A Hanf number for saturation and omission

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Abstract
Suppose $T = (T, T_1, p)$ is a triple of two countable theories in languages $\tau \subset \tau_1$ and a $\tau_1$-type $p$ over the empty set. We show the Hanf number for the property: There is a model $M_1$ of $T_1$ which omits $p$, but $M_1 \restriction \tau$ is saturated is at least the Löwenheim number of second order logic.

Newelski [3] asked to calculate the Hanf number of the following property $P_N$.

Definition 0.1 We say $M_1 \models T$ where $T = (T, T_1, p)$ is a triple of two countable theories in vocabularies $\tau \subset \tau_1$ and $p$ is a $\tau_1$-type over the empty set if $M_1$ is a model of $T_1$ which omits $p$, but $M_1 \restriction \tau$ is saturated. Let $K_T$ denote the $M_1$ which satisfy $T$.

For $K = K_T$ for some $T$ in a countable vocabulary, let $P^f_N(K_T, \lambda)$ hold if $\tau_1$ is countable and for some $M_1$ with $|M_1| = \lambda$, $M_1 \models T$. $P^f_N(K_T, \lambda)$ is the same property restricted to triples where $T_1$ and $T$ are finitely axiomatizable in finite vocabularies.

spec($T$) is the collection of cardinals $\lambda$ such that there is an $M_1$ satisfying $T$ with $|M_1| = \lambda$.

Recall Hanf’s observation [1] that for any such property $P(K, \lambda)$, where $K$ is ranges over a set of classes of models, there is a cardinal $\kappa = H(P)$ such that: if $P(K, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(K, \lambda)$ holds for arbitrarily large $\lambda$. $H(P)$ is called the Hanf number of $P$. E.g. $P(K, \lambda)$ might be the property that $K$ has a model of power $\lambda$. Similarly the Löwenheim number $\ell(P)$ of a set $P$ of classes is the least cardinal $\mu$ such that any class $K \in P$ that has a model has one of cardinal $\leq \mu$.

Theorem 0.2 $H(P^f_N) = \ell(L^{II})$ where $L^{II}$ denotes the collection of sentences of second order logic.

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Since $H(P^N_N) \geq H(P^L_N)$, this shows that the Hanf number in the abstract is at least $\ell(L^{II})$, as asserted.

Jouko Vaananen provided the following summary of the effect of this result by indicating the size of $\ell(L^{II})$. $\ell(L^{II})$ is bigger than the first (second, third, etc) fixed point of any normal function on cardinals that itself can be described in second order logic. For example it is bigger than the first $\kappa$ such that $\kappa = \beth_\kappa$, bigger than the first $\kappa$ such that there are $\kappa$ cardinals $\lambda$ below $\kappa$ such that $\lambda = \beth_\lambda$, etc. It is easy to see that if there are measurable (inaccessible, Mahlo, weakly compact, Ramsey) cardinals, then the Lowenheim number of second order logic exceeds the first of them (respectively, the first inaccessible, Mahlo, weakly compact, Ramsey) (and second, third, etc). So even under $V = L$, the Lowenheim number is bigger than any ‘large’ cardinal that is second order definable and consistent with $V = L$. Such results are discussed in Vaananen’s paper “Hanf numbers of unbounded logics”[4]. A result of Magidor [2] shows the Lowenheim number of second order logic is always below the first supercompact. Vaananen’s paper “Abstract logic and set theory II: Large cardinals” gives lower bounds for the Lowenheim number of equicardinality quantifiers and thus a fortiori for second order logic [5]. In simple terms, if $E(\kappa)$ is the statement that $2^\kappa \geq \kappa^{++}$ then the first $\kappa$ cardinals (if any) such that $E(\kappa)$ holds is less than the Lowenheim number of second order logic. This shows that by forcing we can push the Lowenheim number up at will.

We make the following assumption throughout.

