A Hanf number for saturation and omission

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Abstract

Suppose $\boldsymbol{T} = (T, T_1, p)$ is a triple of two countable theories in vocabularies $\tau \subset \tau_1$ and a τ_1-type p over the empty set. We show the Hanf number for the property: There is a model M_1 of T_1 which omits p, but $M_1 \upharpoonright \tau$ is saturated is essentially equal to the Löwenheim number of second order logic.

Newelski [3] asked to calculate the Hanf number of the following property P_N. In accordance with the original question, we focus on countable vocabularies for the first three sections. We deal with extensions to larger vocabularies in Section 4.

Definition 0.1 *We say* $M_1 \models \boldsymbol{T}$ *where* $\boldsymbol{T} = (T, T_1, p)$ *is a triple of two theories in vocabularies* $\tau \subset \tau_1$*, respectively,* $T \subseteq T_1$ *and* p *is a* τ_1*-type over the empty set if* M_1 *is a model of* T_1 *which omits* p*, but* $M_1 \upharpoonright \tau$ *is saturated. Let* $\boldsymbol{K}_{\boldsymbol{T}}$ *denote the class of models* M_1 *which satisfy* $\boldsymbol{T}$*.*

For $\boldsymbol{K} = \boldsymbol{K}_{\boldsymbol{T}}$ *for some* $\boldsymbol{T}$ *in a countable vocabulary let* $P^c_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ *hold if* $|\tau_1| \leq \aleph_0$ *and for some* M_1 *with* $|M_1| = \lambda$*,* $M_1 \models \boldsymbol{T}$*.* $P^f_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ *is the same property restricted to triples where* T_1 *and* T *are finitely axiomatizable in finite vocabularies and* p *is definable in second order logic.*

Recall Hanf's observation [1] that for any such property $P(\boldsymbol{K}, \lambda)$, where $\boldsymbol{K}$ is ranges over a *set* of classes of models, there is a cardinal $\kappa = H(P)$ such that κ is the least cardinal satisfying: if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. $H(P)$ is called the Hanf number of P. E.g. $P(\boldsymbol{K}, \lambda)$ might be the property that $\boldsymbol{K}$ has a model of power λ. Similarly the Löwenheim number $\ell(P)$ of a set P of classes is the least cardinal μ such that any class $\boldsymbol{K} \in P$ that has a model has one of cardinality $\leq \mu$.

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Theorem 0.2 *Assume the collection of λ with $\lambda^{<\lambda} = \lambda$ is a proper class.* $H(P^f_N) = \ell(L^{II})$ *where* L^{II} *denotes the collection of sentences of second order logic.*

Since $H(P^c_N) \geq H(P^f_N)$, this shows that that the Hanf number in the abstract is at least $\ell(L^{II})$, as asserted. In Section 1 we introduce a variant $\ell^2(L^{II})$ on the Löwenheim number of second order logic which is 'essentially equal' to $\ell(L^{II})$ (i.e. equal modulo a mild set theoretic hypothesis: Assumption 0.3). It is fairly easy to show (Claim 2.9) $\ell^2(L^{II}) \geq H(P^f_N)$ giving the 'essentially equal' of the abstract. We will replace this 'essential equality' with an exact computation and deal with uncountable languages in Section 4.

Jouko Vaananen provided the following summary of the effect of this result by indicating the size of $\ell(L^{II})$. $\ell(L^{II})$ is bigger than the first (second, third, etc) fixed point of any normal function on cardinals that itself can be described in second order logic. For example it is bigger than the first κ such that $\kappa = \beth_\kappa$, bigger than the first κ such that there are κ cardinals λ below κ such that $\lambda = \beth_\lambda$, etc. It is easy to see that if there are measurable (inaccessible, Mahlo, weakly compact, Ramsey, huge) cardinals, then the Lowenheim number of second order logic exceeds the first of them (respectively, the first inaccessible, Mahlo, weakly compact, Ramsey, huge) (and second, third, etc). So even under $V = L$, the Löwenheim number is bigger than any 'large' cardinal that is second order definable and consistent with $V = L$. Such results are discussed in Vaananen's paper "Hanf numbers of unbounded logics"[4]. A result of Magidor [2] shows the Lowenheim number of second order logic is always below the first supercompact. Vaananen's paper "Abstract logic and set theory II: Large cardinals" gives lower bounds for the Lowenheim number of equicardinality quantifiers and thus *a fortiori* for second order logic [5]. In simple terms, if $E(\kappa)$ is the statement that $2^\kappa \geq \kappa^{++}$ then the first κ cardinals (if any) such that $E(\kappa)$ holds is less than the Lowenheim number of second order logic. This shows that by forcing we can push the Lowenheim number up at will.

