Fine Classification of strongly minimal sets
SIU Logic Seminar

John T. Baldwin
on-line from University of Illinois at Chicago

Oct. 15, 2020
Overview

1. Classifying strongly minimal sets by their acl-geometries

2. Quasi-groups and Steiner systems

3. Coordinatization by varieties of algebras

4. Finer Classification: dcl(X), acl(X), sdcl

Thanks to Joel Berman, Gianluca Paolini, Omer Mermelstein, and Viktor Verbovskiy.
Classifying strongly minimal sets and their $acl_1$-geometries
**Definition**

*T* is **strongly minimal** if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor
**Definition**

*T* is **strongly minimal** if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

**Definition**

*a* is in the **algebraic closure** of *B* (*a* ∈ acl(*B*)) if for some \( \phi(x, b) \):

\[ \models \phi(a, b) \] with *b* ∈ *B* and \( \phi(x, b) \) has only finitely many solutions.
Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

**Definition**

A **closure system** is a set $G$ together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

- **A1.** $cl(X) = \bigcup \{ cl(X') : X' \subseteq_{\text{fin}} X \}$
- **A2.** $X \subseteq cl(X)$
- **A3.** $cl(cl(X)) = cl(X)$

$(G, cl)$ is **pregeometry** if in addition:

- **A4.** If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $cl(x) = x$ the structure is called a **geometry**.
The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. Zilber Conjecture
   1. unary (lattice of subspaces distributive) (trivial, discrete, disintegrated)
   2. vector space-like (lattice of subspaces modular)
   3. ‘bi-interpretable’ with an algebraically closed field (non-locally modular)

2. Plus
   1. Flat: Combinatorial class
   2. Are there more?

What was the Zilber conjecture?

Conditions on the acl-geometry imply conditions on the algebra of the structure.
Are there finer measures?
Motive 0: The diversity of *ab initio* strongly minimal sets

*ab initio*: The infinite structure is defined a class $K_0$ of finite structures for a first order theory.

Each of the following may affect properties of an ab initio strongly minimal set that are

- model theoretic
- and/or of wider mathematical interest.

1. the vocabulary $\tau$ including number of relations (sorts)
2. axioms $T_0$ on the class $K_0$ of the finite $\tau$-models
3. the function $\delta$ from $K_0$ to $\mathbb{Z}$
4. the way $K_0$ is determined from $\delta$ and $T_0$
5. the function $\mu$ from ‘good pairs’ to $\mathbb{N}$.

A field guide to Hrushovskii Constructions [Bal]
A Steiner $k$ system is collection of points and lines (block) such that

1. two points determine a line
2. each line has exactly $k$-points.

We work in a vocabulary with one ternary relation $R$ for collinearity. We say the system is $S(2, k, n)$ if there are $n$-points and lines have length $k$. 
Connections with number theory

For which \( n \)'s does an \( S(2, k, n) \) exist?

for 3 point lines:

Necessity:
\[ n \equiv 1 \text{ or } 3 \pmod{6} \] is necessary.

Rev. T.P. Kirkman (1847)
Connections with number theory

For which n’s does an $S(2, k, n)$ exist?
for 3 point lines:

Necessity:
$n \equiv 1 \ or \ 3 \ (mod\ 6)$ is necessary.
Rev. T.P. Kirkman (1847)

Sufficiency:

$n \equiv 1 \ or \ 3 \ (mod\ 6)$ is sufficient.
(Bose $6n + 3$, 1939), Skolem ( $6n + 1$, 1958)
## Linear Spaces

### Definition (1-sorted)

The vocabulary $\tau$ contains a single ternary predicate $R$, interpreted as collinearity.

$K_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems:

1. $R$ is a predicate of sets (hypergraph)
2. Two points determine a line

$K^*$ includes infinite linear spaces.

