# Nonsplitting extensions

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

February 14, 2004

The existence of nonsplitting extensions of a Galois type $p \in \mathrm{ga}-\mathrm{S}(M)$ to a model $M'$ of the same cardinality such that $M'$ is universal over $M$ is obtained in [2] and, in more detail, in [3]. Finding such extensions where $M'$ has larger cardinality requires a different technique. The argument here uses only 'relative saturation' properties of models of the form $EM(\mu^{<\omega}, \Phi)$, which are proved in [1]. The application to categoricity transfer requires the characterization of saturated models as having this form, which is also in [1]. The Corollary is from [2] but this argument is significantly shorter.

By $\mathrm{aut_N}(\mathbb{M})$ we mean the set of automorphisms of $\mathbb{M}$ which fix $N$ pointwise.

**Lemma 1** *Let $M = EM(\mu^{<\omega}, \Phi) = EM(I, \Phi)$ and $p \in S(M)$. Suppose $p$ does not split over $M_0 = EM(I_0, \Phi)$ with $I_0 \subset I$. Then there is an extension of $p$ to $N = EM(J, \Phi)$ $(J = \lambda^{<\omega})$ which does not split over $M_0$.*

Proof. Let $\tau(\boldsymbol{a}) \in N$ realize $p$. Without loss of generality $(\boldsymbol{a} \cap M) \subset I_0$. Now extend $J$ to $J'$ by adding a finite sequence $\boldsymbol{a'}$ with $a'_i$ in the same $I$-cut as $a_i$ but $a'_i$ greater than any element of $J$ in that cut. Now since $\boldsymbol{a}$ and $\boldsymbol{a'}$ realize the same type in the language of orders over $I$, a compactness argument shows there is an extension $J''$ of $J$ and an automorphism $f$ of $J''$ which fixes $I$ and maps $\boldsymbol{a}$ to $\boldsymbol{a'}$. Thus, $\alpha = \tau(\boldsymbol{a'})$ realizes $p$. Consider any $N_1, N_2$ in $N$ and a $\boldsymbol{K}$-map $h$ over $M_0$ which maps $N_1$ to $N_2$; by homogeneity of the monster, $h$ extends to an $\eta \in \mathrm{aut}(\mathbb{M})$. Let $p' = \mathrm{tp}(\tau(\boldsymbol{a'})/N)$. We will show there is a map $\hat{\eta} \in \mathrm{aut}_{\mathrm{N_2}}(\mathbb{M})$ witnessing that $h(p' \restriction N_1) = p' \restriction N_2$; that is $\hat{\eta}$ maps $\eta(\alpha)$ to $\alpha$. Choose $K \subseteq J$ such that $N_1, N_2 \prec_{\boldsymbol{K}} N_3 = EM(K, \Phi)$. By the 'relative saturation' of $I$ in $J$ there is an order isomorphism $g$ fixing $I_0$ and mapping $K$ to $K' \subseteq I$. Moreover, wolog we take the domain of $g$ to be $I_0K\boldsymbol{a'}$ and $g$ fixes $\boldsymbol{a'}$. Thus $g$ extends to a $\gamma \in \mathrm{aut}(\mathbb{M})$, such that $\gamma$ fixes $EM(I_0\boldsymbol{a}, \Phi)$. In particular, $\gamma$ fixes $\alpha M_0$. Let $N'_1$ denote $\gamma(N_1)$ and $N'_2$ denote $\gamma(N_2)$. Then $\eta' = \gamma\eta\gamma^{-1} \in \mathrm{aut}_{\mathrm{M_0}}(\mathbb{M})$ takes $N'_1$ to $N'_2$. Since $p$ does not split over $M_0$, there is an $\hat{\eta}' \in \mathrm{aut}_{\mathrm{N'_2}}(\mathbb{M})$, taking $\eta'(\alpha)$ to $\alpha$. We claim $\gamma^{-1}\hat{\eta}'\gamma$ is the required $\eta$; clearly this map fixes $N_2$. But since $\gamma$ fixes $\alpha$, $\gamma^{-1}\hat{\eta}'\gamma(\eta(\alpha)) = \gamma^{-1}\hat{\eta}'\gamma(\eta(\gamma^{-1}\alpha))$ which is $\gamma(\hat{\eta}'(\eta'(\alpha)))$ which equals $\alpha$ by the choice of $\hat{\eta}'$ and since $\gamma$ fixes $\alpha$.

Modulo the easy fact from [2] that types over saturated models don't split over a smaller model, we have proved:

**Corollary 2** *Suppose an aec with amalgamation and joint embedding is categorical in $\lambda$. If $M \prec_{\boldsymbol{K}} N$ are saturated models in $\boldsymbol{K}$ with $|M| < |N| \leq \lambda$ and $p \in \mathrm{ga}-\mathrm{S}(M)$, $p$ has a nonsplitting extension to $\mathrm{ga}-\mathrm{S}(N)$.*

## References

[1] J.T. Baldwin. Ehrenfreuht-Mostowski models in abstract elementary classes. submitted, 200?

[2] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394.

[3] Monica VanDieren. Categoricity in abstract elementary classes with no maximal models.