A Hanf number for saturation and omission: the superstable case

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Abstract

Suppose $\boldsymbol{t} = (T, T_1, p)$ is a triple of two theories in vocabularies $\tau \subset \tau_1$ with cardinality λ and a τ_1-type p over the empty set. We show the Hanf number for the property: 'There is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated' is less than $\beth_{(2^{(2^\lambda)^+})^+}$ if T is superstable.

We showed in [2] that with no stability restriction the Hanf number is essentially equal to the Löwenheim number of second order logic.

Hanf observed [4] that if one asks for each $\boldsymbol{K}$ in a set of classes of structures, 'Does $\boldsymbol{K}$ have arbitrarily large members?', there is a cardinal κ (the sup of the maxima of the bounded $\boldsymbol{K}$) such that any class with a member at least of cardinality κ has arbitrarily large models. In many cases this bound κ can be calculated (For a countable first order theory, it is $\aleph_0$.) In this paper we call a Hanf number for a family $\mathcal{K}$ of classes *calculable* if it is bounded by a function that can be computed by an arithmetic function in ZFC (See Definition 0.1.) and if not it is *incalculable*.

The following definition is more abstract than needed for this paper but we include it for comparison with other works where other Hanf functions are shown to be not calculable.

Definition 0.1 *1. A function f (a class-function from cardinals to cardinals) is* strongly calculable *if f can (provably in ZFC) be defined in terms of cardinal addition, multiplication, exponentiation, and iteration of the $\beth$ function.*

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2. *A function f is* calculable *if it is (provably in ZFC) eventually dominated by a strongly calculable function. If not, it is* incalculable.

We extend our work on Newelski's [6] question about calculating the Hanf number of the following property:

Definition 0.2 *1. We say $M_1 \models \boldsymbol{t}$ where $\boldsymbol{t} = (T, T_1, p) = (T_{\boldsymbol{t}}, T_{1,\boldsymbol{t}}, p_{\boldsymbol{t}})$ is a triple of two theories T, T_1 in vocabularies $\tau \subset \tau_1$, respectively, such that $|\tau_1| \leq \lambda$, $T \subseteq T_1$ and p is a τ_1-type over the empty set if M_1 is a model of T_1 which omits p, but $M_1 \restriction \tau$ is saturated.*

2. Let $\boldsymbol{N}_\lambda$ denote[1] *the set of $\boldsymbol{t}$ with $|\tau_1| = \lambda$. Then $H(\boldsymbol{N}_\lambda)$ denotes the Hanf number of $\boldsymbol{N}_\lambda$, $H(\boldsymbol{N}_\lambda)$ is least so that if $\boldsymbol{t} \in \boldsymbol{N}_\lambda$ has a model of cardinality $H(\boldsymbol{N}_\lambda)$ it has arbitrarily large models.*

3. The Hanf number of a logic $\mathcal{L}$ (e.g. $L_{\kappa^+,\kappa}$) is the least cardinal μ such that if an $\mathcal{L}$-sentence has model in cardinal μ, then it has arbitrarily large models.

Under mild set theoretic hypotheses, we showed in [2] that $H(\boldsymbol{N}_\lambda)$ equals the Löwenheim number of second order logic, which is incalculable. In Section 1 we restrict the question by requiring that the theory T be superstable case; the number is then easily calculable in terms of Beth numbers.

The phenomena that stability considerations can greatly lower Hanf number estimates was earlier explored in [5]. Work in preparation extends the current context to strictly stable theories. References of the form X.x.y are to [7].

Much of this paper depends on a standard way of translating between sentences in languages of the form $L_{\lambda,\omega}(\tau)$ and first order theories in an expanded vocabulary τ that omit a family of types. This translation dates back to [3]; a short explanation of the process appears in Chapter 6.1 of [1]. Chapter VII.5 of [7] is an essential reference for this paper. There, these (equivalent) Hanf numbers of sentences and associated pair of a family of types and theory are calculated using the 'well-ordering number of a class'. We begin with a slight rewording of Definition VII.5.1 of [7], using language from [3].