### Groupoids and semigroups

1. A groupoid (magma) is a set $A$ with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancellation (Latin Square)
3. A Steiner quasigroup satisfies

   $$x \circ x = x, \quad x \circ y = y \circ x, \quad x \circ (x \circ y) = y.$$
existentially closed Steiner Systems

**Barbina-Casanovas**

[BC1x] Consider the class $\tilde{K}$ of finite structures $(A, R)$ which are the graphs of a Steiner quasigroup.

1. $\tilde{K}$ has ap and jep and thus a limit theory $T_{sqg}$.
2. $T_{sqg}$ has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and **locally finite**;
   4. $T_{sqg}$ has $TP_2$ and $NSOP_1$. 
Strongly minimal linear spaces

Fact
Suppose \((M, R)\) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary
If \((M, R)\) is a strongly minimal linear space for some \(k\), it is a Steiner \(k\)-system.
Ab Initio Strongly minimal Steiner Systems

**Definition**

A Steiner \((2, k, v)\)-system is a linear system with \(v\) points such that each line has \(k\) points.

**Theorem (Baldwin-Paolini)[BP20]**

For each \(k \geq 3\), there are an uncountable family \(T_\mu\) of strongly minimal \((2, k, \infty)\) Steiner-systems.

There is no infinite group definable in any \(T_\mu\). More strongly, Associativity is forbidden.
Hrushovski construction for linear spaces

$K_0^*$ denotes the collection of finite linear spaces in the vocabulary $\tau = \{R\}$.

A line in a linear space is a maximal $R$-clique $L(A)$, the lines based in $A$, is the collections of lines in $(M, R)$ that contain 2 points from $A$.

**Definition: Paolini’s $\delta$**

[Pao21] For $A \in K_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$K_0$ is the $A \in K_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovski functions based on the cardinality of maximal cliques.
Definition

Let $A, B, C \in K_0$.

1. $C$ is a 0-\textit{primitive extension} of $A$ if $C$ is minimal with $\delta(C/A) = 0$.

2. $C$ is good over $B \subseteq A$ if $B$ is minimal contained in $A$ such that $C$ is a 0-\textit{primitive extension} of $B$. We call such a $B$ a base.

In Hrushovski’s examples the base is unique. But not in linear spaces.

$\alpha$ is the isomorphism type of $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Instances of $\alpha$ determine a line in linear spaces.
The $\mu$ function

Context

Let $\mathcal{U}$ be a collection of functions $\mu$ assigning to every isomorphism type $\beta$ of a good pair $C/B$ in $K_0$:

(i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;

(ii) a number $\mu(\beta) \geq 1$, if $\beta = \alpha$

$T_\mu$ is the theory of a strongly minimal Steiner $(\mu(\alpha) + 2)$-system

If $\mu(\alpha) = 1$, $T_\mu$ is the theory of a Steiner triple system, bi-interpretable with a Steiner quasigroup.
Key features of constructed model

\[ A \leq M \text{ if } A \subseteq B \subseteq M \text{ implies } \delta(A/B) \geq 0 \]

When \((K_0, \leq)\) has joint embedding and amalgamation there is unique countable generic.

Realization of good pairs

1. A good pair \(A/B\) well-placed by \(D\) in a model \(M\), if \(B \subseteq D \leq M\) and \(A\) is 0-primitive over \(D\).

2. For any good pair \((A/B)\), \(\chi_M(A/B)\) is the maximal number of disjoint copies of \(A\) over \(B\) appearing in \(M\).

If \(C/B\) is well-placed by \(D \leq M\), \(\chi_M(B, C) = \mu(B/C)\).

