

# COMPLETE $\mathcal{L}_{\omega_1, \omega}$ -SENTENCES WITH MAXIMAL MODELS IN MULTIPLE CARDINALITIES

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ABSTRACT. In [BKS15] examples of incomplete sentences are given with maximal models in more than one cardinality. The question was raised whether one can find similar examples of complete sentences. In this paper we give examples of complete  $\mathcal{L}_{\omega_1, \omega}$ -sentences with maximal models in more than one cardinality; indeed in countably many cardinalities. The key new construction is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence with arbitrarily large models but with  $(\kappa^+, \kappa)$  models for every  $\kappa$ .

We unite ideas from [BFKL13, BKL14, Hjo02, Kni77] to find complete sentences with maximal models in two cardinals. There have been a number of papers finding complete sentences of  $\mathcal{L}_{\omega_1, \omega}$  characterizing cardinals beginning with Baumgartner, Malitz and Knight in the 70's, refined by Laskowski and Shelah in the 90's and crowned by Hjorth's characterization of all cardinals below  $\aleph_{\omega_1}$  in the 2002. These results have been refined since. But this is the first paper finding complete sentences with maximal models in two or more cardinals.

Our arguments combine and extend the techniques of building atomic models by Fraissé constructions using disjoint amalgamation, pioneered by Laskowski-Shelah and Hjorth, with the notion of homogeneous characterization and tools from Baldwin-Koerwien-Laskowski. This paper combines the ideas of Hjorth and Knight with specific techniques from [BFKL13, BKL14, Sou14, Sou13] and many proofs are adapted from these sources.

## Structure of the paper:

In Section 1, we explain the merger techniques for combining sentences that homogeneously characterize one cardinal (possibly in terms of another) to get a single complete sentence with maximal models in prescribed cardinalities.

Section 2 contains the main technical construction of the paper: the existence of a complete sentence with a unary predicate that has  $(\kappa^+, \kappa)$  models for every  $\kappa$ . From this construction and the tools of Section 1, we present examples of  $\mathcal{L}_{\omega_1, \omega}$ -sentences with maximal models in  $\kappa$  and  $\kappa^+$  and no larger models.

In Section 3 we present examples of  $\mathcal{L}_{\omega_1, \omega}$ -sentences with maximal models in  $\kappa$  and  $\kappa^\omega$  and no larger models. The argument can be generalized to maximal models in  $\kappa$  and  $\kappa^{\aleph_\alpha}$ , for all countable  $\alpha$ .

Finally in Section 4, if  $\kappa$  is a homogeneously characterizable cardinal, we present an  $\mathcal{L}_{\omega_1, \omega}$ -sentence with maximal models in cardinalities  $2^{\aleph_0}, 2^{\aleph_1}, \dots, 2^\kappa$  and no models larger than  $2^\kappa$ .

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## 1. THE GENERAL CONSTRUCTION

In this section, for a cardinal  $\kappa$  that admits homogeneous characterization, we construct a complete sentence  $\phi_\kappa$  of  $\mathcal{L}_{\omega_1, \omega}$  that has maximal models in  $\kappa$  and  $\kappa^+$  and no larger models. The proof applies the notion of a receptive model from [BFKL13] and merges a sentence homogeneously characterizing  $\kappa$  with a complete sentence encoding the old idea of characterizing  $\kappa^+$  by a  $\kappa$ -like order. This template is extended to functions other than successor in later sections.

We require a few preliminary definitions.

**Definition 1.1.** *Assume  $\lambda \leq \kappa$  are infinite cardinals,  $\phi$  is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence in a vocabulary that contains a unary predicate  $P$ , and  $\mathcal{M}$  is the (unique) countable model of  $\phi$ . We say*

- (1) *a model  $\mathcal{N}$  of  $\phi$  is of type  $(\kappa, \lambda)$ , if  $|\mathcal{N}| = \kappa$  and  $|P^{\mathcal{N}}| = \lambda$ ;*
- (2) *For a countable structure  $\mathcal{M}$ ,  $P^{\mathcal{M}}$  is a set of absolute indiscernibles for  $\mathcal{M}$ , if  $P^{\mathcal{M}}$  is infinite and every permutation of  $P^{\mathcal{M}}$  extends to an automorphism of  $\mathcal{M}$ .*
- (3)  *$\phi$  homogeneously characterizes  $\kappa$ , if*
  - (a)  *$\phi$  has no model of size  $\kappa^+$ ;*
  - (b)  *$P^{\mathcal{M}}$  is a set of absolute indiscernibles for  $\mathcal{M}$ , and*
  - (c) *there is a maximal model of  $\phi$  of type  $(\kappa, \kappa)$ .*

The next notation is useful for defining mergers. We slightly broaden the notion of ‘receptive’ from [BFKL13] by requiring some sorts of the ‘guest sentence’ to restrict to  $U$  while others are new sorts in the final vocabulary.