We make the following assumption throughout.

Assumption 0.3 *Assume the collection of λ with $\lambda^{<\lambda} = \lambda$ is a proper class.*

This assumption follows from GCH, but if GCH fails badly the only such cardinals are strongly inaccessible. The key point for our use of the condition is that $\lambda^{<\lambda} = \lambda > |\tau_T| + \aleph_0$ is a sufficient condition for the existence of a saturated model in λ of a complete theory T; if T is unstable $\lambda^{<\lambda} = \lambda$ is also necessary. We will explore this issue for stable theories, in the absence of this condition, elsewhere. In Section 1 we review some properties of second order logic and show the equality of two 'Löwenheim numbers' in our context. In Section 2, we state two technical results, prove one, and deduce Theorem 0.2 from them. In Section 3, we prove the more difficult technical result. In Section 4, we code syntax more carefully and obtain a uniform equivalence for vocabularies of all cardinalities.

Newelski's question arose in the study of the model theory of groups and the existence of groups with bounded orbits. The authors acknowledge very fruitful discussions with Jouko Väänänen and Tapani Hyttinen concerning the material of this paper.

1 Some Second Order Logic

By (pure) second order logic, L^{II}, we mean the logic with individual variables and variables for relations of all arities but no non-logical constants. The atomic formulas are equalities between variables and expressions $X(\mathbf{x})$ where X is an n-ary relation variable and $\mathbf{x}$ is an n-tuple of individual variables. Note that a structure A for this logic is simply a set so is determined entirely by its cardinality. But we use the full semantics; the n-ary relation variables range over all n-ary relations on A.

We put our restriction to $\lambda = \lambda^{<\lambda}$ in a more general setting. In general for any class $\boldsymbol{K}$ of models write $\operatorname{spec}(\boldsymbol{K})$ for the collection of λ such that there is a model in $\boldsymbol{K}$ with cardinality λ. We describe some technical variants for the second order case that are relevant here.

Definition 1.1 *Let ψ be a sentence of second order logic.*

1. $\operatorname{spec}^1(\psi) = \{\lambda : \lambda \models \psi\}$.
2. $\operatorname{spec}^2(\psi) = \{\lambda : \lambda = \lambda^{<\lambda} \wedge \lambda \models \psi\}$

Note that there is a sentence χ in second order logic which has a model of size λ if and only if $\lambda^{<\lambda} = \lambda$. Namely, let χ assert there is an extensional relation R on sets such that each element denotes, via R, a set of smaller cardinality than the universe and each such set is coded by R. We will generally write $\lambda^{<\lambda} = \lambda$ to denote this sentence.

Definition 1.2 *Define H^2 and ℓ^2 to be the Hanf and Lowenheim numbers with respect to* spec^2.

We'll write ℓ^1 for ℓ and H^1 for H where it is convenient for comparison. Note the following easy transformations in second order logic.

Fact 1.3 *Fix $\phi \in L^{II}$.*

1. $\operatorname{spec}^1(\phi)$*:*
   - *(a) There is a $\phi_1 \in L^{II}$ with* $\min(\operatorname{spec}(\phi)) < \min(\operatorname{spec}(\phi_1))$.
   - *(b) If* $\operatorname{spec}(\phi)$ *is bounded and nonempty there is a $\phi_2 \in L^{II}$ with* $\operatorname{spec}(\phi_2)$ *bounded and nonempty and* $\sup(\operatorname{spec}(\phi)) < \sup(\operatorname{spec}(\phi_2))$.
2. $\operatorname{spec}^2(\phi)$*:*
   - *(a) There is a $\phi_3 \in L^{II}$ with* $\min(\operatorname{spec}(\phi)) < \min(\operatorname{spec}(\phi_3))$ *and if $\lambda \models \phi_3$, $\lambda^{<\lambda} = \lambda$.*
   - *(b) If* $\operatorname{spec}^2(\phi)$ *is bounded and nonempty there is a $\phi_4 \in L^{II}$ with* $\operatorname{spec}^2(\phi_4)$ *bounded and nonempty and* $\sup(\operatorname{spec}^2(\phi_4)) < \sup(\operatorname{spec}^2(\phi_4))$.