Definition 0.3 *1. The Morley number $\mu(\lambda, \kappa)$ is the least cardinal μ such that if a first order theory T is a vocabulary of cardinality λ has a model in cardinality μ which omits a family of κ types over the empty set, it has arbitrarily large such models.*

2. The well-ordering number $\delta(\lambda, \kappa)$ is the least ordinal α such that if a first order theory T is a vocabulary τ of cardinality λ, which includes a symbol $<$ has a model in which omits a family κ types over the empty set $<$ is well ordering of type α, then there is such a model where $<$ is not a well-order.

The connection between these two notions is in section VII of [7].

Fact 0.4 *1. If $\kappa > 0$, $\mu(\lambda, \kappa) = \beth_{\delta(\lambda,\kappa)}$.*

[1] Thus, 'there is an $M \in \boldsymbol{N}_\lambda$ with cardinality κ' replaces the more cumbersome notation in [2], '$P^\lambda_N(\boldsymbol{K}_{\boldsymbol{t}}, \kappa)$ holds'.

2. *For every infinite cardinal θ, $H(L_{\theta^+,\omega}) \leq \mu(\theta, 1) < \beth_{(2^\theta)^+}$.*

Proof. Item 1 is VII.5.4 of [7]. Recall that Lopez-Escobar and Chang (e.g. [3]) showed how to code sentences of $L_{\lambda^+,\omega}$ as first order theories omitting types and even (as in proof of VII.5.1.4) a single type. $H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+}$ is now clear from Theorems VII.5.4 and VII.5.5.7 of [7].

1 Computing $H(N_\lambda^{ss})$

1.1 Introduction

We study the following notions in this section.

Definition 1.1 *Let $\boldsymbol{N}_\lambda^{ss}$ denote the set*[2] *of $\boldsymbol{t}$ with $|\tau_1| = \lambda$ with the additional requirement that $T_{\boldsymbol{t}}$ is a superstable theory. Now we have the natural notion of the Hanf number, $H(\boldsymbol{N}_\lambda^{ss})$ for this set: If $\boldsymbol{t} \in \boldsymbol{N}_\lambda^{ss}$ has a model of cardinality $\geq H(\boldsymbol{N}_\lambda^{ss})$, it has arbitrarily large models.*

We will prove the following theorem:

Theorem 1.2

$$H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss}) < H(L_{(2^\lambda)^+,\omega}) < \beth_{(2^{(2^\lambda)^+})^+}.$$

The first and fourth of these inequalities are immediate from Fact 0.4.2 taking θ first as λ and then as 2^λ.

In Subsection 1.2, we give a rather involved proof that $\beth_{(2^\lambda)^+}$ is strictly less than $H(\boldsymbol{N}_\lambda^{ss})$; together with the first inequality, this implies immediately that $H(L_{\lambda^+,\omega}) < H(\boldsymbol{N}_\lambda^{ss})$. Note that less than or equal, $H(L_{\lambda^+,\omega}) \leq H(\boldsymbol{N}_\lambda^{ss})$ is straightforward. Just set $\boldsymbol{t}$ as (T_0, T_1, p) where T_0 is pure equality and (T_1, p) encode a given sentence $\psi \in L_{\lambda^+,\omega}$. Then T_0 is superstable and every model is saturated, so we have the desired interpretation.

The second and third inequalities are in Subsections 1.2 and 1.3, respectively.

1.2 The Second Inequality

To show $H(L_{\lambda^+,\omega}) < H(\boldsymbol{N}_\lambda^{ss})$, we actually show

Theorem 1.3 $H(L_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss})$.

As noted the first inequality in Theorem 1.2.1 is standard. The following Lemma will be key to showing $\beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss})$. We will construct a sequence of $\boldsymbol{K}_\alpha \in \boldsymbol{N}_\lambda^{ss}$ which code sentences of $L_{\lambda^+,\omega}$ which, in turn, code increasingly large

[2] Technically, this is not a set since a vocabulary is a sequence of relation symbols and we could use different names for the symbols; this pedantry can be avoided in at least two ways: restrict the symbols to come from a specified set; return to Tarski's convention of discussing not vocabularies but similarity types, the equivalence classes of enumerated vocabularies such that the ith symbol has aritity n_i.

well-orders. Then in Theorem 1.10, we will combine the $\boldsymbol{K}_\alpha$ into a single $\boldsymbol{K} \in \boldsymbol{N}^{ss}_\lambda$ which codes the supremum of these well-orders. The proof of Lemma 1.4 takes several steps.