**Notation 1.2.** *Fix a vocabulary  $\tau$  containing unary predicates  $V, U$  and a binary relation symbol  $P$ . The sentence  $\theta_0$  says  $V$  and  $U$  partition the universe and  $P$  is a projection of  $V$  onto  $U$ .*

*Let  $\tau_1$  extend  $\tau$  and let  $\theta$  be a complete  $\tau_1$ -sentence of  $\mathcal{L}_{\omega_1, \omega}$  that implies  $\theta_0$ . Fix a vocabulary  $\tau'$  disjoint from  $\tau_1$  that contains a unary predicate  $Q$ , and let  $\psi$  an arbitrary (possibly incomplete)  $\tau'$ -sentence of  $\mathcal{L}_{\omega_1, \omega}$ . Let  $\tau_2$  contain the symbols of  $\tau_1 \cup \tau'$  except for  $Q$ .*

- *If  $U$  defines an infinite absolutely indiscernible set in the countable model of  $\theta$ , we call the pair  $(\theta, U)$  receptive. We call  $\theta$  receptive if there is a  $U$  such that  $(\theta, U)$  is receptive and in that case we also call the countable model of  $\theta$  a receptive model.*
- *The merger  $\chi_{\theta, U, \psi, Q}$  of the pair  $(\theta, U)$  is the conjunction of  $\theta$  and  $\psi^{U, Q}$ , where  $\psi^{U, Q}$  is the result of substituting  $U$  for  $Q$  in  $\psi$ . Thus  $\chi_{\theta, U, \psi, Q}$  is a  $\tau_2$ -sentence.*
- *If in all models  $\mathcal{N}$  of  $\psi$ ,  $Q^{\mathcal{N}}$  is the domain of  $\mathcal{N}$ , then we will drop  $Q$  and write  $\chi_{\theta, U, \psi}$ .*
- *If  $\mathcal{M} \models \theta$  and  $\mathcal{N} \models \psi$ , the merger model  $(\mathcal{M}, \mathcal{N})$  denotes a model of  $\chi_{\theta, U, \psi, Q}$  where the elements of  $Q^{\mathcal{N}}$  have been identified with the elements of  $U^{\mathcal{M}}$ , which is the intersection of  $\mathcal{M}$  and  $\mathcal{N}$ .*

*$\mathcal{M}$  will be called the host model and  $\mathcal{N}$  the guest model.*

Note that if  $\phi$  and  $P$  homogeneously characterize some  $\kappa$ , then the countable model of  $\phi$  is receptive. Fact 1.3 extends the argument in [BFKL13] to reflect our more general notion of merger.

**Fact 1.3.** *Let  $(\theta, U)$  be receptive and  $\psi$  a sentence of  $\mathcal{L}_{\omega_1, \omega}$ .*

- (1) *The merger  $\chi_{\theta, U, \psi, Q}$  is a complete sentence if and only if  $\psi$  is complete.*
- (2) *There is a 1-1 isomorphism preserving function between isomorphism types of the countable models of  $\psi$  and the isomorphism types of countable models of the merger  $\chi_{\theta, U, \psi}$ .*

- (3) If there is a model  $M_0$  of  $\theta$  with  $|M_0| = \lambda_0$  and  $|U^{M_0}| = \rho$  and also a model  $M_1$  of  $\psi$  with  $|M_1| = \lambda_1$  and  $|Q^{M_1}| = \rho$ , then there is a model of  $\chi_{\theta,U,\psi,Q}$  with cardinality  $\max(\lambda_0, \lambda_1)$ .

**Remark 1.4.** The proof of 1) in [BFKL13] is a bit quick. The completeness also depends on absolute indiscernability. Let  $\mathcal{N}$  and  $\mathcal{N}'$  be countable models of  $\psi$ . Then  $Q(\mathcal{N}) \cong_{\tau'} Q(\mathcal{N}')$ . By absolute indiscernability that automorphism extends to a  $\tau_1$  automorphism of any  $\mathcal{M}$  being merged with  $\mathcal{N}$  or  $\mathcal{N}'$ .

In Section 2 we prove:

**Theorem 1.5.** *There is a complete  $\mathcal{L}_{\omega_1,\omega}$ -sentence<sup>1</sup>  $\phi_M$  with a unary predicates  $X, Y$  and binary predicate  $<$  such that:*

- (1)  $\phi_M$  has arbitrarily large models.
- (2) If  $\mathcal{M} \models \phi_M$ ,  $|\mathcal{M}| \leq |X^{\mathcal{M}}|^+$ .
- (3) For each  $\kappa$ , there exist a model of  $\phi_M$  of type  $(\kappa^+, \kappa)$ .

Idea of the Proof: We will construct (via a generalized Fraïssé construction) a sentence  $\phi_M$  whose models behave as follows. The sort  $Y$  is linearly ordered by  $<$ . Each element  $y$  of  $Y$  determines a function  $g(-, y) : X \rightarrow Y$  so that for  $y \in Y$ ,  $g(-, y)$  maps  $X$  onto the initial segment below  $y$ . The mapping is finite-to-one and so bounds the size of any initial segment by  $|X|$ . The full proof is in Section 2.

Using this result we show if  $\kappa$  is homogeneously characterizable, we can construct a complete sentence of  $\mathcal{L}_{\omega_1,\omega}$  that has maximal models in  $\kappa$  and  $\kappa^+$  and no larger models. Before we proceed with the particular example we need to prove the following theorem.

**Theorem 1.6.** *Let  $\kappa$  be a homogeneously characterizable cardinal. Then there exists an  $\mathcal{L}_{\omega_1,\omega}$ -sentence  $\chi$  in a vocabulary with a new unary predicate symbol  $B$ , such that  $(\chi, B)$  is receptive,  $\chi$  homogeneously characterizes  $\kappa$  and  $\chi$  has maximal models with  $(|\mathcal{M}|, |B^{\mathcal{M}}|) = (\kappa, \lambda)$ , for all  $\lambda \leq \kappa$ .*

*Proof.* Fix a receptive pair  $(\theta, U)$  such that  $\theta$  homogeneously characterizes  $\kappa$ . Define a new vocabulary  $\tau = \{A, B, p\}$  where  $A, B$  are unary predicates and  $p$  is a binary predicate. Let  $\phi$  be the conjunction of: (a)  $A, B$  partition the universe and (b)  $p$  is a total function from  $A$  onto  $B$  such that each  $p^{-1}(x)$  is infinite. In the countable model of  $\phi$ ,  $B$  is a set of absolute indiscernibles.

