Russell's Paradox

John T. Baldwin and Olivier Lessmann
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60680

August 4, 1998

Russell's paradox is based on examples like this: 'Consider the collection of barbers who shave exactly those men who do not shave themselves.' Suppose there is a barber in this collection who does not shave himself; then by the definition of the collection, he must shave himself. On the other hand, no barber in the collection can shave himself.

Bertrand Russell's discovery of this paradox in 1903 dealt a blow to one of his fellow mathematicians. In the late 1800s Gottlob Frege tried to develop a foundation for all of mathematics using symbolic logic. He established a correspondence between formal expressions (such as '$x = 2$') and mathematical properties (such as 'even numbers'). In Frege's development, one could freely use any property to define further properties.

Russell's paradox demonstrated a fundamental limitation of such a system. In modern terms, it is best described in terms of sets, using so-called 'set-builder' notation. For example, we can describe the collection of numbers 4, 5 and 6 by saying that x is the collection of integers n which are greater than 3 and less than 7; we write this formally as $x = \{n : n \text{ is an integer and } 3 < n < 7\}$. The objects don't have to be numbers. We might let $y = \{x : x \text{ is a male resident of the United States }\}$. Seemingly, any description of x could fill the space after the colon. But Russell (and independently, Ernst Zermelo) noticed that $x = \{a : a \text{ is not in } a\}$ leads to a contradiction in the same way as the description of the barber. Is x itself in the set x? Either answer leads to a contradiction.

When Russell discovered this paradox, Frege immediately saw that it had a devastating effect for his system. He was unable to resolve the paradox and there have been many further attempts in the last century to avoid it.

Russell's own answer was to elaborate a 'theory of types.' The problem in the paradox, he reasoned, is that we are confusing a description of sets of numbers with a description of sets of sets of numbers. So Russell introduced a hierarchy of objects: numbers, sets of numbers, sets of sets of numbers, etc. This system served as vehicle for the first formalizations of the foundations of mathematics and is still used in some philosophical investigations and in branches of computer science.

Zermelo's solution to Russell's paradox is to replace the axiom: 'for every formula $A(x)$ there is a set $y = \{x : A(x)\}$' by the axiom, 'for every formula $A(x)$ and every set b there is a set $y = \{x : x \in b \text{ and } A(x)\}$.'

What became of the effort to develop a logical foundation for all of mathematics? Mathematicians now recognize that the field can be formalized in the so-called Zermelo-Fraenkel set theory. The formal language contains symbols $\in$ for in, $=$ for equality and $\emptyset$ to denote the set with no elements. So one can write formulas such as $B(x)$: if y is in x then y is empty. In set-builder notation we could write this as $y = \{x : x = \emptyset\}$ or more simply as $y = \{\emptyset\}$. Russell's paradox becomes: let $y = \{x : x \notin x\}$, is $y \in y$?

Russell's and Frege's correspondence on Russell's discovery of the paradox can be found in "From Frege to Godel, a Source Book in Mathematical Logic, 1879-1931," edited by Jan van Heijenoort, Harvard University Press, 1967.