Primitive singleton

\(\alpha\) is the isomorphism class of the good pair \((\{a, b\}, \{c\})\) with \(R(a, b, c)\).
Stages of the construction

1. $K_0^*: \text{all finite linear } \tau\text{-spaces}.$
2. $K_0 \subseteq K_0^*: \delta(A) \text{ hereditarily } \geq 0.$
3. $K_\mu \subseteq K_0: \chi_M(A, B) \leq \mu(A, B) \text{ } \mu \text{ bounds the number of disjoint realizations of a ‘good pair’}$
   Now Fraïssé.
4. For $\mu \in \mathcal{U}$, $K_\mu$ is the collection of $M \in K_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair $(A, B)$.
5. $X$ is $d$-closed in $M$ if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0.$).
6. Let $K_\mu^d$ consist of those $M \in K_\mu$ such that $M \leq N$ and $N \in \hat{K}_\mu$ implies $M$ is $d$-closed in $N$.

$T_\mu$ is the common theory of the models in $K_\mu^d$. 
Main existence theorems

When \((K_0, \leq)\) has joint embedding and amalgamation there is unique countable generic.

**Theorem: Paolini [Pao21]**

There is a generic model for \(K_0\); it is \(\omega\)-stable with Morley rank \(\omega\).

**Theorem (Baldwin-Paolini)[BP20]**

For any \(\mu \in \mathcal{U}\), there is a generic strongly minimal structure \(G_\mu\) with theory \(T_\mu\).

If \(\mu(\alpha) = k\), all lines in any model of \(T_\mu\) have cardinality \(k + 2\). Thus each model of \(T_\mu\) is a Steiner \(k\)-system and \(\mu(\alpha)\) is a fundamental invariant.

Proof follows Holland’s [Hol99] variant of Hrushovski’s original argument.

New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. \(\alpha\)).
Coordinatization by varieties of algebras
Coordinatizing Steiner Systems

Weakly coordinatized
A collection of algebras $V$ "weakly coordinatizes" a class $S$ of $(2, k)$-Steiner systems if
1. Each algebra in $V$ definably expands to a member of $S$
2. The universe of each member of $S$ is the underlying system of some (perhaps many) algebras in $V$.

Many results showing arithmetical conditions for the existence of $(S(t, k, n))$ that could be weakly coordinatized. Conclusion: $t$ better be 2.
Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras $V$ "weakly coordinatizes" a class $S$ of $(2, k)$-Steiner systems if

1. Each algebra in $V$ definably expands to a member of $S$
2. The universe of each member of $S$ is the underlying system of some (perhaps many) algebras in $V$.

Many results showing arithmetical conditions for the existence of $(S(t, k, n))$ that could be weakly coordinatized. Conclusion: $t$ better be 2.

Coordinatized

A collection of algebras $V$ "coordinatizes" a class $S$ of $(2, k)$-Steiner systems if

in addition the algebra operation is definable in the Steiner system.
Euclid used the Archimedean axiom to define proportionality of segments.

Descartes used Euclid’s theorem of the 4th proportional to define multiplication.

Hilbert inverted the procedure to define multiplication (and so interpret a field) in any Desarguesian plane and then define proportionality.

Thus Descartes ‘weakly coordinatizes’ any plane (over ‘Euclidean’ subfields of the reals)

Hilbert coordinatizes all Euclidean planes (by Euclidean fields)
2 VARIABLE IDENTITIES

Definition
A variety is **binary** if all its equations are 2 variable identities: [Eva82]

Definition
Given a (near)field \((F, +, \cdot, -, 0, 1)\) of cardinality \(q = p^n\) and an element \(a \in F\), define a multiplication \(\ast\) on \(F\) by

\[ x \ast y = y + (x - y)a. \]

An algebra \((A, \ast)\) satisfying the 2-variable identities of \((F, \ast)\) is a **block algebra** over \((F, \ast)\) (a quasigroup).
Motive 1

Key fact: weak coordinatization [Ste64, Eva76]
If $V$ is a variety of binary, idempotent algebras and each block of a Steiner system $S$ admits an algebra from $V$ then so does $S$.

Consequently
If $V$ is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality $k$, each $A \in V$ determines a Steiner $k$-system. (The 2-generated subalgebras.) And for prime power $k$, each strongly minimal Steiner $k$-system admits a weak coordinatization by $V$.