Lemma 1.4 *Fix a sentence $\phi \in L_{\lambda^+,\omega}$ in a vocabulary of cardinality at most λ, which has a model of cardinality at least 2^λ but for some (least) $\mu = \mu_\phi$ has no model of cardinality $\geq \mu_\phi$.*

1. *There is an associated $\psi = \psi_\phi \in L_{\lambda^+,\omega}$ and a class $\boldsymbol{K}_\psi$ of models of ψ such that $\boldsymbol{K}_\psi$ satisfies Requirements 1.5 and 1.6, and each model of $\boldsymbol{K}_\psi$ has cardinality χ for some χ with $\lambda \leq \chi < \mu$.*

2. *Moreover, for each such $\boldsymbol{K}_\psi$ we can define a $\boldsymbol{t} = \boldsymbol{t}_\psi \in \boldsymbol{N}^{ss}_\lambda$ such that the following are equivalent.*

   (a)
   $$\mathrm{spec}(\boldsymbol{K}_\psi) = \{\chi : \lambda \leq \chi < \mu\}.$$

   (b)
   $$\mathrm{spec}(\boldsymbol{t}_\psi) = \{\chi : 2^\lambda \leq \chi < \mu\}.$$

Proof. First we construct $\psi = \psi_\phi$ and $\boldsymbol{K}_\psi$. Recall ϕ is in some vocabulary σ of cardinality at most λ. The vocabulary τ_* of ψ_ϕ adds to σ, unary predicates P, Q_0, Q_1, a binary relation R, a unary function G, and λ constants c_i.

Here is a sketch of the argument. By omitting a type we can guarantee P^M has cardinality λ. Q_1^M is a family of subsets of P^M. As we describe immediately after Notation 1.9, by the saturation of the reduct to a model of a superstable theory, we can guarantee that every subset of P^M is represented in Q_1^M. Via a pairing function on P^M, we can show every subset of Q_1^M with cardinality $< \lambda$ can be coded; hence we can 'say' $<^M$ well-orders Q_1^M.

Requirement 1.5 *ψ asserts that*

1. *the symbols of σ are defined only on Q_0 and if $M \models \psi$, and N is the restriction of M to Q_0^M, $N \restriction \sigma \models \phi$;*

2. *the c_i are all in P and exhaust P; P, Q_0, Q_1 partition the universe;*

3. *R is a extensional binary relation between P and Q_1; thus the elements of Q_1 code subsets of P.*

4. *G is a 1-1 function from Q_1 into Q_0.*

For any $M \models \psi$, for $b \in Q_1^M$, let $u_M(b) = \{i : M \models R(c_i, b)\}$. To complete the definition of $\boldsymbol{K}_\psi$, we add the additional requirement (not axiomatizable in $L_{\lambda^+,\omega}$):

Requirement 1.6 *For $M \in \boldsymbol{K}_\psi$, for every $w \subset \lambda$ there is a $b \in Q_1^M$ such that $u_M(b) = w$.*

Conditions 1) and 4) of Requirement 1.5 guarantee each member of $\boldsymbol{K}_\psi$ has cardinality less than μ. Condition 2) implies $|P^M| = \lambda$. Conditions 2) and 3) guarantee that Q_1^M has cardinality at most 2^λ. Condition 4) guarantees that $|Q_0^M| \geq |Q_1^M|$. And Requirement 1.6 asserts $|Q_1^M|$ is at least 2^λ.

Definition 1.7 *$\boldsymbol{K}_\psi$ is the class of τ_*-structures satisfying Requirements 1.5 and 1.6.*

We have constructed ψ from ϕ proving Lemma 1.4.1. Now we construct $\boldsymbol{t}_\psi$. to satisfy Lemma 1.4.2. As in Fact 0.4 $\psi \in L_{\lambda^+,\omega}$ by a first order theory T_ψ of cardinality λ and omitting a single type: $\{P(x), x \neq c_i : i < \lambda\}$.