Now merge  $\theta$  and  $\phi$  by identifying  $U$  and  $A$ . The merger  $\chi = \chi_{\theta,U,\phi,A}$  is a complete sentence which does not have any models of size  $\kappa^+$ . Let  $\mathcal{M}$  be a maximal model of  $\theta$  with  $U^{\mathcal{M}}$  of size  $\kappa$ , and  $\mathcal{N}$  a model of  $\phi$  of type  $(\kappa, \lambda)$ , for some  $\lambda \leq \kappa$ . Then the merger model  $(\mathcal{M}, \mathcal{N})$  is a maximal model of  $\chi$  with  $|(\mathcal{M}, \mathcal{N})| = \kappa$  and  $|B^{(\mathcal{M}, \mathcal{N})}| = \lambda$ , which proves the result  $\square$

A word of caution: In the countable model of  $\theta$ , the predicate  $U$  defines a set of absolute indiscernibles, and the same is true for the countable model of  $\phi$  and  $B$ . So, we started with two models and two sets of absolute indiscernibles. In the merger  $\chi_{\theta,U,\phi,A}$ , the absolute indiscernibles of the host model (model of  $\theta$ ) are used to bound the size of  $A$  from the guest model (model of  $\phi$ ). Moreover, the predicate  $B$  from the guest model defines a set of absolute indiscernibles in the merger model too, and it may be small.

Now, combining Theorem 1.6 and Theorem 1.5 we conclude

**Corollary 1.7.** *Let  $\kappa$  be a homogeneously characterizable cardinal. Then there is a complete sentence that characterizes  $\kappa$  and has maximal models in  $\kappa$  and  $\kappa^+$ .*

*In particular there is a complete sentence with maximal models in  $\aleph_1$  and  $\aleph_2$ .*

<sup>1</sup>The subscript  $M$  on  $\phi_M$  is residue of the proof; we have only one such sentence.

*Proof.* Let  $\phi_\kappa^*$  be the merger  $\chi_{\phi_\kappa, U, \phi_M, X}$ , where the pair  $(\phi_\kappa, U)$  witnesses the homogeneous characterizability of  $\kappa$ , and the pair  $(\phi_M, X)$  is given by Theorem 1.5. By Fact 1.3,  $\phi_\kappa^*$  is complete. Suppose  $(\mathcal{M}, \mathcal{N}) \models \phi_\kappa^*$  and  $\mathcal{M}$  is a maximal model of  $\phi_\kappa$  of size  $\kappa$ . If  $|U^\mathcal{M}| < \kappa$ , then  $|X^\mathcal{M}| < \kappa$ , and by Theorem 1.5 (2)  $|\mathcal{N}| \leq \kappa$ . So, the merger model is maximal and has size  $\kappa$ .

On the other hand, if  $|U^\mathcal{M}| = \kappa$ , then choose a model  $\mathcal{N}$  of  $\phi_M$  of type  $(\kappa^+, \kappa)$ , which is maximal among models of  $\phi_M$  with  $X = U^\mathcal{M}$ . By Theorem 1.5, there is no  $(\kappa^{++}, \kappa)$ -model of  $\phi_M$  and therefore, there must be such a maximal  $(\kappa^+, \kappa)$  model. The merger model  $(\mathcal{M}, \mathcal{N})$  is again a maximal model of  $\phi_\kappa^*$ , but this time it has size  $\kappa^+$ .  $\square$

Particular examples of homogeneously characterizable cardinals are given by [Bau74], [Hjo02], [Sou12], [Sou13], [Sou14], [BKL14].

**Theorem 1.8.** *If  $\kappa$  is homogeneously characterizable, then the same holds true*

- (1) for<sup>2</sup>  $2^\kappa$ ;
- (2) for<sup>3</sup>  $\kappa^\omega$ ;
- (3) for<sup>4</sup>  $\kappa^{\aleph_\alpha}$ , for all countable ordinals  $\alpha$ .

**Theorem 1.9** (Theorem 4.29, [Sou13]). *If  $\aleph_\alpha$  is a characterizable cardinal, then  $2^{\aleph_{\alpha+\beta}}$  is homogeneously characterizable, for all  $0 < \beta < \omega_1$ .*

Finally a result of slightly different character; we note a direct proof of a sentence  $\phi_n$  that homogeneously characterizes  $\aleph_n$  ( $n > 0$ ) and has  $(\aleph_n, \aleph_k)$  models for  $k \leq n$ .

**Theorem 1.10** ([BKL14]). *For each  $n \in \omega$ , there is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi_n$  such that*

- $\phi_n$  homogeneously characterizes  $\aleph_n$  with absolute indiscernibles in a predicate  $P$ ;  
and
- for each  $k \leq n$ , there is a maximal model  $N_k$  of  $\phi_n$  of type  $(\aleph_n, \aleph_k)$ .

Since in this last example, the complete sentence<sup>5</sup> has maximal models of type  $(\aleph_n, \aleph_k)$ , for all  $k \leq n$  there is no need to appeal to Theorem 1.6 for the proof of Corollary 1.7.

## 2. A COMPLETE SENTENCE WITH ONLY $(\kappa^+, \kappa)$ -MODELS FOR ALL $\kappa$

Knight [Kni77] constructed the first example of a complete sentence characterizing  $\aleph_1$ . We will vary that idea to get the result announced.

