A Hanf number for saturation and omission

John T. Baldwin*
University of Illinois at Chicago

Saharon Shelah
Hebrew University of Jerusalem
Rutgers University

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Abstract

Suppose $\boldsymbol{T} = (T, T_1, p)$ is a triple of two countable theories in languages $\tau \subset \tau_1$ and a τ_1-type p over the empty set. We show the Hanf number for the property: There is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated is at least the Löwenheim number of second order logic.

Newelski [3] asked to calculate the Hanf number of the following property P_N.

Definition 0.1 *We say $M_1 \models \boldsymbol{T}$ where $\boldsymbol{T} = (T, T_1, p)$ is a triple of two countable theories in vocabularies $\tau \subset \tau_1$ and p is a τ_1-type over the empty set if M_1 is a model of T_1 which omits p, but $M_1 \restriction \tau$ is saturated. Let $\boldsymbol{K_T}$ denote the M_1 which satisfy $\boldsymbol{T}$.*

For $\boldsymbol{K} = \boldsymbol{K_T}$ for some $\boldsymbol{T}$ in a countable vocabulary, let $P^c_N(\boldsymbol{K_T}, \lambda)$ hold if τ_1 is countable and for some M_1 with $|M_1| = \lambda$, $M_1 \models \boldsymbol{T}$. $P^f_N(\boldsymbol{K_T}, \lambda)$ is the same property restricted to triples where T_1 and T are finitely axiomatizable in finite vocabularies.

$\mathrm{spec}(\boldsymbol{T})$ *is the collection of cardinals λ such that there is an M_1 satisfying $\boldsymbol{T}$ with $|M_1| = \lambda$,*

Recall Hanf's observation [1] that for any such property $P(\boldsymbol{K}, \lambda)$, where $\boldsymbol{K}$ is ranges over a *set* of classes of models, there is a cardinal $\kappa = H(P)$ such that: if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. $H(P)$ is called the Hanf number of P. E.g. $P(\boldsymbol{K}, \lambda)$ might be the property that $\boldsymbol{K}$ has a model of power λ. Similarly the Löwenheim number $\ell(P)$ of a set P of classes is the least cardinal μ such that any class $\boldsymbol{K} \in P$ that has a model has one of cardinal $\leq \mu$.

Theorem 0.2 $H(P^f_N) = \ell(L^{II})$ *where L^{II} denotes the collection of sentences of second order logic.*

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Since $H(P^c_N) \geq H(P^f_N)$, this shows that that the Hanf number in the abstract is at least $\ell(L^{II})$, as asserted.

Jouko Vaananen provided the following summary of the effect of this result by indicating the size of $\ell(L^{II})$. $\ell(L^{II})$ is bigger than the first (second, third, etc) fixed point of any normal function on cardinals that itself can be described in second order logic. For example it is bigger than the first κ such that $\kappa = \beth_\kappa$, bigger than the first κ such that there are κ cardinals λ below κ such that $\lambda = \beth_\lambda$, etc. It is easy to see that if there are measurable (inaccessible, Mahlo, weakly compact, Ramsey) cardinals, then the Lowenheim number of second order logic exceeds the first of them (respectively, the first inaccessible, Mahlo, weakly compact, Ramsey) (and second, third, etc). So even under $V = L$, the Löwenheim number is bigger than any 'large' cardinal that is second order definable and consistent with $V = L$. Such results are discussed in Vaananen's paper "Hanf numbers of unbounded logics"[4]. A result of Magidor [2] shows the Lowenheim number of second order logic is always below the first supercompact. Vaananen's paper "Abstract logic and set theory II: Large cardinals" gives lower bounds for the Lowenheim number of equicardinality quantifiers and thus *a fortiori* for second order logic [5]. In simple terms, if $E(\kappa)$ is the statement that $2^\kappa \geq \kappa^{++}$ then the first κ cardinals (if any) such that $E(\kappa)$ holds is less than the Lowenheim number of second order logic. This shows that by forcing we can push the Lowenheim number up at will.