These transformations imply:

Fact 1.4 *1. $H^1(L^{II}), H^2(L^{II}), \ell^1(L^{II}), \ell^1(L^{II})$ are strong limit cardinals.*

2. *There is no sentence attaining any of these values exactly. (E.g., there is no* $\phi \in L^{II}$ *with* $\sup(\mathrm{spec}(\phi)) = H^1(L^{II})$*.)*
3. *For either spectrum,* $\ell^i(L^{II}) = \sup\{\min\{\mathrm{spec}^i(\phi)\} : \phi \in L^{II}$ *has a model*$\}$ *and similarly* $H^i(L^{II}) = \sup\{\sup\{\mathrm{spec}^i(\phi)\} : \phi \in L^{II}$ *is bounded*$\}$.

Note that any logic satisfying Fact 1.3 will also satisfy Fact 1.4. We use this observation without comment in studying infinitary second order logics in Section 4.

Using Assumption 0.3 we can show:

Lemma 1.5 $H(L^{II}) = H^2(L^{II})$, $\ell(L^{II}) = \ell^2(L^{II})$

Proof. One direction is easy. For every sentence ψ of second order logic, there is a sentence ψ^* such that:

$$\mathrm{spec}^2(\psi) = \mathrm{spec}^1(\psi^*).$$

ψ^* just expresses the conjunction of ψ with $\lambda^{<\lambda} = \lambda$. Recall Fact 1.4.3 Since every 2-spectrum is a 1-spectrum $\ell^2(L^{II}) \leq \ell^1(L^{II})$ and $H^2(L^{II}) \leq H^1(L^{II})$.

But the opposite inequality also holds. Let ϕ be a sentence with a non-empty 2-spectrum. Let $f(\lambda)$ denote the least $\mu > \lambda$ with $\mu^{<\mu} = \mu$. It is easy to construct for each second order sentence ϕ a sentence ϕ^* such that

$$\mathrm{spec}(\phi^*) = \mathrm{spec}^2(\phi^*) = \{f(\lambda) : \lambda \in \mathrm{spec}(\phi)\}.$$

Clearly the map $\phi \mapsto \phi^*$ shows $\ell^2(L^{II}) \geq \ell^1(L^{II})$ and $H^2(L^{II}) \geq H^1(L^{II})$.

$\square_{1.5}$

2 The main result

Recall our notation from Definition 0.1.

Notation 2.1 *We will write* $\boldsymbol{T}$ *(possibly with subscripts) for a triple* (T, T_1, p)*. The expression '*$\boldsymbol{T}$ *has a model in* λ*' means there is a model of* T_1 *with cardinality* λ *that omits* p *and whose reduct to* $L(T) = \tau$ *is saturated.*

Convention 2.2 *When* τ_1 *is finite we consider it to be a subset of* ω*. Thus the set of first order* τ_1*-sentences is recursive and we can code them as natural numbers.*

We concentrate first on $P^f_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ from Definition 0.1. We need some additional coding to handle non-finitely axiomatizable theories and consider this generalization in Section 4. We prove Theorem 2.5 in Section 3. We begin by clarifying a notion from Definition 0.1.

Definition 2.3 *A type* p *in a vocabulary satisfying Convention 2.2 is* definable in second order logic *if we can code the type as a subset* A_p *of* ω *so that in the vocabulary with constant symbol* 0 *and relation symbol* S *there is a second order sentence* ψ *and second order formula* $\varphi(x)$ *such that for the first cardinal* λ *which satisfies* ψ *letting* M *be* $(\lambda, 0, S)$*, with* 0 *is interpreted as* 0*,* S *as successor on the natural numbers then* $A_p = \{n : M \models \varphi(n)\}$.

Now for convenience we restrict our triples to those satisfying the convention. Formally:

Notation 2.4 *$\mathcal{T}^f$ denotes the set of triples $\boldsymbol{T}$ as in Definition 2.1 such that τ_1 satisfies Convention 2.2, T is finitely axiomatizable and p is second order definable.*

Theorem 2.5 *For every second order sentence ϕ, there is a triple $\boldsymbol{T}_\phi \in \mathcal{T}^f$ such that if $\lambda^{<\lambda} = \lambda$, then the following are equivalent:*

1. *$\boldsymbol{T}_\phi$ has a model in λ.*
2. *ϕ has a model in every cardinal strictly less than λ.*

Lemma 2.6 *For every $\boldsymbol{T} \in \mathcal{T}^f$ there is a second order $\phi_{\boldsymbol{T}}$, such that $\phi_{\boldsymbol{T}}$ has a model in λ if and only if $\boldsymbol{T}$ has a model in λ.*

Proof. Recalling the restrictions involved in $\mathcal{T}^f$, it is easy to write a second order sentence θ such that $M \models \theta$ if and only if $M \models T_1$, M omits p and $M\restriction\tau$ is saturated. $\Box_{2.6}$

We could strengthen Lemma 2.6 by restricting the second order quantification to sets of size strictly less than the size of the model, but that is not important here. We now deduce Theorem 0.2 from Theorem 2.5 and Lemma 2.6. We use the following notation.