**Lemma 2.1.** *There is a structure  $A = (Q, <, g_n)$  in the vocabulary  $\langle <, g_n \rangle$ , where  $<$  is the usual dense order on  $Q$  and the  $g_n$  are unary, such that for each  $n$  and  $x$ ,  $g_n(x) < x$  and for each  $x$  the set of  $g_n(x)$  is the set of predecessors of  $x$ . Moreover this structure has a proper elementary extension isomorphic to itself. Thus  $A$  is extendible. If  $\phi_A$  is the Scott sentence of  $A$ ,  $\phi_A$  characterizes  $\aleph_1$ .*

The proof of this lemma involves an intricate construction of functions  $f_a$  for  $a \in \mathcal{P}_\omega(A)$ ; we now produce a similar structure by a Fraissé style construction, proving Theorem 1.5. We follow the idea of Hjorth in replacing functions  $f_n$  by a uniform  $f(x, y)$  indexed in the model. This sentence will have arbitrarily large models with two sorts  $X, Y$  such that the

<sup>2</sup>Baumgartner; see also Theorem 3.4 of [Sou13]

<sup>3</sup>Theorem 3.6, [Sou14]

<sup>4</sup>Corollary 5.6, [Sou12]

<sup>5</sup>The proof that these  $(\aleph_n, \aleph_k)$  models exist requires the use of both frugal amalgamation and an amalgamation which allows identification. We say a class has frugal amalgamation if for every amalgamation triple  $A, B, C$  there is an amalgam on the union of the domains with no identifications. See [BKS09].

sort of  $|Y| \leq |X|^+$ . This construction will allow us in Theorem 2.17 to construct  $(\kappa^+, \kappa)$  models of  $\phi_M$ . The predicates  $G_n$  are used to enforce local finiteness.

We now describe the sentence  $\phi_M$ ; the remainder of the subsection completes the proof of Theorem 1.5.

**Construction 2.2.** Let  $\tau$  contain binary  $<$ , unary  $X, Y$  and ternary  $g(x, y, z)$  and  $(n+2)$ -ary relation symbols  $G_n(x_0, \dots, x_{n-1}, y, z)$  on  $X^n \times Y^2$ .

$\mathbf{K}_0$  is the collection of finite structures such  $X$  and  $Y$  are disjoint,  $<$  linearly orders  $Y$ ,  $g(x, y, z)$  is a *total* function from  $X \times Y$  into  $Y$  such that  $g(x, y, z)$  implies  $z \leq y$  and for each  $y \in Y$ , letting  $W_y = \{z \in Y | z \leq y\}$ , the function  $g_y = g(-, y) : X \rightarrow W_y$  is onto. We often write  $g(x, y) = z$  for  $g(x, y, z)$ .

We want to guarantee that in each model for each  $z < y \in Y$  there is an  $n$ ,  $0 < n < \omega$ , such that the set  $\{x \in X | g(x, y) = z\}$  has size  $n$ . In other words, the function  $g(-, y)$  is finite-to-1 when restricted to the set  $\{x \in X | g(x, y) < y\}$ . Notice that while  $g(x, y)$  is allowed to take the value  $y$ , the finite-to-1 restriction does not apply to the set  $\{x \in X | g(x, y) = y\}$ .

For each  $z < y \in Y$ , we use the  $(n+2)$ -ary relation symbols  $G_n(x_0, \dots, x_{n-1}, y, z)$  on  $X^n \times Y^2$  to indicate that the set  $\{x_0, \dots, x_{n-1}\}$  equals the set  $\{x \in X | g(x, y) = z\}$ , which is of size  $n$ . The relations  $G_n$  satisfy the following:

The universal closure of each formula of the following form(s):

$$G_n(x_0, \dots, x_{n-1}, y, z) \rightarrow \bigwedge_{i < j < n} x_i \neq x_j;$$

$$G_n(x_0, \dots, x_{n-1}, y, z) \rightarrow (z < y) \wedge \bigwedge_{i < n} g(x_i, y, z);$$

$$G_n(x_0, \dots, x_{n-1}, y, z) \wedge g(w, y, z) \rightarrow \bigvee_{i < n} w = x_i;$$

and for every permutation  $\pi : n \rightarrow n$ ,

$$G_n(x_0, \dots, x_{n-1}, y, z) \leftrightarrow G_n(x_{\pi(0)}, \dots, x_{\pi(n-1)}, y, z)$$

and each sentence of the form:

$$\forall y \forall z < y \bigvee_{0 < n < \omega} \exists x_0 \exists x_1 \exists x_2 \dots \exists x_{n-1}, G_n(x_0, x_1, \dots, x_{n-1}, y, z);$$

Note that since in every member of  $\mathbf{K}_0$  and for every  $y$  there is an  $x$  with  $g(x, y) = y$ , we will never have to check this condition in an amalgamation. The following easy lemma will be applied several times.

**Lemma 2.3.** *If  $A \in \mathbf{K}_0$  and  $m \in \omega$ , there is a structure  $B \in \mathbf{K}_0$  with  $A$  a substructure of  $B$  and  $|X^B \setminus X^A| \geq m$ .*

*Proof.* Adjoin  $m$  elements to  $X^A$  and let  $g^B(x, y) = y$ , for all new  $x$  and each  $y \in Y^A$ . For each  $n$  exactly the same elements satisfy  $G_n$  in  $A$  and  $B$ .  $\square$

**Definition 2.4.** *We call  $(A, B)$  a good pair, if  $A \subset B$ ,  $A, B \in \mathbf{K}_0$  and  $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$ .*

**Lemma 2.5.**  *$\mathbf{K}_0$  has the disjoint amalgamation property under substructure.*

*Proof.* Suppose  $A$  is a substructure of  $B$  and  $C$ . Applying Lemma 2.3 to  $B$  with  $m = |Y^B \setminus Y^A|$  and to  $C$  with  $m = |Y^C \setminus Y^A|$  we may assume  $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$  and  $|X^C \setminus X^A| \geq |Y^C \setminus Y^A|$  (and they intersect in  $A$ ). Extend the partial order on  $Y^B \cup Y^C$  by linearly ordering the  $a_i, b_j$  that are in an interval  $(c_i, c_j)$  in any way that preserves the ordering of the a's and of the b's.