We make the following assumption throughout.

Assumption 0.3 *Assume the set of λ with $\lambda^{<\lambda} = \lambda$ is a proper class.*

This assumption follows from GCH, but if GCH fails badly the only such cardinals are strongly inaccessible. The key point for our use of the condition is that $\lambda^{<\lambda} = \lambda$ is a sufficient condition for the existence of a saturated model in λ. In Section 1 we review some properties of second order logic and show the equality of two 'Löwenheim numbers'; this equality demonstrates the assumption is harmless in our context. In Section 2, we state two technical results, prove one, and deduce Theorem 0.2 from them. In Section 3, we prove the more difficult technical result. Newelski's question arose in the study of the model theory of groups and the existence of groups of bounded order.

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1 Some Second Order Logic

By (pure) second order logic, we mean the logic with individual variables and variables for relations of all arities. The atomic formulas are equalities between variables and expressions $X(\mathbf{x})$ where X is an n-ary relation and $\mathbf{x}$ is an n-tuple of variables. Note that a structure A for this logic is simply a set so is determined entirely by its cardinality. But we use the full semantics; the n-ary relation variables range over all n-ary relations on A.

We put our restriction to $\lambda = \lambda^{<\lambda}$ in a more general setting. In general for any class $\boldsymbol{K}$ of models write $\mathrm{spec}(\boldsymbol{K})$ for the collection of λ such that there is a model in

$\boldsymbol{K}$ with cardinality λ. We describe some technical variants for the second order case that are relevant here.

Definition 1.1 *Let ψ be a sentence of second order logic.*

1. $\mathrm{spec}^1(\psi) = \{\lambda : \lambda \models \psi\}$.
2. $\mathrm{spec}^2(\psi) = \{\lambda : \lambda = \lambda^{<\lambda} \wedge \lambda \models \psi\}$

Note that there is a sentence χ in second order logic which has a model size λ if and only if $\lambda^{<\lambda} = \lambda$. Namely, let χ assert there is an extensional relation R on sets such that each element denotes, via R, a set of smaller cardinality than the universe and each such set is coded by R. We will generally write $\lambda^{<\lambda} = \lambda$ to denote this sentence.

Definition 1.2 *Define H^2 and ℓ^2 to the be Hanf and Lowenheim numbers with respect to* spec^2.

We can show

Lemma 1.3 $H(L^{II}) = H^2(L^{II})$ *and* $\ell(L^{II}) = \ell^2(L^{II})$

Proof. We prove the Lówenheim number assertion. One direction is easy. For every sentence ψ of second order logic, there is a sentence ψ^* such that:

$$\mathrm{spec}^2(\psi) = \mathrm{spec}^1(\psi^*).$$

ψ^* just expresses the conjunction of ψ with $\lambda^{<\lambda} = \lambda$. Recalling that for either spectrum $\ell^i(L^{II}) = \sup\{\min\{\mathrm{spec}^i(\phi)\} : \phi \in (L^{II}\}$, since every 2-spectrum is a 1-spectrum $\ell^2(L^{II}) \leq \ell^1(L^{II})$.

But the opposite inequality also holds. Let ϕ be a sentence with a non-empty 2-spectrum. Then let ϕ^* express $(\exists U)(\phi^U \wedge \lambda^{<\lambda} = \lambda)$. Now ϕ^* has a non-empty 2-spectrum so it has a model M with cardinality $|M| = \kappa \leq \ell^2(L^{II})$. But then ϕ has a model with cardinality $|U^M| \leq \kappa \leq \ell^2(L^{II})$. So $\ell^1(L^{II}) \leq \ell^2(L^{II})$.

A similar argument works for Hanf numbers. $\square_{1.3}$

2 The main result

We prove Theorem 2.2 in Section 3. Recall our notation from Definition 0.1.