**Assumption 0.3** Assume the set of $\lambda$ with $\lambda^{<\lambda} = \lambda$ is a proper class.

This assumption follows from GCH, but if GCH fails badly the only such cardinals are strongly inaccessible. The key point for our use of the condition is that $\lambda^{<\lambda} = \lambda$ is a sufficient condition for the existence of a saturated model in $\lambda$. In Section 1 we review some properties of second order logic and show the equality of two ‘Löwenheim numbers’; this equality demonstrates the assumption is harmless in our context. In Section 2, we state two technical results, prove one, and deduce Theorem 0.2 from them. In Section 3, we prove the more difficult technical result. Newelvis question arose in the study of the model theory of groups and the existence of groups of bounded order.

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## 1 Some Second Order Logic

By (pure) second order logic, we mean the logic with individual variables and variables for relations of all arities. The atomic formulas are equalities between variables and expressions $X(x)$ where $X$ is an $n$-ary relation and $x$ is an $n$-tuple of variables. Note that a structure $A$ for this logic is simply a set so is determined entirely by its cardinality.

We put our restriction to $\lambda = \lambda^{<\lambda}$ in a more general setting. In general for any class $K$ of models write $\text{spec}(K)$ for the collection of $\lambda$ such that there is a model in
\( K \) with cardinality \( \lambda \). We describe some technical variants for the second order case that are relevant here.

**Definition 1.1** Let \( \psi \) be a sentence of second order logic.

1. \( \text{spec}^1(\psi) = \{ \lambda : \lambda \models \psi \} \).
2. \( \text{spec}^2(\psi) = \{ \lambda : \lambda = \lambda^{<\lambda} \land \lambda \models \psi \} \)

Note that there is a sentence \( \chi \) in second order logic which has a model size \( \lambda \) if and only if \( \lambda^{<\lambda} = \lambda \). Namely, let \( \chi \) assert there is an extensional relation \( R \) on sets such that each element denotes, via \( R \), a set of smaller cardinality than the universe and each such set is coded by \( R \). We will generally write \( \lambda^{<\lambda} = \lambda \) to denote this sentence.

**Definition 1.2** Define \( H^2 \) and \( \ell^2 \) to be the Hanf and Lowenheim numbers with respect to \( \text{spec}^2 \).

We can show

**Lemma 1.3** \( H(L^{II}) = H^2(L^{II}) \) and \( \ell(L^{II}) = \ell^2(L^{II}) \)

Proof. We prove the L"owenheim number assertion. One direction is easy. For every sentence \( \psi \) of second order logic, there is a sentence \( \psi^* \) such that:

\[
\text{spec}^2(\psi) = \text{spec}^1(\psi^*).
\]

\( \psi^* \) just expresses the conjunction of \( \psi \) with \( \lambda^{<\lambda} = \lambda \). Recalling that for either spectrum \( \ell^i(L^{II}) = \sup\{\min\{\text{spec}^i(\phi)\} : \phi \in (L^{II})\} \), since every 2-spectrum is a 1-spectrum \( \ell^2(L^{II}) \leq \ell^1(L^{II}) \).

But the opposite inequality also holds. Let \( \phi \) be a sentence with a non-empty 2-spectrum. Then let \( \phi^* \) express \((\exists U)(\phi^U \land \lambda^{<\lambda} = \lambda)\). Now \( \phi^* \) has a non-empty 2-spectrum so it has a model \( M \) with cardinality \( |M| = \kappa \leq \ell^2(L^{II}) \). But then \( \phi \) has a model with cardinality \( |U^M| \leq \kappa \leq \ell^1(L^{II}) \). So \( \ell^1(L^{II}) \leq \ell^2(L^{II}) \).

A similar argument works for Hanf numbers. \( \square \)

2 The main result

We prove Theorem 2.2 in Section 3. Recall our notation from Definition 0.1.