Notation 2.7 $\mathrm{Spec}(\boldsymbol{T})$ *is the collection of cardinals λ such that there is an M_1 satisfying $\boldsymbol{T}$ with $|M_1| = \lambda$.*

We have not established that the Hanf and Lowenheim numbers for the P_N satisfy Fact 1.4.2 or Fact 1.4.2. This complicates the argument for the following two results.

Claim 2.8 *$H(P^f_N) \leq \ell^2(L^{II})$ where L^{II} denotes second order logic.*

Proof. Lemma 2.6 shows that for any $\boldsymbol{T} \in \mathcal{T}^f$, there is a $\phi_{\boldsymbol{T}} \in L^{II}$ with $\mathrm{spec}(\boldsymbol{T}) = \mathrm{spec}(\phi_{\boldsymbol{T}})$. Suppose for contradiction that $H(P^f_N) > \ell^2(L^{II})$. Then there is a triple $\mathbf{T} \in \mathcal{T}^f$ such that $\sup(\mathrm{spec}(\mathbf{T})) \geq \ell^2(L^{II})$.

Let $\mathbf{C} = \{\mu : \mu = \mu^{<\mu}\}$. Choose $\psi_{\mathbf{T}} \in L^{II}$ so that $\mu \models \psi_{\mathbf{T}}$ iff for every infinite cardinal $\kappa \in \mathbf{C} \cap \mu$ there is $\theta \in [\kappa, \mu) \cap \mathrm{spec}(\mathbf{T})$.

Let $\lambda_{\mathbf{T}}$ be the minimal element of $\mathbf{C} \cap \mathrm{spec}(\mathbf{T})^c$. Then $\lambda_{\mathbf{T}} \geq \ell^2(L^{II})$. For any $\boldsymbol{T}$, the definition of P^f_N guarantees that if $\lambda^{<\lambda} = \lambda, \mu > \lambda$ and some $M \models \boldsymbol{T}$ has cardinality μ then some $N \models \boldsymbol{T}$ has cardinality λ (take a saturated elementary submodel). Thus, $\mu \models \psi_{\mathbf{T}}$ if and only if every $\kappa \in \mathbf{C} \cap \mu$ belongs to $\mathrm{spec}(\mathbf{T})$.

Now $\mathrm{spec}(\psi_{\mathbf{T}})$ is exactly $\{\mu : \mu \leq \lambda_{\mathbf{T}}\}$, whence $\mathrm{spec}(\neg\psi_{\mathbf{T}})$ is $\{\mu : \mu > \lambda_{\mathbf{T}}\}$. So the Löwenheim number of $\neg\psi_{\mathbf{T}}$ is $(\lambda_{\mathbf{T}})^+ > \ell^2(L^{II})$ and this contradiction completes the proof.

$\Box_{2.8}$y

Lemma 2.9 *$H(P^f_N) \geq \ell^2(L^{II})$ where L^{II} denotes second order logic.*

Proof. Suppose for contradiction that there is a second order sentence ψ such that $\lambda_0 = \min(\mathrm{spec}^2(\psi)) \geq H(P^f_N)$. By the definition of spec^2, $\lambda_0^{<\lambda_0} = \lambda_0$. Let $\hat{\psi}$ express $(\exists U)(\psi^U \wedge |U|^{|U|} = |U|)$. We apply Theorem 2.5 to $\neg(\hat{\psi})$. Note that $\hat{\psi}$ is true on all cardinals $\geq \lambda_0$ and false on all $\mu < \lambda_0$. By Theorem 2.5, $\lambda_0 \models \boldsymbol{T}_{\neg(\hat{\psi})}$ and $\lambda_0 \geq H(P^f_N)$. So $\boldsymbol{T}_{\neg(\hat{\psi})}$ and therefore $\neg(\hat{\psi})$ has arbitrarily large models. But $\neg(\hat{\psi})$ has no models larger than λ_0. This contradiction yields the theorem. $\square_{2.9}$

In the next section we prove the crucial Theorem 2.5. In the last section we remove the restrictions to finitely axiomatizable theories and countable languages.

3 Essential Lemmas

Now we prove Theorem 2.5. For convenience, we list here the two vocabularies. We describe the axioms of T and T_1 below.

Notation 3.1 *1. τ contains unary predicates Q_1, Q_2, a binary relation R and partial binary functions F and F_2. It contains two constant symbols c_0, c_ω and a unary function symbol g.*

2. τ_1 adds a unary predicate Q_0 and a binary relation $<_1$.