Notation 2.1 *We will write $\boldsymbol{T}$ (possibly with subscripts) for a triple (T, T_1, p). The expression '$\boldsymbol{T}$ has a model in λ' means there is a model of T_1 with cardinality λ that omits p and whose reduct to $L(T) = \tau$ is saturated*

Theorem 2.2 *For every second order sentence ϕ, there is a triple $\boldsymbol{T}_\phi$ in a finite vocabulary such that if $\lambda^{<\lambda} = \lambda$, then the following are equivalent:*

1. *$\boldsymbol{T}_\phi$ has a model in λ.*
2. *ϕ has a model in every cardinal less than λ.*

Note that the following extends from finitely axiomatizable to 'arithmetic' by coding a model of arithmetic in the second order sentence. And it easy to see that the theory constructed in Theorem 2.2 is recursive.

Lemma 2.3 *For every $\boldsymbol{T}$, with finitely axiomatizable T_1, there is a second order $\phi_{\boldsymbol{T}}$, such that $\phi_{\boldsymbol{T}}$ has a model in λ if and only if $\boldsymbol{T}$ has a model in λ.*

Since T_1 is finitely axiomatizable, it is easy to write second order sentence θ such that if $M \models \theta$, $M \models T_1$, M omits p and $M \restriction \tau$ is saturated. $\square_{2.3}$

We now deduce Theorem 0.2 from these two results.

Claim 2.4 $H(P_N^f) \leq \ell^2(L^{II})$ *where L^{II} denotes second order logic.*

Proof. Lemma 2.3 shows that for any $\boldsymbol{T}$, there is a $\phi_{\boldsymbol{T}}$ with $\mathrm{spec}(\boldsymbol{T}) = \mathrm{spec}(\phi_{\boldsymbol{T}})$. Suppose for contradiction that $H(P_N^f) > \ell^2(L^{II})$. Then there is a triple $\boldsymbol{T}$ with a bounded spectrum and the bound is greater than $\ell^2(L^{II})$. But then, $\neg\phi_{\boldsymbol{T}}$ has a model and $\min \mathrm{spec}(\neg\phi_{\boldsymbol{T}}) > \ell^2(L^{II})$.This contradicts the definition of the Löwenheim number.

$\square_{2.4}$

Lemma 2.5 $H(P_N^f) \geq \ell^2(L^{II})$ *where L^{II} denotes second order logic.*

Proof. Suppose for contradiction that there is a second order sentence ψ such that $\lambda_0 = \min(\mathrm{spec}^2(\psi)) > H(P_N^f)$. We apply Theorem 2.2 to $\neg\psi^*$ (as defined the proof of Lemma 1.3). On the one hand ψ^* has a model in cardinality λ_0 where $\lambda_0^{<\lambda_0} = \lambda_0$ and $\neg\psi$ is true on all $\mu < \lambda_1$. By Theorem 2.2, $\lambda_0 \models \boldsymbol{T}_{\neg\psi}$ and $\lambda_0 \geq H(P_N^f)$. So $\boldsymbol{T}_{\neg\psi}$ and therefore $\neg\psi^*$ has arbitrarily large models. But $\neg\psi$ has no models larger than λ_0. This contradiction yields the theorem. $\square_{2.5}$

Remark 2.6 *We could slightly more easily prove*

$$H(P_N^f) \leq \ell^2(L^{II}) \leq H(P_N^c),$$

which gives our answer to Newelski's question but is not quite as sharp. That is, if we had just required $\boldsymbol{T}_\phi$ in Theorem 2.2 to be in a countable language rather than finitely axiomatizable, this would have no effect on the proof of Lemma 2.5 and it would have simplified the proof of Theorem 2.2 since we could have worked with countably many constants and omitted the function g.

3 Essential Lemmas

Now we prove Theorem 2.2. For convenience, we list here the two vocabularies. We describe the axioms of T and T_1 below.