Question 1
Can this coordinatization be definable in the strongly minimal $(M, R)$? Is there even a definable binary function?
Forcing a prime power

**Theorem**

If a $k$-Steiner system is weakly coordinatized, $k$ is a prime power $q^n$.

Proof: As, if an algebra $A$ is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?
Strongly minimal block algebras \((M, R, \ast)\)

**Theorem: Baldwin**

For every prime power \(q\) there is a strongly minimal Steiner \(q\)-system \((M, R)\) whose theory is interpretable in a strongly minimal block algebra \((M, R, \ast)\).

We modify the collection of \(R\)-structures \(K_{\mu}\) to a collection \(K_{\mu'}\) of \((R, \ast)\) structures so that the generic is a strongly minimal quasigroup that induces a Steiner system.

**Theorem: Baldwin-Verbovskiy**

But for \(k > 3\) the coordinatization CAN NOT BE defined in the strongly minimal \((M, R)\).

In searching for a proof we find a related problem.
Finer Classification: dcl(X), acl(X), sdcl
Motive 2

Definition

A theory $T$ admits elimination of imaginaries if for every model $M$ of $T$, for every formula $\varphi(x, y)$ and for every $\bar{a} \in M^n$ there exists $\bar{b} \in M^m$ such that

$$\{ f \in \text{aut}(M) \mid f\bar{b} = \text{id}_b \} = \{ f \in \text{aut}(M) \mid f(\varphi(M, \bar{a}, )) = \varphi(M, \bar{a}) \}.$$ 

Let $T_\mu$ be theory of the basic Hrushovski example.

Question 2

Does $T_\mu$ admit elimination of imaginaries?

Baizanov, Verbovskiy
\textbf{dcl\textsuperscript{*} and definability of functions}

\textbf{*-closure}

\textit{dcl}(X) and \textit{acl}(X) are the definable and algebraic closures of set $X$. $b \in \text{dcl}\textsuperscript{*}(X)$ means $b \notin \text{dcl}(U)$ for any proper subset of $X$. 

\textbf{Fact}

Let $I$ be two independent points in $M$. If $\text{dcl}\textsuperscript{*}(I) = \emptyset$ then no binary function is $\emptyset$-definable in $M$. That is, $\text{dcl}$ is generically $2$-trivial: if $a, b$ are independent $\text{dcl}(\{a, b\}) = \text{dcl}(a) \cup \text{dcl}(b)$. 

John T. Baldwin on-line from University of Illinois at Chicago 

Fine Classification of strongly minimal sets SIU Logic Seminar 

Oct. 15, 2020 26 / 47
**dcl* and definability of functions**

*-closure

dcl\((X)\) and acl\((X)\) are the definable and algebraic closures of set \(X\).

\(b \in \text{dcl}^* (X)\) means \(b \notin \text{dcl} (U)\) for any proper subset of \(X\).

**Fact**

Let \(l\) be two independent points in \(M\).

If \(\text{dcl}^*(l) = \emptyset\) then no binary function is \(\emptyset\)-definable in \(M\).

That is, \(\text{dcl}\) is *generically 2-trivial*: if \(a, b\) are independent

\(\text{dcl}\{a, b\} = \text{dcl}(a) \cup \text{dcl}(b)\).
Finite Coding

Definition

A finite set \( F = \{\bar{a}_1, \ldots, \bar{a}_k\} \) of tuples from \( M \) is said to be coded by \( S = \{s_1, \ldots, s_n\} \subset M \) over \( A \) if

\[
\sigma(F) = F \iff \sigma|_S = \text{id}_S \quad \text{for any } \sigma \in \text{aut}(M/A).
\]

We say \( T = \text{Th}(M) \) has the finite set property if every finite set of tuples \( F \) is coded by some set \( S \) over \( \emptyset \).