The vocabulary τ is a set of λ unary predicates P_i. The theory T asserts the $\langle P_i : i < \lambda\rangle$ are an independent family of unary predicates on the entire model. The theory T is superstable as it is formulated in a language with only unary predicates.

The vocabulary τ_1 adds the rest of τ_* (from the proof of Lemma 1.4) and additionally a binary function F and a unary function H.

Requirement 1.8 *T_1 asserts:*

1. *The axioms of T_ψ.*
2. $(\forall x)[Q_1(x) \rightarrow (P_i(x) \leftrightarrow c_i R x)]$.
3. *If $M \models T_1$, F maps $M \times M$ to M so that*
   (a) *for each x, $F(x, \_)$ is a 1-1 function from M to M.*
   (b) $(\forall x)(\forall y)[P_i(x) \rightarrow P_i(F(x, y))]$.
4. *H is a function that preserves each P_i and maps into Q_1.*

Notation 1.9 *The crucial type $p_{\boldsymbol{t}}$ is $\{P(x), x \neq c_i : i < \lambda\}$. Omitting p guarantees that structure satisfies (the translation of) ψ (by the choice of T_ψ and $p_{\boldsymbol{t}}$).*

Requirement 1.6 is guaranteed by the saturation: For every $w \subset \lambda$, the type

$$q_w(y) = \{P_i(y)^{\text{ if } i \in w} : i < \lambda\}$$

is consistent and so realized by some b_w. Then, H guarantees there is a realization $H(b_w)$ satisfying Q_1. Requirement 1.8.2 implies $R(c_i, H(b_w))$ if and only if $i \in w$. So $M \restriction \tau_*$ satisfies Requirement 1.6 and so is in $\boldsymbol{K}_\psi$.

Conversely, each τ_*-model N of $\boldsymbol{K}_\psi$ of cardinality less than μ expands to a τ_1-structure N' satisfying $\boldsymbol{t}_\psi$ of the same cardinality in the obvious way. It omits $p_{\boldsymbol{t}}$ by Requirement 1.5. Moreover, $N' \restriction \tau$ is saturated. All that is needed for this is that each conjunction of the P_i is realized $|N|$-times; this is guaranteed by Requirement 1.8.3 on F. $\square_{1.4}$

Lemma 1.10 $\beth_{(2^\lambda)^+} < H(\boldsymbol{N}_\lambda^{ss})$.

Proof: Fix a sentence $\psi \in L_{\lambda^+,\omega}(\tau)$ with $|\tau| = \lambda$ and with a model in 2^λ and no larger. Construct $\boldsymbol{t}_\psi$ by Lemma 1.4. We now construct an extension $\boldsymbol{t}^*_\psi = (T^*, T, p)$. Without loss of generality, we can expand the vocabulary and add a pairing function pr on P, a new index sort I, a linear ordering $<$ of P such that $c_\alpha < c_\beta$ if and only if $\alpha < \beta$ and a ternary relation W. Let $W \subset I \times Q_1 \times Q_1$ and for each $a \in I$ assert that $W(a, x, y)$ (written $x <_a y$) defines a well ordering on Q_1. For this, note that each $x \in Q_1$ defines a family U^x_y for $y \in P$, $U^x_y = \{z : \mathrm{pr}(y, z)Rx\}$. By Requirement 1.6, there is an $e_{x,y} \in Q_1$ such that $U^x_y = \{z : zRe_{x,y}\}$. Now to say $W(a, \_, \_)$ defines a well ordering on Q_1, write

$$\neg(\exists x \in Q_1)(\forall y_1 < y_2 < c_\omega)e_{x,y_2} <_a e_{x,y_1}.$$

Moreover, we can say

$$(\exists a)(\exists z \in Q_1)(\forall x \in Q_1)(\forall y \in P)e_{x,y_1} <_a z.$$

That is, there is a well ordering $<_a$ of cofinality greater than λ. The well-orderings on Q_1 induces a well-ordering $a <^* b$ on I by $W(a, \_, \_)$ is an initial segment of $W(b, \_, \_)$. Now let J be a partial function from I to Q_1 and assert that each $<^*$-initial segment of I is embeddible in Q_1. So $|I| \leq (2^\lambda)^+$. Since there is a model of ψ with cardinality 2^λ and our expansion codes every well ordering of that model, there is an $M \in \boldsymbol{t}$ with I^M well-ordered of type $(2^\lambda)^+$.