Notation 3.1

1. *τ contains unary predicates Q_1, Q_2, a binary relation R and partial binary functions F and F_2. It contains two constant symbols c_0, c_ω and a unary function symbol g.*

2. *τ_1 adds a unary predicate Q_0 and a binary relation $<_1$.*

Remark 3.2 (Proof Sketch) For each second order ϕ, we construct a triple $\boldsymbol{T}_\phi$. But most of the construction is independent of the particular ϕ and so we first construct a theory T_1 which does not depend on ϕ. The vocabulary τ will contain unary predicates Q_1, Q_2. The axioms will assert that Q_1, Q_2 partition the universe. Q_0 is in τ_1. Omission of the type p will guarantee that $Q_0 \subset Q_1$ is countable. Omission of the type in a model M of T_1 whose τ-reduct $\aleph_1$-saturated and some coding involving the partial order $<_0$ in τ will guarantee that in $Q_1(M)$ is well-ordered by a relation symbol $<_1$ in τ_1. A relation symbol R in τ will code subsets of Q_1 by elements of Q_2. Thus first order quantification on Q_2 will encode second order quantification on Q_1. In particular, we can code a given second order sentence ϕ and thus extend T_1 to T_ϕ. But the encoding will be 'correct' only on subsets whose every subset is coded in Q_2. But if $\mu < \lambda$ and M is λ-saturated, μ is a $<_1$-initial segment Q_1. Since $\mu < \lambda$ each subset of μ is coded by a type of size μ so the encoded semantics is correct and μ is a model of ϕ.

Proof of Theorem 2.2. We gradually introduce the vocabulary and theory explaining the use of various predicates as they are introduced; we repeat a bit of the proof sketch. Below we say certain conditions hold to mean they hold in any model of T. In particular, τ contains unary predicates Q_1, Q_2 that partition the universe.

There is a binary relation $<_0$, which is a partial order of Q_1. There is a partial function F mapping $Q_1 \times Q_1$ into Q_1. We write F_a for the partial function from Q_1 into Q_1 indexed by a. The partial order $<_0$ satisfies: $a \leq_0 b$ implies $F_a \subset F_b$.

We have two further properties of F. F_{c_0} is the empty function. For every $a \in Q_1$ and every $c \in Q_1$, if $c \notin \operatorname{dom} F_a$, then there are $b, d \in Q_1$ with $a <_0 b$ and $F_b = F_a \cup \{\langle c, d\rangle\}$.

Further there is a pairing function F_2 on Q_1 and an extensional relation R between Q_1 and Q_2 so that each element of Q_2 codes a subset of Q_1 via R. We write U_b for $\{a : R(b, a)\}$ (for $a \in Q_1$ and $b \in Q_2$).

T asserts that Q_1 is preserved by g, that g is a permutation, and $Q_1(c_0)$.

The set of $\{U_a : a \in Q_2\}$ is closed under Boolean operations and if U_b is such a set so is $F_a(U_b)$ for any $a \in Q_1$. For each $a \in Q_1$, there is $b \in Q_2$ such that $U_b = \{c : c <_1 a\}$.

Now we turn to the description of τ_1 and T_1. In τ_1, there is a unary relation Q_0 such that $Q_0 \subset Q_1$ and T_1 asserts Q_0 is preserved by g and c_0, c_ω are in Q_0. Thus, each $g^i(c_0) \in Q_0$. Further, there is a binary τ_1-relation $<_1$, which is a linear order of Q_1 and such that on Q_1, $x <_1 g(x)$ and $x < c_\omega$ implies $g(x) < c_\omega$. Thus, $\langle g^i(c_0) : i < \omega\rangle \cup \{c_\omega\}$ name countably many elements of Q_1 which are $<_1$- ordered in order type $\omega + 1$. T_1 further asserts $(Q_1, <_1)$ is 'pseudo-well-ordered' in the following sense. For every $a \in Q_2$, if U_a is non-empty, it has a $<_1$-least element.