Let  $X^D = X^B \cup X^C$  and  $Y^D = Y^B \cup Y^C$ . The goal is to define  $g^D(x, y)$  on the pairs  $(x, y) \in (X^B \setminus X^A) \times (Y^C \setminus Y^A) \cup (X^C \setminus X^A) \times (Y^B \setminus Y^A)$  to obtain an amalgam  $D \in \mathbf{K}_0$ .

For each  $y \in Y^C \setminus Y^A$ , let  $W_y^B = \{z \in Y^B \setminus Y^A \mid z \leq y\}$ . Define  $g^D(x, y)$  so that for each  $y \in Y^C \setminus Y^A$  and each  $x \in X^B \setminus X^A$ ,  $g^D(x, y) \in W_y^B$  and each element in  $W_y^B$  is mapped to by at least one  $x \in X^B \setminus X^A$ . This is possible by the application above of Lemma 2.3. If  $W_y^B$  is empty, define  $g^D(x, y) = y$ .

Symmetrically, for each  $x \in X^C \setminus X^A$  and each  $y \in Y^B \setminus Y^A$ , define  $g^D(x, y)$  to be an element in  $W_y^C = \{z \in Y^C \setminus Y^A \mid z \leq y\}$ , so that each element in  $W_y^C \subset Y^C \setminus Y^A$  is mapped to at least once. Again, if the set is empty, define  $g^D(x, y) = y$ .

Since no newly defined triple  $x, y_1, y_2$  that satisfies  $g(x, y_1) = y_2$  has both  $y_1$  and  $y_2$  in  $C$ , or both  $y_1$  and  $y_2$  in  $B$ , we can extend the definition of the  $G_n^D$ 's consistent with the new assignments of  $g$  while extending  $G_n^B$  and  $G_n^C$  and so making  $D \in \mathbf{K}_0$ .  $\square$

We summarize the results of this construction. We call a structure  $\mathbf{K}_0$ -generic if it is homogeneous and universal for  $\mathbf{K}_0$ -structures and is a union of finite structures.

**Theorem 2.6.** *There is a countable  $\mathbf{K}_0$ -generic (and atomic) model  $M$  (with Scott sentence  $\phi_M$ ), which is an increasing union of members of  $\mathbf{K}_0$  which are each closed under  $g$ . Thus,  $M$  is locally (but not uniformly locally) finite. Moreover  $M \models (\forall y_1, y_2)[y_1 < y_2 \rightarrow (\exists x)g(x, y_2) = y_1]$  since each model in the union does.*

*Thus, if  $N \models \phi_M$ ,  $Y^N$  is a dense linear order and for every  $y \in Y^N$ ,  $|X^N| \geq |\{y' \in Y^N : y' < y\}|$ . In particular, if  $|X^N| \leq \kappa$ , then the size of  $Y^N$  is bounded by  $\kappa^+$ .*

*Proof.* Since the  $G_n$  are only defined on models of cardinality at least  $n$ , there are only countably many models in  $\mathbf{K}_0$ , so the amalgamation property (and trivially joint embedding) guarantee the existence of  $\mathcal{M}$ .  $\square$

**Corollary 2.7.** *There is an  $(\aleph_1, \aleph_0)$ -model of  $\phi_M$ .*

*Proof.* We borrow the technique of Lemma 1.10 of [BFKL13]. Expand the vocabulary  $\tau$  to  $\tau'$  by adding a new unary predicate  $Q$ . Consider the class  $\mathbf{K}'$  of finite  $\tau'$ -structures  $N$  such that:

- (1)  $N \upharpoonright \tau \in \mathbf{K}_0$ ;
- (2)  $X^N \subseteq Q^N$ ; and
- (3)  $Y^N \cap Q^N$  is an initial segment of  $Y^N$ .

By virtue of  $\mathbf{K}_0$  satisfying disjoint amalgamation, the same is also true for  $\mathbf{K}'$ . We claim that the generic model  $M'$  for  $\mathbf{K}'$  has a proper  $\tau$ -substructure  $M_0$  with universe  $Q^{M'}$ ,  $M_0 \upharpoonright \tau \cong M' \upharpoonright \tau$ , and  $X^{M_0} = X^{M'}$ .

Since  $M'$  is  $\mathbf{K}'$ -generic, both  $M' \upharpoonright \tau$  and  $M_0 \upharpoonright \tau$  must be  $\mathbf{K}_0$ -generic. By Theorem 2.6, they are isomorphic. The fact that  $X^{M_0} = X^{M'}$  follows from condition (2).

Now, consider the question whether  $M_0$  is a proper submodel of  $M'$ . It is immediate that  $M_0, M'$  agree on  $<$ , and  $X^{M_0} = X^{M'}$  implies that they agree on the  $G_n$ 's too. The only case that needs to be considered is if there exist  $x \in X^{M_0}$  and  $y_1 \in Y^{M_0}$ , and  $g^{M_0}(x, y_1)$  does

not equal  $g^{M'}(x, y_1)$ . But this can happen only if  $g^{M'}(x, y_1) \notin Y^{M_0}$ , i.e.  $g^{M'}(x, y_1) \notin Q^{M'}$ . In this case,  $g^{M'}(x, y_1) < y_1$ ,  $y_1 \in Q^{M'}$  and  $g^{M'}(x, y_1) \notin Q^{M'}$ , contradicting condition (3).