If \( \text{dcl}^*(I) = \emptyset \), \( T \) does not have the finite set property.
Fact: Elimination of imaginaries

A theory $T$ admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no; combinatorial: no (here)

Fact

*If $T$ admits weak elimination of imaginaries then $T$ satisfies the finite set property if and only $T$ admits elimination of imaginaries.*

Since every strongly minimal theory has weak elimination of imaginaries, we have

*If a strongly minimal $T$ has no definable binary functions it does not admit elimination of imaginaries.*
Fix $I$ as two independent points in the generic model $M$ of $T_\mu$.

**2 groups**

Let $G\{I\}$ be the set of automorphisms of $M$ that fix $I$ setwise and $G_I$ be the set of automorphisms of $M$ that fix $I$ pointwise.

Note that $\text{dcl}^*(X)$ consists of those elements are fixed by $G_I$ but not by $G_X$ for any $X \subsetneq I$.

**symmetric definable closure**

The *symmetric definable closure* of $X$, $\text{sdcl}(X)$, is those $a$ that are fixed by every $g \in G\{X\}$.

$b \in \text{sdcl}^*(X)$ exactly when $b \in \text{sdcl}(S)$ but $b \notin \text{sdcl}(U)$ for any proper subset $U$ of $X$. 


No definable binary function/elimination of imaginaries: Sufficient

Lemma

Let \( I = \{a_0, a_1\} \) be an independent set with \( I \leq M \) and \( M \) is a generic model of a strongly minimal theory.

1. If \( \text{sdcl}^*(I) = \emptyset \) then \( I \) is not finitely coded.
2. If \( \text{dcl}^*(I) = \emptyset \) then \( I \) is not finitely coded and there is no parameter free definable binary function.
No definable binary function/elimination of imaginaries

Theorem (B-Verbovskiy)

Suppose $T_\mu$ has only a ternary predicate (3-hypergraph) $R$. If $T_\mu$ is either in

1. Hrushovski's original family of examples
2. or one of the B-Paolini Steiner systems

and also satisfies:

1. $\mu \in \mathcal{U}$
2. If $\delta(B) = 2$, then $\mu(B/C) \geq 3$ except
3. $\mu(\alpha) \geq 2$ (for linear spaces)

If $I$ is an independent pair $A \leq M \models T_\mu$, then

(i) $\text{dcl}^*(I) = \emptyset$
(ii) $T_\mu$ does not admit elimination of imaginaries.
Counterexample: $\text{dcl}^*(l) \neq \emptyset$

There are examples (Verbovskiy) of the Hrushovski construction and linear spaces with $\mu(C/B) = 2$ and $|B| = 2$ and $\text{dcl}^*(l) \neq \emptyset$.

$A_1^1$ and $A_2^1$ are isomorphic and primitive over $A^0$. $A_1^2$ is 0-primitive over $B$ and can be mapped into $A^0A_1^1A_2^1$ over $B$ taking $a_1$ to $a$. Obviously this is not an isomorphism over $A^0$. 
Definition

\( A \subseteq M \) is \( G \)-normal if

1. \( A \leq M \)
2. \( A \) is \( G \)-invariant
3. \( A \subset_{<\omega} \text{acl}(I) \).

Fact

There are \( G \)-decomposable sets. Namely for any finite \( U \) with \( d(U/I) = 0 \),

\[ A = \text{icl}(I \cup G(U)) \]
Constructing a $G$-decomposition

Linear Decomposition
Constructing a \( G \)-decomposition

Linear Decomposition

Tree Decomposition

Prove by induction on levels that \( \text{dcl}^*(I) = \emptyset \). (\( \text{sdcl}^*(I) = \emptyset \))
First thought: All petals move

To show $\text{dcl}^*(X) = 0$:

Trap: Not all petals are moved by $G$
First thought: All petals move

To show $\text{dcl}^*(X) = 0$:

Trap: Not all petals are moved by $G$
Dual Induction

Lemma

Assume that $\hat{T}_\mu$ satisfies $\mu(A/B) \geq 3$ for any good pair $(A/B)$, where $\delta(B) = 2$. For $m \geq 1$,

1. $\dim_m$: $d(E) = 2$ for any $G_I$-invariant set $E \subseteq A^m$, which is not a subset of $A^0$.