**Notation 2.1** We will write \( T \) (possibly with subscripts) for a triple \( (T, T_1, p) \). The expression ‘\( T \) has a model in \( \lambda \)’ means there is a model of \( T_1 \) with cardinality \( \lambda \) that omits \( p \) and whose reduct to \( L(T) = \tau \) is saturated

**Theorem 2.2** For every second order sentence \( \phi \), there is a triple \( T_{\phi} \) in a finite vocabulary such that if \( \lambda^{<\lambda} = \lambda \), then the following are equivalent:

1. \( T_{\phi} \) has a model in \( \lambda \).
2. \( \phi \) has a model in every cardinal less than \( \lambda \).
Note that the following extends from finitely axiomatizable to ‘arithmetic’ by coding a model of arithmetic in the second order sentence. And it easy to see that the theory constructed in Theorem 2.2 is recursive.

**Lemma 2.3** For every $T$, with finitely axiomatizable $T_1$, there is a second order $\phi_T$, such that $\phi_T$ has a model in $\lambda$ if and only if $T$ has a model in $\lambda$.

Since $T_1$ is finitely axiomatizable, it is easy to write second order sentence $\theta$ such that if $\mathcal{M} |\theta$, $\mathcal{M} |\phi_{T_1}$, $\mathcal{M}$ omits $p$ and $M |_{\tau}$ is saturated. □

**Claim 2.4** $H(P_f) \leq \ell^2(L^{II})$ where $L^{II}$ denotes second order logic.

Proof. Lemma 2.3 shows that for any $T$, there is a $\phi_T$ with $\text{spec}(T) = \text{spec}(\phi_T)$. Suppose for contradiction that $H(P_f^T) > \ell^2(L^{II})$. Then there is a triple $T$ with a bounded spectrum and the bound is greater than $\ell^2(L^{II})$. But then, $\neg \phi_T$ has a model and $\min \text{spec}(\neg \phi_T) > \ell^2(L^{II})$. This contradicts the definition of the Löwenheim number. □

**Lemma 2.5** $H(P_c) \geq \ell^2(L^{II})$ where $L^{II}$ denotes second order logic.

Proof. Suppose for contradiction that there is a second order sentence $\psi$ such that $\lambda_0 = \min(\text{spec}^2(\psi)) > H(P_c^\psi)$. We apply Theorem 2.2 to $\neg \psi^*$ (as defined the proof of Lemma 1.3). On the one hand $\psi^*$ has a model in cardinality $\lambda_0$ where $\lambda_0 < \lambda_0 = \lambda_0$ and $\neg \psi$ is true on all $\mu < \lambda_1$. By Theorem 2.2, $\lambda_0 \models T_{\neg \psi}$ and $\lambda_0 \geq H(P_c^\psi)$. So $T_{\neg \psi}$ and therefore $\neg \psi^*$ has arbitrarily large models. But $\neg \psi$ has no models larger than $\lambda_0$. This contradiction yields the theorem. □

**Remark 2.6** We could slightly more easily prove

$$H(P_c^T) \leq \ell^2(L^{II}) \leq H(P_f^T),$$

which gives our answer to Newelski’s question but is not quite as sharp. That is, if we had just required $T_{\phi}$ in Theorem 2.2 to be in a countable language rather than finitely axiomatizable, this would have no effect on the proof of Lemma 2.5 and it would have simplified the proof of Theorem 2.2 since we could have worked with countably many constants and omitted the function g.

### 3 Essential Lemmas

Now we prove Theorem 2.2. For convenience, we list here the two vocabularies. We describe the axioms of $T$ and $T_1$ below.

**Notation 3.1** 1. $\tau$ contains unary predicates $Q_1, Q_2$, a binary relation $R$ and partial binary functions $F$ and $F_2$. It contains two constant symbols $c_0, c_\omega$ and a unary function symbol $g$. 

2. $\tau_1$ adds a unary predicate $Q_0$ and a binary relation $<_1$.