Now expand the vocabulary still further by adding binary predicates $S(x, y)$ and ternary predicates $R(x, y, z)$. Assert in a new theory T^{**} that the $S(a, x)$ for $x \in I^M$ are disjoint if the order type of a with respect to $<^*$ is a successor ordinal and if it is a limit ordinal $S(a, x) \leftrightarrow (\exists b)[b <^* a \wedge S(b, x)]$. Further, for a the $<^*$-successor of b , $R(a, y, z)$ is an extensional relation coding subsets of $S(b, x)$ by elements $S(a, x)$. Let $\boldsymbol{t}^{**}$ be (T^{**}, T, p). Then $\boldsymbol{t}^{**}$ has a model in $\beth_{(2^\lambda)^+}$ but no larger, as required.

$\square_{1.10}$

Note that since we have a well-order of I^M of order type (2^λ), we can get even larger bounds on the Hanf number by replacing the well-order on M by one of higher ordinality.

This completes the proof of the second inequality in Theorem 1.2; we pass to the third.

1.3 The third inequality

Lemma 1.11 $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

We first show $H(\boldsymbol{N}^{ss}_\lambda) \leq H(L_{(2^\lambda)^+,\omega})$ by constructing a map from $\boldsymbol{t} \in \boldsymbol{N}^{ss}_\lambda$ to $\psi_{\boldsymbol{t}} \in L_{(2^\lambda)^+,\omega}$; this construction depends heavily on the superstability hypothesis. Then we use some observations on Hanf numbers to show the inequality is strict: $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

Lemma 1.12 *For each* $\boldsymbol{t} = (T, T_1, p) \in \boldsymbol{N}^{ss}_\lambda$, *there is a* τ_2 *extending* τ_1 *with* $|\tau_1| = |\tau_2| = \lambda$ *and a* $\psi \in L_{(2^\lambda)^+,\omega}$ *such that* $\mathrm{spec}(\boldsymbol{t}) = \mathrm{spec}(\psi)$.

Proof. In preparation consider a fixed saturated model M of cardinality $2^{|T|}$ of T.

To form τ_2, we add to τ_1 constants $\langle c_\alpha : \alpha < (2^\lambda)\rangle$ and as described below $2n+1$-ary functions H_n and function symbols $G_{n,m}$ indexing maps from N to N^m by $n+m$ tuples.

Notation 1.13 *1. We write $\mathbf{y}_1 \equiv_{\mathbf{x}_1,\mathbf{x}_2} \mathbf{y}_2$ to mean for every $\phi(\mathbf{v},\mathbf{w})$, $\phi(\mathbf{x}_1,\mathbf{y}_1) \leftrightarrow \phi(\mathbf{x}_2,\mathbf{y}_2)$.*

2. $\mathbb{F}(\mathbf{x})$ is the collection (for $i < 2^\lambda$) of m-ary finite equivalence relations $E_i(\mathbf{x};\mathbf{y},\mathbf{z})$ over $\mathbf{x}$.

We need these notions below.

Definition 1.14 *1. Recall that a model N is $F^a_{\kappa(T)}$-saturated (also called a-saturated and ϵ-saturated if each strong type over a set of size less $\kappa(T)$ is realized. For superstable theories $F^a_{\kappa(T)}$-saturated is just $F^a_{\aleph_0}$-saturated (each strong type over a finite set is realized).*

2. A model N is strongly ω-homogeneous, if any two finite sequences that realize the same type over the empty set are automorphic in N.

Fact 1.15 (III.3.10.2) *If a model M of a stable theory is $F^a_{\kappa(T)}$-saturated and for each set of infinite indiscernibles $\boldsymbol{I}$ in M there is an equivalent set of indiscernibles $\boldsymbol{I}'$ in M that has cardinality $|M|$, then M is saturated.*

Notation 1.16 *Now let $\psi_{\boldsymbol{t}} \in L_{(2^\lambda)^+,\omega}(\tau_1)$ assert of a model N:*