2. $\text{moves}_m$: No $A^m_{f,k}$ is $G_I$-invariant.

Key step

$\dim_m$ and $\text{moves}_{m+1}$ implies $\dim_{m+1}$. 
How $\mu$ matters

$G\{I\}$-decomposition permits removal of the hypothesis:
If $\delta(B) = 2$, then $\mu(B/C) \geq 3$
and elimination of imaginaries still fails.

Theorem

[BV20] Let $T_\mu$ be the theory of the basic Hrushovski construction or a strongly minimal Steiner system
Let $I = \{a, b\}$ be a pair of independent points.

1. $\text{sdcl}^*(I) = \emptyset$.
2. Further, if for any good pair $C/B$,
   $\delta(B) = 2$ implies $\mu(C/B) \geq 3$,
   then $T_\mu$ is generically-discrete ($\text{dcl}^*(I) = \emptyset$).

Consequently, no such $T$ admits elimination of imaginaries.

In particular the Steiner systems $(M, R)$ do not interpret quasigroups.
The $n$-ample hierarchy

$n$-Ample for $1 \leq n < \omega$ ([Eva11, BMPZ14] requires the existence of a sequence of tuple with a certain specified interaction of $acl^eq$ and forking in a stable first order theory $T$.

Thus, a property of the entire theory rather than of the $acl$-geometry of particular strongly minimal sets.
Non-trivial flat strongly minimal sets

Flat: dimension computed by inclusion-exclusion.

Non-trivial flat and so 1-ample but not 2-ample

1. Strongly minimal theories
   (i) no binary function (generically 2-discrete)
   (ii) no commutative binary function (no elim imag)
   (iii) definable binary functions exist
       (a) strongly minimal quasigroups: \((M, R, \star)\) [Bal20] and an example of Hrushovski [Hru93, Proposition 18]
       (b) strongly minimal theory that eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)

2. Almost strongly minimal theories
   (i) Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
   (ii) almost strongly minimal buildings [Ten00a, Ten00b]

2-ample but not 3-ample sm sets (not flat) [MT19]
The Ample Hierarchy of stable theories
Further Questions

1. generic $n$-discrete and arbitrary finite relational language
2. explore the consequences in combinatorics and universal algebra
3. How do the examples of Andrews in computable model theory fit into the scheme? How does the Muller-Tent example fit in the Zilber classification?
References I

John T. Baldwin.  
A field guide to Hrushovski constructions.  

John T. Baldwin.  
Some projective planes of Lenz Barlotti class I.  

John T. Baldwin.  
Strongly minimal Steiner Systems II: Coordinatization and Strongly Minimal Quasigroups.  


References III


D. Evans.  
An introduction to ampleness.  

Kitty Holland.  
Model completeness of the new strongly minimal sets.  

E. Hrushovski.  
A new strongly minimal set.  
M. Mermelstein.  
Geometry preserving reducts of hrushovskis non-collapsed construction.  

I. Muller and K. Tent.  
Building-like geometries of finite morley rank.  
DOI: 10.4171/JEMS/912.

Gianluca Paolini.  
New $\omega$-stable planes.  
to appear.

D.J.S. Robinson.  
*A Course in the Theory of Groups*.  
Springer-Verlag, 1982.
S. Świerczkowski.
Algebras which are independently generated by every \( n \) elements.  

Sherman K Stein.
Homogeneous quasigroups.  

Katrin Tent.
A note on the model theory of generalized polygons.  

Katrin Tent.
Very homogeneous generalized polygons of finite Morley rank.  