**Remark 3.2 (Proof Sketch)** For each second order $\phi$, we construct a triple $T_\phi$. But most of the construction is independent of the particular $\phi$ and so we first construct a theory $T_1$ which does not depend on $\phi$. The vocabulary $\tau$ will contain unary predicates $Q_1, Q_2$. The axioms will assert that $Q_1, Q_2$ partition the universe. $Q_0$ is in $\tau_1$. Omission of the type $p$ will guarantee that $Q_0 \subset Q_1$ is countable. Omission of the type in a model $M$ of $T_1$ whose $\tau$-reduct $\aleph_1$-saturated and some coding involving the partial order $<_0$ in $\tau$ will guarantee that in $Q_1(M)$ is well-ordered by a relation symbol $<_1$ in $\tau_1$. A relation symbol $R$ in $\tau$ will code subsets of $Q_1$ by elements of $Q_2$. Thus first order quantification on $Q_2$ will encode second order quantification on $Q_1$. In particular, we can code a given second order sentence $\phi$ and thus extend $T_1$ to $T_\phi$. But the encoding will be `correct' only on subsets whose every subset is coded in $Q_2$. But if $\mu < \lambda$ and $M$ is $\lambda$-saturated, $\mu$ is a $<_1$-initial segment $Q_1$. Since $\mu < \lambda$ each subset of $\mu$ is coded by a type of size $\mu$ so the encoded semantics is correct and $\mu$ is a model of $\phi$.

Proof of Theorem 2.2. We gradually introduce the vocabulary and theory explaining the use of various predicates as they are introduced; we repeat a bit of the proof sketch. Below we say certain conditions hold to mean they hold in any model of $T$. In particular, $\tau$ contains unary predicates $Q_1, Q_2$ that partition the universe.

There is a binary relation $<_0$, which is a partial order of $Q_1$. There is a partial function $F$ mapping $Q_1 \times Q_1$ into $Q_1$. We write $F_a$ for the partial function from $Q_1$ into $Q_1$ indexed by $a$. The partial order $<_0$ satisfies: $a <_0 b$ implies $F_a \subset F_b$.

We have two further properties of $F$. $F_{00}$ is the empty function. For every $a \in Q_1$ and every $c \in Q_1$, if $c \notin \text{dom } F_a$, then there are $b, d \in Q_1$ such that $a <_0 b$ and $F_b = F_a \cup \{(c, d)\}$.

Further there is a pairing function $F_2$ on $Q_1$ and an extensional relation $R$ between $Q_1$ and $Q_2$ so that each element of $Q_2$ codes a subset of $Q_1$ via $R$. We write $U_b$ for $\{a : R(b, a)\}$ (for $a \in Q_1$ and $b \in Q_2$).

$T$ asserts that $Q_1$ is preserved by $g$, that $g$ is a permutation, and $Q_1(c_0)$.

The set of $\{U_a : a \in Q_2\}$ is closed under Boolean operations and if $U_b$ is such a set so is $F_b(U_b)$ for any $a \in Q_1$. For each $a \in Q_1$, there is $b \in Q_2$ such that $U_b = \{c : c <_1 a\}$.

Now we turn to the description of $\tau_1$ and $T_1$. In $\tau_1$, there is a unary relation $Q_0$ such that $Q_0 \subset Q_1$ and $T_1$ asserts $Q_0$ is preserved by $g$ and $c_0, c_\omega$ are in $Q_0$. Thus, each $g^i(c_0) \in Q_0$. Further, there is a binary $\tau_1$-relation $<_{11}$, which is a linear order of $Q_1$ and such that on $Q_1$, $x <_{11} g(x)$ and $x <_{11} c_\omega$ implies $g(x) < c_\omega$. Thus, $(g^i(c_0) : i < \omega) \cup \{c_\omega\}$ name countably many elements of $Q_1$ which are $<_1$-ordered in order type $\omega + 1$. $T_1$ further asserts $(Q_1, <_{11})$ is `pseudo-well-ordered' in the following sense. For every $a \in Q_2$, if $U_a$ is non-empty, it has a $<_{11}$-least element.

The type $p$ asserts $Q_0(x)$ and $x$ is not a $g^i(c_0)$.

**Claim 3.3** If a model $M$ of $T_1$ is such that its reduct to $\tau$ is an $\aleph_1$-saturated model of $T$ but $M$ omits $p$, $(Q_1, <_{11})$ is a well-ordering in $M$. 