So, we proved all three conditions on  $M_0 \upharpoonright \tau$  and  $M' \upharpoonright \tau$ . Taking a countable increasing chain of atomic models, we get an  $(\aleph_1, \aleph_0)$  model  $M^* \models \phi_M$ .  $\square$

**Definition 2.8.**  $C \in \hat{\mathbf{K}}_0$  if

- (1) for every finite  $A \subset C$  there exists some finite substructure  $A'$  of  $C$  with  $A \subset A'$  and  $A' \in \mathbf{K}_0$ .
- (2)  $(Y^C, <)$  is a linear order without endpoints.

Condition 1) is often expressed by saying  $C$  is locally finite. The following lemma is a crucial observation.

**Lemma 2.9.** If  $C \in \hat{\mathbf{K}}_0$ , then  $g_y^C$  maps onto  $W_y^C = \{z \in C : z \leq y\}$ .

*Proof.* Let  $z \in W_y^C$ . By local finiteness there is a finite  $A$  such that  $g_y^A$  maps onto  $W_y^A$ ; in particular there is an  $x \in A$  with  $g^A(x, y) = g^C(x, y) = z$ .  $\square$

The structure  $N$  is called  $\mathbf{K}_0$ -rich, or just *rich*, if for every  $A \subset N$  which is in  $\mathbf{K}_0$  and every extension  $B \in \mathbf{K}_0$  of  $A$ , there is an embedding of  $B$  into  $N$  over  $A$ . If  $N \in \hat{\mathbf{K}}_0$  and  $N$  is rich then standard arguments show  $N \models \phi_M$ .

Our first goal is to prove that  $\phi_M$  has model in all cardinalities. Then we will merge this sentence with a complete sentence  $\phi$  that homogeneously characterizes some cardinal  $\kappa$  by identifying  $X$  with the set of absolute indiscernibles of  $\phi$ . This will result in a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence that has maximal models in two cardinalities:  $\kappa$  and  $\kappa^+$ .

In the following argument we write  $g^C(x, y)$  to emphasize when we are using the existing model  $C$  to define the extended function  $g^D(x, -)$ .

**Theorem 2.10.** Let  $C \in \hat{\mathbf{K}}_0$  be infinite. Let  $A, B \in \mathbf{K}_0$ ,  $A \subset B, C$ . Then the triple  $(A, B, C)$  can be disjointly amalgamated to a structure  $D \in \hat{\mathbf{K}}_0$  with  $|X^D| = |X^C|$  and  $|Y^D| = |Y^C|$ .

*Proof.* As in the proof of Lemma 2.5, extend the partial order on  $Y^B \cup Y^C$  to a total order without endpoints. Without loss of generality by Lemma 2.3 assume that  $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$ . The goal is to extend  $g^C$  and  $g^B$  to a function  $g^D$  on all pairs  $(x, y) \in (X^B \setminus X^A) \times (Y^C \setminus Y^A) \cup (X^C \setminus X^A) \times (Y^B \setminus Y^A)$  so that  $D \in \hat{\mathbf{K}}$ . In stage 1, we almost achieve this goal. But we may need a further finite extension of  $D$  to  $D'$  and this is stage 2 of the construction.

In making this extension no new values are assigned to  $g(a, b) = c$  where both  $b, c$  are in  $Y^C$ , or both  $b, c$  are in  $Y^B$ . So there is no danger of injury to any predicate  $G_n(\mathbf{x}, b, c)$  with  $b, c \in B$ , or  $b, c \in C$ .

**Stage 1:** We organize the case structure by the possible locations of an  $(x, y)$  on which  $g^D$  must be defined. The first case is  $(x, y) \in X^B \setminus X^A \times Y^C \setminus Y^A$ . For fixed  $y$  and each  $x$  define  $g^D(x, y)$  to be an element of the set  $W_y^B = \{z \in Y^B \setminus Y^A \mid z \leq y\}$ , ensuring that each element in  $W_y^B$  is mapped to at least once. This is possible given the assumption  $|X^B \setminus X^A| \geq |Y^B \setminus Y^A|$ . If  $W_y^B$  is empty, let  $g^D(x, y) = y$ . Since only finitely many extensions  $g^D(x, -)$ 's have been defined and all of them take values outside of  $C$ , it is trivial to define the  $G_n^D$ . Note that since each  $z \in Y^C$ ,  $z \leq y$ , is already in the range of  $g_y^C$ ,  $g_y^D$  maps onto  $\{z \in Y^D \mid z \leq y\}$ .

The second case of stage 1 is  $(x, y) \in X^C \setminus X^A \times Y^B \setminus Y^A$ . It requires more work, since we must define  $g^D(x, y)$  for infinitely many  $x$  and finitely many  $y$  without violating the

constraints on the  $G_n$ . We choose a surrogate  $y_0 > Y^B$  with  $y_0 \in Y^C \setminus Y^A$  that we use to define  $g^D(x, y)$  for all  $(x, y)$  satisfying this case. For every  $x \in X^C \setminus X^A$  and  $y \in Y^B \setminus Y^A$ , we define  $g^D(x, y)$  by three subcases.