Remark 3.2 (Proof Sketch) For each second order ϕ, we construct a triple $\boldsymbol{T}_\phi$. But most of the construction is independent of the particular ϕ and so we first construct a theory T_1 which does not depend on ϕ. The vocabulary τ will contain unary predicates Q_1, Q_2. The axioms will assert that Q_1, Q_2 partition the universe. Q_0 is in τ_1. Omission of the type p will guarantee that $Q_0 \subset Q_1$ is countable. Omission of the type in a model M of T_1 whose τ-reduct $\aleph_1$-saturated and some coding involving the partial order $<_0$ in τ will guarantee that $Q_1(M)$ is well-ordered by a relation symbol $<_1$ in τ_1. A relation symbol R in τ will code subsets of Q_1 by elements of Q_2. Thus first order quantification on Q_2 will encode second order quantification on Q_1. In particular, we can code a given second order sentence ϕ and thus extend T_1 to T_ϕ. But the encoding will be 'correct' only on subsets whose every subset is coded in Q_2. But if $\mu < \lambda$ and M is λ-saturated, μ is a $<_1$-initial segment Q_1. Since $\mu < \lambda$ each subset of μ is coded by a type of size μ so the encoded semantics is correct and μ is a model of ϕ.

Proof of Theorem 2.5. We gradually introduce the vocabulary and theory explaining the use of various predicates as they are introduced; we repeat a bit of the proof sketch. Below we say certain conditions hold to mean they hold in any model of T. We first describe τ and T. In particular, τ contains unary predicates Q_1, Q_2 that partition the universe.

There is a binary relation $<_0$, which is a partial order of Q_1. There is a partial function F mapping $Q_1 \times Q_1$ into Q_1. We write F_a for the partial function from Q_1 into Q_1 indexed by a. The partial order $<_0$ satisfies: $a \leq_0 b$ implies $F_a \subset F_b$.

We have two further properties of F. F_{c_0} is the empty function. For every $a \in Q_1$ and every $e \in Q_1$, if $e \notin \mathrm{dom}\, F_a$, then there are $b, d \in Q_1$ with $a <_0 b$ and $F_b = F_a \cup \{\langle e, d\rangle\}$.

Further there is a pairing function F_2 on Q_1 and an extensional relation R between Q_1 and Q_2 so that each element of Q_2 codes a subset of Q_1 via R. We write U_b for $\{a{:}R(b,a)\}$ (for $a \in Q_1$ and $b \in Q_2$).

T asserts that Q_1 is preserved by g, that g is a permutation, and $Q_1(c_0)$.

The set of $\{U_a : a \in Q_2\}$ is closed under Boolean operations and if U_b is such a set so is $F_a(U_b)$ for any $a \in Q_1$. For each $a \in Q_1$, there is $b \in Q_2$ such that $U_b = \{c{:}c <_1 a\}$.

Secondly, we turn to the description of τ_1 and T_1. In τ_1, there is a unary relation Q_0 such that $Q_0 \subset Q_1$ and T_1 asserts Q_0 is preserved by g and c_0, c_ω are in Q_0. Thus, each $g^i(c_0) \in Q_0$. Further, there is a binary τ_1-relation $<_1$, which is a linear order of Q_1 and such that on Q_1, $x <_1 g(x)$ and $x < c_\omega$ implies $g(x) < c_\omega$. Thus, $\langle g^i(c_0) : i < \omega\rangle \cup \{c_\omega\}$ name countably many elements of Q_1 which are $<_1$- ordered in order type $\omega + 1$. T_1 further asserts $(Q_1, <_1)$ is 'internally well-ordered' in the following sense. For every $a \in Q_2$, if U_a is non-empty, it has a $<_1$-least element.

The type p asserts $Q_0(x)$ and x is not a $g^i(c_0)$ for any $i < \omega$.

Claim 3.3 *If a model M of T_1 is such that its reduct to τ is an $\aleph_1$-saturated model of T but M omits p, $(Q_1, <_1)$ is a well-ordering in M.*

Proof. Suppose there is a countable $<_1$-descending chain $B = \{b_i : i < \omega\}$ in $(Q_1, <_1)$. Using the properties of F, we can define a $<_0$-increasing chain of a_n in Q_1 such that $F_{a_n} = \{\langle c_1, b_1\rangle, \ldots \langle g^n(c_0), b_n\rangle\}$, where the $g^i(c_0)$ are images of c_0 by iterating g. Since the model is $\aleph_1$-saturated there is an $a_\omega \in Q_1$ such that each $F_{a_n} \subset F_{a_\omega}$. But then $B = F_{a_\omega}(\{g^i(c_0) : i < \omega\})$. Note that while the choice of b_i involved the τ_1-symbol $<_1$, the existence of a_ω is by the consistency of a τ-type so the use of saturation is legitimate.

