, the maximum is achieved if $(X, <)$ has a $<$- cofinal sequence of length κ.

Cor:CharPowerset2

Theorem 4.2. *Suppose that ψ is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that homogeneously characterizes κ with absolute indiscernibles in the predicate P. Then there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ψ' that characterizes 2^κ.*

Furthermore for every $\lambda \leq \kappa$, there exists a maximal model of ψ' of size $\max\{\kappa, 2^\lambda\}$.

Proof. By Theorem 1.6, we can assume ψ has maximal models of type (κ, λ), for all $\lambda \leq \kappa$. As in Corollary ??, merge ψ with the complete sentence ϕ from Theorem 4.1 identifying X with P. Let $\psi' = \chi_{\psi,P,\phi,X}$. By Fact 1.3 (1), ψ' is a complete sentence.

Now suppose M is a maximal model of ψ of type (κ, λ). Linearly order P^M so that $(P^M, <)$ is a dense linear order with a λ-cofinal sequence. Then by Theorem 4.1 we get a maximal model of ψ' with cardinality $\max\{\kappa, 2^\lambda\}$. □

Exactly what this says about the cardinality of maximal models depends on the cardinal arithmetic. We just give some sample applications of Theorem 4.2 with various choices of the λ.

Thm:MaximalModels

Theorem 4.3. *Assume κ is a homogeneously characterizable cardinal and let μ be the least cardinal such that $2^\mu \geq \kappa$. Then there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_κ with maximal models in cardinalities 2^λ, for all $\mu \leq \lambda \leq \kappa$.*

Thm:MaximalModels2

Theorem 4.4. *Assume that $2^{\aleph_0} > \aleph_\omega$. For each $n \in \omega$, there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ'_n with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\aleph_n}$.*

Theorem 4.5. *Let $(\kappa_i | i \leq n)$ be an increasing sequence of cardinals such that $cf(\kappa_i) > \aleph_i$. It is consistent that there is a complete sentence with maximal models in cardinalities $\kappa_0, \kappa_1, \ldots, \kappa_n$.*

Proof. By Easton's Theorem, there is model of ZFC where $2^{\aleph_i} = \kappa_i$. Then apply Theorem 4.4. □

Sec:Conclusion

5. Conclusion

The examples from [Sou12] and [Sou14] have the maximal number of models in the cardinals they characterize, namely κ^ω and $\kappa^{\aleph_\alpha}$ respectively. As a consequence, the sentence ϕ^*_κ in Theorem 3.2 has the maximal number of maximal models in κ and κ^ω, and the sentence ϕ^*_κ in Theorem 3.3 has the maximal number of maximal models in κ and $\kappa^{\aleph_\alpha}$.

This motivates the following question.

OpenQuestion1

Open Question 5.1. *Is there a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ which has at least one maximal model in an uncountable cardinal κ, but less than 2^κ many models of cardinality κ?*

In particular, a negative answer to Open Question 5.1 implies a negative answer to the following Open Question 5.2, which was asked in [BKL14] and which in return relates to old conjectures of S. Shelah.

OpenQuestion2

Open Question 5.2 ([BKL14])**.** *Is there a complete $\mathcal{L}_{\omega_1,\omega}$-sentence which characterizes an uncountable cardinal κ and it has less than 2^κ many models in cardinality κ?*

Finally, we want to stress the differences in techniques of this paper from [BKS14]. The main idea behind [BKS14] is certain combinatorial properties of bipartite graphs. Here the main construction is a refinement of the construction from [Kni77] combined with repeated use of sets of absolute indiscernibles. All the examples presented here are *complete* sentences with maximal models in more than one cardinality, which *do not* have arbitrarily large models. In [BKS14] the examples are *incomplete* sentences with maximal models in more than one cardinality, which *do* have arbitrary large models. The following question arises naturally: The JEP- and AP-spectra of the sentences presented in [BKS14] there are precisely calculated. The JEP- and AP-spectra of our examples seem harder to calculate and the question remains open.

Notice: After this paper was submitted, Baldwin and Shelah began the paper 'The Hanf number for extendability and related phenomena'. They construct (under mild set theoretic hypotheses which are expected to be eliminated) a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ with maximal models arbitrarily high below the first measurable. Note that every model above the first measurable has a proper $\mathcal{L}_{\omega_1,\omega}$-elementary extension. In contrast to this result the method discussed in the last paragraph seem to be limited to counterexamples below $\beth_{\omega_1}$. Can one find a sentence ϕ with maximal models bounded somewhere between these bounds? If not, can one explain why there is a such an immense gap? We would have under ZFC + "there exists a measurable cardinal", no compete sentence of $\mathcal{L}_{\omega_1,\omega}$ has arbitrarily large maximal models. But under ZFC + "there are no measurable cardinals", if there is a maximal model of cardinality at least $\beth_{\omega_1}$, then there are arbitrarily large maximal models. But under ZFC + 'no measurable cardinals', our only example with a maximal model of cardinality beyond $\beth_{\omega_1}$ has arbitrarily large maximal models. Does this make the Hanf number for the existence of a maximal model (with no measurable) $\beth_{\omega_1}$ or can more counterexamples be constructed?

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