- (1) if  $g^C(x, y_0) \geq y$ , let  $g^D(x, y) = y$ .
- (2) If  $g^C(x, y_0) < y$  and  $g^C(x, y_0) \notin Y^A$ , let  $g^D(x, y) = g^C(x, y_0)$ .
- (3) If  $g^C(x, y_0) < y$  and  $g^C(x, y_0) \in Y^A$ , let<sup>6</sup>  $g^D(x, y) = y$ .

Each  $g^D(x, \_)$  maps into  $W_y^D = \{z \in D : z \leq y\}$ . Further the range of  $g^D(\_, y)$  contains all of  $W_y^D$ , except for the elements (if any) in the set

$$P_y = \{z \in Y^C \setminus Y^A \mid z < y \text{ and } g^C(x, y_0) = z \rightarrow x \in X^A\}.$$

Indeed, let  $z < y$ . If  $z \in Y^B$ , then there is an  $x$  in  $X^B$  with  $g^B(x, y) = z$  since  $B \in \mathbf{K}_0$ . If  $z \in Y^C \setminus Y^A$ , there exist  $x_1 \in X^C$  with  $g^C(x_1, y_0) = z$  since  $C \in \hat{\mathbf{K}}_0$ . If in addition  $z \notin P_y$ , then  $x_1$  can be assumed to belong to  $X^C \setminus X^A$ . Then  $g^C(x_1, y_0) = g^D(x_1, y) = z$ , by subcase (2). If  $z$  belongs to  $P_y$ , then  $x_1$  is necessarily an element of  $X^A$ , and none of subcases (1)-(3) applies since  $x_1, y \in B$  and  $g^D(x_1, y)$  equals  $g^B(x_1, y)$ . Thus, there is no  $x$  with  $g^D(x, y) = z \in P_y$ .

**Stage 2:** We now ensure that all elements in  $P_y$  are also included in the range of  $g^D(\_, y)$ . For this we add a finite number of new  $x$ 's,  $X_{\text{new}}$ , call the extended domain  $D'$ , and extend  $g^D$  to  $g^{D'}$ . Since  $X^A$  is finite, the same is true for  $P_y$ . We map each  $y \in Y^B \setminus Y^A$  to  $P_y$  using these new elements so that for each  $y \in Y^B \setminus Y^A$  and each  $z \in P_y$ , there exists at least one  $x \in X_{\text{new}}$  such that  $g^{D'}(x, y) = z$ . For every  $y \in Y^C$  and  $x \in X_{\text{new}}$  set  $g^{D'}(x, y) = y$ .

We now define the  $G_n^{D'}$ . In stage 1 we did this for all  $y \in Y^C$  and  $z \in Y^B$  with  $z < y$ . By property (1) of Definition 2.8, for each  $z \in Y^C \setminus Y^A$ , there are only finitely many  $x \in X^C$  that agree that the value of  $g^C(x, y_0)$  is  $z$ . This implies that for each  $y \in Y^B \setminus Y^A$  and each  $z \in (Y^C \setminus Y^A) \setminus P_y$ ,  $z < y$ , there are only finitely many  $x \in X^C \setminus X^A$  such that  $g^{D'}(x, y) = z$ . If  $z \in P_y$ , then there are only finitely many  $x \in X_{\text{new}}$  such that  $g^{D'}(x, y) = z$ . Extend the  $G_n$ 's from  $B$  and  $C$  to  $D'$  as the new assignments of  $g^{D'}$  require.

This completes the construction of the amalgam  $D'$ ; next we show that each of  $B, C$  is a substructure of  $D'$ . That is, the  $G_n$ 's are preserved when we pass from  $B, C$  to  $D'$ . To check from  $B$  to  $D'$ , we must show that for each  $y, z \in Y^B$  there is no  $x \in X_{\text{new}} \cup (X^C \setminus X^A)$ , with  $y > z$  and  $g^{D'}(x, y) = z$  as this would violate  $G_n^B(\mathbf{x}, y, z)$ . If  $x \in X_{\text{new}}$  and  $y \in Y^B \setminus Y^A$ , then  $g^{D'}(x, y) \neq z$  since  $g^{D'}(x, y)$  belongs to  $P_y \subset Y^C \setminus Y^A$ . If  $x \in X_{\text{new}}$  and  $y \in Y^A$ , then  $g^{D'}(x, y) = y \neq z$ . If  $x \in X^C \setminus X^A$ , there do not exist  $y, z \in Y^B$  with  $y > z$  and  $g^{D'}(x, y) = z$ . For such a  $g^{D'}(x, y)$  could not be defined using subcase (2), as in subcase (2)  $z = g^{D'}(x, y) = g^C(x, y_0) \notin Y^A$  (and so certainly not in  $Y^B$  since  $z \in Y^C$ ). So, for any  $y, z \in Y^B$  with  $y > z$ ,  $|\{x : g^{D'}(x, y) = z\}| = |\{x : g^B(x, y) = z\}|$ , which makes  $B$  a substructure of  $D'$ .

To check that the  $G_n$ 's are preserved from  $C$  to  $D'$ , let  $y, z \in Y^C$  with  $y > z$ . We must show there is no  $x \in X_{\text{new}} \cup (X^B \setminus X^A)$  such that  $g^{D'}(x, y) = z$ . If  $x \in X_{\text{new}}$ , then  $g^{D'}(x, y) = y$ . If  $x \in X^B \setminus X^A$ , then  $g^{D'}(x, y) \in W_y^B \subset Y^B \setminus Y^A$  by the first case of Stage 1. Thus, both  $B$  and  $C$  are substructures of  $D'$ .

It remains to prove that  $D' \in \mathbf{K}_0$ . Property (2) is immediate. Property (1) is the one that needs work.