Since M omits p, $\{g^i(c_0) : i < \omega\} = \{a : a <_1 c_\omega\}$ and therefore is coded by an element of Q_2. By the closure properties of the coded sets, $B = U_d$ for some $d \in Q_2$. This contradicts the internal well-ordering of Q_1. $\Box_{3.3}$

Now translate ϕ to the first order formula $\phi^*(v)$ by translating each bound second order variable X to a first order formula in x and v. Replace each occurrence of $X(z)$ by $R(z, v) \wedge R(z, x)$. This translation has the following consequence. (This is immediate for monadic second order but we included a pairing function F_2 on Q_1 so it extends to arbitrary sentences.)

Fact 3.4 *If $M \models T$, $a \in Q_2(M)$ and each subset of U_a is coded by an element of $Q_2(M)$, then $M \models \phi^*(a)$ if and only $U_a(M) \models \phi$.*

Add the following axiom to T_1 to obtain T_ϕ

$$(\forall u)(\forall w)[((\forall z)R(z, w) \leftrightarrow z <_1 u) \rightarrow \phi^*(w)].$$

Claim 3.5 *If $\mu < \lambda = \lambda^{<\lambda}$ and M is model of ϕ with cardinality λ that omits p but whose reduct to τ is saturated then $\mu \models \phi$.*

Conversely, if ϕ is true on all $\mu < \lambda = \lambda^{<\lambda}$, there is a model M_1 of T_ϕ with cardinality λ that omits p but whose reduct to τ is saturated.

Proof. Since $\mu < \lambda$, μ is an initial segment of Q_1 so $\mu = \{a \in Q_1 : R(y, d)\}$ for some $d \in Q_2$. But then each subset Y of μ gives rise to a type $q_Y(x)$:

$$\{R(y, d)\} \cup \{R(y, x) : y \in Y\} \cup \{\neg R(y, x) : y \notin Y\}.$$

For each Y the τ-type $q_Y(x)$ has cardinality less than λ and so is realized by saturation. We finish by Fact 3.4.

For the converse, well-order Q_1 by $<_1$ in order type λ. Add in Q_2 a code for each subset of cardinality $< \lambda$. Let the F_a list the partial functions of cardinality less than λ from Q_1 to Q_1 and let $<_0$ denote the natural partial ordering on Q_1 induced by inclusion of the named functions. Since ϕ is true below λ, each infinite initial segment in λ defines a model of ϕ and the definition of T_ϕ shows that we have a saturated model of T when we take the reduct to τ. Finally, let Q_0 include exactly the first ω elements of Q_1.

$\square_{3.5}$

Letting $\boldsymbol{T}_\phi$ be the triple (T, T_ϕ, p) we have a triple satisfying Theorem 2.5. $\square_{2.5}$

4 The exact strength

In this section we remove the restrictions to finitely axiomatizable theories and countable languages. We prove actual equality of the Hanf number studied here (for any theory) with a Löwenheim number of second order logic; the cost is that we must move into infinitary second order logic and allow relation constants (i.e. predicate symbols other than equality) in the vocabulary of the second order sentence.

Instead of Theorem 2.5 we could slightly more easily prove

$$H(P^f_N) \leq \ell^2(L^{II}) \leq H(P^c_N),$$

which gives our answer to Newelski's question but is not quite as sharp. That is, if we had just required $\boldsymbol{T}_\phi$ in Theorem 2.5 to be in a countable language rather than finitely axiomatizable, this would have no effect on the proof of Lemma 2.9 and it would have simplified the proof of Theorem 2.5 since we could have worked with countably many constants and omitted the function g. Similarly the arguments of Sections 2 and 3 extend from finitely axiomatizable to 'arithmetic' by coding a model of arithmetic in the second order sentence. And it is easy to see that the theory constructed in Theorem 2.5 is recursive.

This observation is generalized in Theorem 4.11 to remove the restrictions on axiomatizability. The key idea is to see that we can use the same ideas as in Section 3 to code the syntax of infinitary second order logic by a triple $\boldsymbol{T}$.

We extend our notion of second order logic in two ways. First we allow infinite conjunctions and strings of quantifiers. Secondly we now allow some relation constants instead of dealing with 'pure' second order logic.

Definition 4.1 *1. L_θ denotes the θ stage in the construction of the inner model L.*

2. Let $L_{\theta^+,\kappa}(II)$ denote second order logic allowing strings of second order quantifiers of cardinality $< \kappa$ and conjunctions and disjunctions of cardinality $\leq \theta$.