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Proof. Suppose there is a countable $<_1$-descending chain $B = \{b_i : i < \omega\}$ in $(Q_1, <_1)$. Using the properties of $F$, we can define a $<_0$-increasing chain of $a_n$ in $Q_1$ such that $F_{a_n} = \{(c_1, b_1), \ldots, (g^n(c_0), b_n)\}$, where the $g^n(c_0)$ are images of $c_0$ by iterating $g$. Since the model is $\aleph_1$-saturated there is an $a_\omega \in Q_1$ such that each $F_{a_n} \subset F_{a_\omega}$. But then $B = F_{a_\omega}(\{g^n(c_0) : i < \omega\})$. Note that while the choice of $b_i$ involved the $\tau_1$-symbol $<_1$, the existence of $a_\omega$ is by the consistency of a $\tau$-type so the use of saturation is legitimate.

Since $M$ omits $p$, $\{g^n(c_0) : i < \omega\} = \{a : a <_1 c_\omega\}$ and therefore is coded by an element of $Q_2$. By the closure properties of the coded sets, $B = U_d$ for some $d \in Q_2$. This contradicts the pseudo-well-ordering of $Q_1$. □

Now translate $\phi$ to the first order formula $\phi^*(v)$ by translating each second bound variable $X$ to a first order formula in $x$ and $v$. Replace each occurrence of $X(z)$ by $R(z, v) \land R(z, x)$. This translation has the following consequence. (This is immediate for monadic second order but we included a pairing function $F_2$ on $Q_1$ so it extends to arbitrary sentences.)

**Fact 3.4** If $M \models T$, $c \in Q_2(M)$ and each subset of $U_c$ is coded by an element of $Q_2(M)$, then $M \models \phi^*(c)$ if and only $U_c(M) \models \phi$.

Add the following axiom to $T_1$ to obtain $T_\phi$

$$\forall u \exists w[(\forall z) R(z, w) \leftrightarrow z <_1 u] \rightarrow \phi^*(w)].$$

**Claim 3.5** If $\mu < \lambda = \lambda^{<\lambda}$ and $M$ is model of $T_\phi$ with cardinality $\lambda$ that omits $p$ but whose reduct to $\tau$ is saturated then $\mu \models \phi$.

Conversely, if $\phi$ is true on all $\mu < \lambda = \lambda^{<\lambda}$, there is a model $M_1$ of $T_\phi$ with cardinality $\lambda$ that omits $p$ but whose reduct to $\tau$ is saturated.

Proof. Since $\mu < \lambda$, $\mu$ is an initial segment of $Q_1$ so $\mu = \{a \in Q_1 : R(y, d)\}$ for some $d \in Q_2$. But then each subset $Y$ of $\mu$ gives rise to a type $q_Y(x)$:

$$\{R(y, d)\} \cup \{R(y, x) : y \in Y\} \cup \{\neg R(y, x) : y \notin Y\}.$$ 

For each $Y$ in the $\tau$-type $q_Y(x)$ has cardinality less than $\lambda$ and so us realized by saturation. We finish by Fact 3.4.

For the converse, well-order $Q_1$ by $<_1$ in order type $\lambda$. Add in $Q_2$ a code for each subset of cardinality $<_\lambda$. Let the $F_a$ list the partial functions of cardinality less than $\lambda$ from $Q_1$ to $Q_1$ and let $<_0$ denote the natural partial ordering on $Q_1$ induced by inclusion of the named functions. Since $\phi$ is true below $\lambda$, each infinite initial segment in $\lambda$ defines a model of $\phi$ and the definition of $T_\phi$ shows that we have a saturated model of $T$ when we take the reduct to $\tau$. Finally, let $Q_0$ include exactly the first $\omega$ elements of $Q_1$.

Letting $T_\phi$ be the triple $(T, T_\phi, p)$ we have a triple satisfying Theorem 2.2.
References