1. *The specified $p = p_{\boldsymbol{t}}$ is omitted.*
2. *The complete diagram of M where the c_α enumerate M. (M is the saturated model of cardinality $2^{|T|}$ specified at the beginning of the proof.)*
3. *$N \restriction \tau$ is strongly ω-homogeneous. (Add $2n+1$-ary functions H_n satisfying if $\boldsymbol{a} \equiv \mathbf{b}$, $(\lambda z)H_n(\boldsymbol{a},\mathbf{b},z)$ is a τ-automorphism taking $\boldsymbol{a}$ to $\mathbf{b}$. This is expressible since having the same type over the empty set is expressible in $L_{(2^\lambda)^+,\omega}(\tau)$.)*
4. *For each $n<\omega, m<\omega$ there is an $n+2m$-ary function G such that G witnesses that for any n-tuple $\boldsymbol{a}$ and m-tuple $\mathbf{b}$, if $\mathrm{stp}(\mathbf{b}/\boldsymbol{a})$ is realized infinitely often then it is realized $|N|$-times. Formally, $N \models \psi_{\boldsymbol{t}}$:*

$$[(\forall \mathbf{x}\mathbf{y}) \bigwedge_{n<\omega} (\exists^{\geq n}\mathbf{z})(\bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z}) \in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x},\mathbf{z},\mathbf{y}))] \rightarrow \bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z}) \in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x}, G(\mathbf{x},\mathbf{z},w),\mathbf{y}))$$

where for every $\mathbf{x},\mathbf{z}$ $\lambda w G(\mathbf{x},\mathbf{z},w)$ is a 1-1 map from N into N^m.

Proof of $H(\boldsymbol{N}^{ss}_\lambda) \le H(L_{(2^\lambda)^+,\omega})$: Suppose $\boldsymbol{t} \in \boldsymbol{N}^{ss}_\lambda$, $\psi_{\boldsymbol{t}}$ is constructed to satisfy Notation 1.16, and $N \models \psi_{\boldsymbol{t}}$. Since $|N| = |N\restriction\tau_1|$, it suffices to show $N\restriction\tau_1 \models \boldsymbol{t}$. Clearly N omits $p_{\boldsymbol{t}}$ and $N\restriction\tau$ is superstable; in particular it is an elementary extension of $F^a_{\aleph_0}$-saturated the model M. We must show $N\restriction\tau$ is saturated. But N is strongly ω-homogeneous by Notation 1.16.2. So each consistent strong type p over an n-element sequence $\boldsymbol{a} \in N$ is realized by $H_n^{-1}(\boldsymbol{a}, \mathbf{b}, \mathbf{c})$ where $\mathbf{b} \in M$ satisfies $\boldsymbol{a} \equiv \mathbf{b}$ and $\mathbf{c} \models H_n(\boldsymbol{a}, \mathbf{b}, q)$ (where the H_n transforms a strong type over $\boldsymbol{a}$ to one over $\mathbf{b}$ in the natural manner. Thus, N is $F^a_{\aleph_0}$-saturated so we may apply Fact 1.15. Every infinite indiscernible set J in $N\restriction\tau$ is based on a finite $\mathbf{d}$. That is, there is a strong type p_J over $\mathbf{d}$ such that J contains infinitely many realizations of p. Now the conditions on G of Notation 1.16.4 guarantee that p_J is realized N times in N as required. $\square_{1.12}$

Now we strengthen the inequality $H(\boldsymbol{N}^{ss}_\lambda) \le H(L_{(2^\lambda)^+,\omega})$ to a strict one.

Claim 1.17 $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$.

Proof. VII.5.4 and VII.5.5.1 of [7] shows for any μ, $\mathrm{cf}(H(L_{\mu^+,\omega})) \ge \mu^+$; in particular, $\mathrm{cf}(H(L_{(2^\lambda)^+,\omega}) \ge (2^\lambda)^+$. But there are at most 2^λ-classes in $\boldsymbol{N}^{ss}_\lambda$ and Lemma 1.12 implies that the supremum of the spec of each is less than $H(L_{(2^\lambda)^+,\omega})$. Thus, $H(\boldsymbol{N}^{ss}_\lambda) < H(L_{(2^\lambda)^+,\omega})$. $\square_{1.17}$

Note that our result is even stronger than advertized. The coding theory T is not merely superstable but weakly minimal with trivial forking.

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