3. *$\mathcal{T}^\theta$ denotes the set of triples $\boldsymbol{T}$ as in Definition 2.1 (except allowing $|\tau_1| \leq \theta$) such that τ_1 satisfies $\tau_1 \subset L_\theta$, T is finitely axiomatizable and p is second order definable.*

Remark 4.2 *Again using Assumption 0.3, note that the Löwenheim number of $L_{\theta^+,\kappa}(II)$ is a strong limit cardinal of cofinality $> \theta$ and is an accumulation point of $\{\mu : \mu = \mu^{<\mu}\}$.*

Notation 4.3 *We denote by $L(II, \tau_*)$ the second order logic in the vocabulary τ_* consisting of unary predicates P and P_1 and a binary relation R_1.*

Notation 4.4 *For $\boldsymbol{K} = \boldsymbol{K}_{\boldsymbol{T}}$, let $P^\kappa_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ hold if $|\tau_1| \leq \kappa$ and for some M_1 with $|M_1| = \lambda$, $M_1 \models \boldsymbol{T}$.*

We now show how to code the Löwenheim number of sentences ϕ of $L_{\theta^+,\kappa}(II)$ by using a sentence in $\psi \in L(II, \tau_*)$ and a set A_ϕ of ordinals. We begin with a more general lemma concerning the coding of transitive sets.

Definition 4.5 *Let X be a transitive subset of $H(\theta^+)$ with cardinality θ such that $\{\emptyset\} \subset X$. We say that X* is coded by *$A \subseteq \theta$ if there is an injection f from X into θ such that:*

$$A = \{\mathrm{pr}(f(a), f(b)) : a \in b \in X\}.$$

Here pr *is the standard pairing function on ordinals.*

Now we show that the coding of X by A does *not* depend on the choice of f.

Lemma 4.6 *If $A \subseteq \theta$ codes X_1 by f_1 and X_2 by f_2 then $X_1 = X_2$.*

Proof. We first argue that for any X, A if A codes X by f then f is a bijection between X and $B = \{\alpha : \mathrm{pr}(\alpha, \beta) \in A \vee \mathrm{pr}(\beta, \alpha) \in A\}$. The nontriviality conditions on X guarantee that f maps into B; f is 1-1 by definition of f and onto B by the definition of B.

Applying this remark to f_1, f_2, we see $g = f_2^{-1} f_1$ is a bijection from X_1 onto X_2. But then f is an isomorphism with respect to ϵ as it is easy to check (from the definition of coding) that for any $y_1, z_1 \in X_1$ with $y_2 = g(y_1)$ and $z_2 = g(z_1)$, $y_1 \in z_1$ if and only if $y_2 \in z_2$. That is, $y_1 \in z_1$ if and only $\mathrm{pr}(f_1(y_1), f_1(z_1)) \in A$ if and only $\mathrm{pr}(f_2(y_2), f_2(z_2)) \in A$ if and only $y_2 \in z_2$. But then since $\emptyset \in X_1 \cap X_2$, ϵ-induction yields that $X_1 = X_2$. $\square_{4.6}$

In the following $H(\mu)$ denotes the set of all sets whose transitive closure has cardinality less than μ. We use implicitly that $\mu^{<\mu} = \mu$ implies that μ is regular and so $(H(\mu), \epsilon)$ satisfies all axioms of ZFC except power set.

Lemma 4.7 *If $\tau \in L_\theta$ then each $\phi \in L_{\theta^+,\kappa}(II)(\tau)$ has at most θ subformulas, each in* tc(ϕ)*, and* tc$(\phi) \cup \{\phi\}$ *is coded (in the sense of Definition 4.5) by a set $A_\phi \subset \theta$.*

Proof. The standard construction of $\phi \in L_{\theta^+,\kappa}(II)(\tau)$ yields that each formula is in $H(\theta^+)$, each subformula of $\phi \in \mathrm{tc}(\phi)$, and ϕ has at most θ subformulas. Define A_ϕ by Definition 4.5. $\square_{4.5}$

The function G in Definition 4.8.2c simply formalizes the normal definition of truth.