The type p asserts $Q_0(x)$ and x is not a $g^i(c_0)$.

Claim 3.3 *If a model M of T_1 is such that its reduct to τ is an $\aleph_1$-saturated model of T but M omits p, $(Q_1, <_1)$ is a well-ordering in M.*

Proof. Suppose there is a countable $<_1$-descending chain $B = \{b_i : i < \omega\}$ in $(Q_1, <_1)$. Using the properties of F, we can define a $<_0$-increasing chain of a_n in Q_1 such that $F_{a_n} = \{\langle c_1, b_1\rangle, \ldots \langle g^n(c_0), b_n\rangle\}$, where the $g^i(c_0)$ are images of c_0 by iterating g. Since the model is $\aleph_1$-saturated there is an $a_\omega \in Q_1$ such that each $F_{a_n} \subset F_{a_\omega}$. But then $B = F_{a_\omega}(\{g^i(c_0) : i < \omega\})$. Note that while the choice of b_i involved the τ_1-symbol $<_1$, the existence of a_ω is by the consistency of a τ-type so the use of saturation is legitimate.

Since M omits p, $\{g^i(c_0) : i < \omega\} = \{a : a <_1 c_\omega\}$ and therefore is coded by an element of Q_2. By the closure properties of the coded sets, $B = U_d$ for some $d \in Q_2$. This contradicts the pseudo-well-ordering of Q_1. $\square_{3.3}$

Now translate ϕ to the first order formula $\phi^*(v)$ by translating each second bound order variable X to a first order formula in x and v. Replace each occurrence of $X(z)$ by $R(z, v) \wedge R(z, x)$. This translation has the following consequence. (This is immediate for monadic second order but we included a pairing function F_2 on Q_1 so it extends to arbitrary sentences.)

Fact 3.4 *If $M \models T$, $c \in Q_2(M)$ and each subset of U_c is coded by an element of $Q_2(M)$, then $M \models \phi^*(c)$ if and only $U_c(M) \models \phi$.*

Add the following axiom to T_1 to obtain T_ϕ

$$(\forall u)(\forall w)[((\forall z)R(z, w) \leftrightarrow z <_1 u) \rightarrow \phi^*(w)].$$

Claim 3.5 *If $\mu < \lambda = \lambda^{<\lambda}$ and M is model of T_ϕ with cardinality λ that omits p but whose reduct to τ is saturated then $\mu \models \phi$.*

Conversely, if ϕ is true on all $\mu < \lambda = \lambda^{<\lambda}$, there is a model M_1 of T_ϕ with cardinality λ that omits p but whose reduct to τ is saturated.

Proof. Since $\mu < \lambda$, μ is an initial segment of Q_1 so $\mu = \{a \in Q_1 : R(y, d)\}$ for some $d \in Q_2$. But then each subset Y of μ gives rise to a type $q_Y(x)$:

$$\{R(y, d)\} \cup \{R(y, x) : y \in Y\} \cup \{\neg R(y, x) : y \notin Y\}.$$

For each Ym the τ-type $q_Y(x)$ has cardinality less than λ and so us realized by saturation. We finish by Fact 3.4.

For the converse, well-order Q_1 by $<_1$ in order type λ. Add in Q_2 a code for each subset of cardinality $< \lambda$. Let the F_a list the partial functions of cardinality less than λ from Q_1 to Q_1 and let $<_0$ denote the natural partial ordering on Q_1 induced by inclusion of the named functions. Since ϕ is true below λ, each infinite initial segment in λ defines a model of ϕ and the definition of T_ϕ shows that we have a saturated model of T when we take the reduct to τ. Finally, let Q_0 include exactly the first ω elements of Q_1.

$\square_{3.5}$

Letting $\boldsymbol{T}_\phi$ be the triple (T, T_ϕ, p) we have a triple satisfying Theorem 2.2.

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