Definition 4.8 *1. For every sentence $\phi \in L_{\theta^+,\kappa}(II)(\tau)$ and every vocabulary $\tau \in L_\theta$ we define $A_\phi \subseteq \theta$ to be the set which codes* $\mathrm{tc}(\{\phi, \tau\}) \cup \{\phi, \tau\}$ *in the sense of Definition 4.5.*

2. *We define a sentence in $\psi \in L(II, \tau_*)$ to assert the following:*
   (a) *(P^M, P_1^M, R_1^M) is a well ordering with $P_1^M \subseteq P^M$*
   (b) *$(M, R_1^M) \approx (H(\mu), \epsilon)$ for some μ with $\mu^{<\mu} = \mu$.*
   (c) *There is a function G which defines truth on subsets b of M of sentences of $L_{\theta^+,\kappa}(II)$: $G(b, \phi) = 0$ for false and 1 for true.*
   (d) *$G^M(b, \phi) = 0$ if $M \models |b| < |M|$.*

The goal of the following lemma is to compute the Löwenheim number of $L_{\theta^+,\kappa}(II)$. Since it is certainly greater than θ, we may assume $\lambda^{<\lambda} > \theta$.

Lemma 4.9 *Fix $\kappa \leq \theta^+$, $\phi \in L_{\theta^+,\kappa}(II)$ and A_ϕ, $\psi \in L(II, \tau_*)$ satisfying Definition 4.8. For any cardinal $\lambda = \lambda^{<\lambda} > \theta$, the following are equivalent.*

1. *ϕ has no model of card $< \lambda$.*
2. *There is a model (M, P^M, P_1^M, R_1^M) of ψ with cardinality λ such that (P^M, R_1^M) has order type θ and*

$$A_\phi = \{\alpha < \kappa : \text{for some } a \in P_1^M \subseteq P^M, \alpha = \mathrm{otp}(\{b \in P^M : bR_1^M a\}, R_1^M)\}.$$

Proof. Suppose 2). Then $|M| = \lambda$, (P^M, R_1^M) has order type θ, and A_ϕ is the image of P_1^M under an isomorphism from (P^M, R_1^M) to θ. By the choice of A_ϕ , the model M correctly recognizes the formula ϕ and the function G^M correctly represents truth in M by Definition 4.8 2b and 2c. So ϕ fails on all subsets of M with cardinality $< \lambda$ by Definition 4.8 2d. Thus 2) implies 1). Clearly if 1) holds we can construct a model M satisfying 2).

$\square_{4.9}$

Definition 4.10 *For ψ defined as in Lemma 4.9,* $\mathrm{spec}(\psi, \theta, A_\phi)$ *is the set of the cardinalities of models M of ψ with $(P^M, P_1^M, R_1^M) \approx (\theta, <, A_\phi)$.*

Theorem 4.11 *For any cardinal θ, the following four cardinals are equal.*

1. *λ_1 is the Hanf number of P_N^θ.*
2. *λ_2 is the Löwenheim number of $L_{\theta^+,\omega}(II)$.*

3. *λ_3 is the Löwenheim number of $L_{\theta^+,\theta^+}(II)$.*

4. *$\lambda_4 = \sup\{\mathrm{spec}(\psi,\theta,A_\phi) : \psi \in L(II,\tau_*)$, $\phi \in L_{\theta^+,\theta^+}(II)$,* and *$A_\phi \subset \theta$* such that $\mathrm{spec}(\psi,\theta,A_\phi)$ is bounded$\}$.

Proof. We chose the logic $L_{\theta^+,\omega}$ precisely so $\lambda_1 \leq \lambda_2$ (by a proof like that of Lemma 2.6 but now we have conjunctions of cardinality θ) and clearly $\lambda_2 \leq \lambda_3$. Lemma 4.9 yields:

$$\{\min(\mathrm{spec}^2(\phi)) : \phi \in L_{\theta^+,\theta^+}\} \subseteq \{\sup(\mathrm{spec}^2(\theta,\psi,A_\phi)) : \phi \in L_{\theta^+,\theta^+} \text{ is bounded}\}.$$

(We can replace ϕ by a ϕ^* whose only model is the model of ϕ with minimum cardinality to guarantee the containment.) Thus, $\lambda_3 \leq \lambda_4$.

The proof that $\lambda_4 \leq \lambda_1$ is obtained by modifying the proof of Theorem 2.5. Add to the vocabulary in the T_ϕ from the proof in section 3 of Theorem 2.5, symbols P, P_1, R_1 and use the same coding ideas to guarantee that $P_1 \subseteq P$ and both are well-ordered by R_1. Thus, for $\phi \in L_{\theta^+,\theta^+}(II)$ we can construct $\boldsymbol{T}_\phi$, encoding the second order sentence $\psi \in L(II,\tau_*)$ defined in Definition 4.8 and where the type p also codes that ${P_1}^M \approx A_\phi$ so that the two spectra are related as in Theorem 2.5. This yields $\lambda_4 \leq \lambda_1$ by slightly modifying the argument for Lemma 2.9.

$\square_{4.11}$

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