Let  $D_0$  be a finite substructure of  $D'$ . Without loss of generality assume that  $X_{\text{new}} \cup B \cup \{y_0\} \subset D_0$ . Applying property (1) on  $C$ , there exists a substructure  $D_1$  of  $C$  with  $D_1 \in \mathbf{K}_0$

<sup>6</sup>Since each  $g^B(\_, y)$  is onto, there must be an  $x_1 \in X^B$ ,  $g^B(x_1, y) = g^C(x, y_0)$  holds; following the recipe in 2) would make  $G_n^D(\mathbf{x}, y, z)$  differ from  $G_n^B(\mathbf{x}, y, z)$ .

and  $D_0 \cap C \subset D_1$ . We claim that  $D_2 = D_0 \cup D_1$  is a substructure of  $D'$  and is in  $\mathbf{K}_0$ , which proves the result.

We must show  $D_2$  is closed under  $g$ . We follow the same case structure as in the construction.

In the first case of Stage 1, where  $(x, y) \in X^B \setminus X^A \times Y^{D_1} \setminus Y^A$ ,  $g^D(x, y)$  was defined to be either an element  $w \in W_y^B \subset Y^B \setminus Y^A$  or  $y$ . In either event it is in  $D_2$ .

Consider now the second case of Stage 1:  $(x, y) \in X^{D_1} \setminus X^A \times Y^B \setminus Y^A$ . In either subcase (1) or (3)  $g^{D'}(x, y) = y$  and the result is immediate. In subcase (2) suppose  $g^{D'}(x, y) = g^C(x, y_0) = z'$ . Then  $z' \in D_1$  since  $x, y_0$  are in the substructure  $D_1$ . Any other  $(x, y)$  with  $x \notin X_{\text{new}}$  are either both in  $B$  or both in  $D_1$  so closure under  $g^{D'}$  follows since  $B$  and  $D_1$  are in  $\mathbf{K}_0$ .

And finally we consider the elements added in stage 2. If  $(x, y) \in X_{\text{new}} \times Y^{D_1}$ , then  $g^{D'}(x, y) = y \in D_2$ . If  $(x, y) \in X_{\text{new}} \times Y^B$ , then  $g^{D'}(x, y) \in P_y \subset g^C(X^A, y_0)$ . Since  $A \cup \{y_0\} \subset D_1$  and  $D_1$  is closed under  $g^C$ ,  $g^C(X^A, y_0)$  is a subset of  $D_1$ .

We have shown  $D_2$  is closed under  $g^{D'}$ . It remains to prove that  $G_n^{D'} \upharpoonright D_2 = G_n^{D_2}$  and  $D_2 \in \mathbf{K}_0$ .

Suppose  $z < y \in Y^{D_2}$ ; we must show  $G^{D'}(\_, y, z) \upharpoonright D_2 = G^{D_2}(\_, y, z)$ . If both  $y, z$  are in  $B$  or both are in  $C$ , we showed this in showing  $B, C$  substructures of  $D'$  (since  $D_1$  is given to be a substructure of  $C$ ).

Again we follow the case structure of the construction.

If  $y \in Y^{D_1} \setminus Y^A$  and  $z \in Y^B \setminus Y^A$ ,  $y > z$ , then  $g^{D'}(x, y) = z$  was defined in Stage 1 of the construction for some  $x \in X^B \setminus X^A$ . So  $\{x : x \in D' \ \& \ g^{D'}(x, y) = z\} = \{x : x \in B \ \& \ g^{D'}(x, y) = z\}$  as required.

If  $y \in Y^B \setminus Y^A$ ,  $z \in Y^{D_1} \setminus Y^A$ ,  $y > z$  and  $g^{D'}(x, y) = z$  for some  $x$ , then the value was defined either using subcase (2) in Stage 1 or using the new elements  $X_{\text{new}}$  in Stage 2. If under subcase (2), there are only finitely many  $x \in X^C \setminus X^A$  such that  $g^C(x, y_0) = z$ . Since  $z, y_0 \in D_1$  and  $D_1 \in \mathbf{K}_0$ , it follows that all these  $x$ 's belong to  $D_1$  and thus to  $D_2$ . If  $z \in P_y$ , the Stage 2 construction showed  $\{x : x \in D' \ \& \ g^{D'}(x, y) = z\} \subset X_{\text{new}} \subset D_2$ .

Overall,  $\{x : g^{D_2}(x, y) = z\} = \{x : g^D(x, y) = z\}$  and  $D_2$  is a substructure of  $D$ .

Finally we must argue that  $D_2 \in \mathbf{K}_0$ . But we constructed it so that  $g^D$  is total and each  $g_y^{D'} = g_y^{D_2}$  maps onto  $W_y^{D_2}$ , so  $D_2 \in \mathbf{K}_0$  and hence  $D' \in \hat{\mathbf{K}}$ .  $\square$

**Corollary 2.11.** *There are models of  $\phi_M$  in all infinite cardinalities.*

*Proof.* Proceed by induction. The countable case has been established. Let  $N_0$  be a structure of size  $\kappa$  that satisfies  $\phi_M$ . Construct inductively a sequence of models  $N_\alpha \in \hat{\mathbf{K}}$  for  $\alpha < \kappa^+$ , so that every  $\alpha < \kappa^+$ ,  $N_{\alpha+1}$  is the amalgam given by Theorem 2.10 of  $N_\alpha$  and some  $B \in \mathbf{K}_0$  over a finite substructure  $A \in \mathbf{K}_0$  of  $N_\alpha$  and take unions at limits. Organize the induction so that for every  $\alpha < \kappa^+$ , every finite  $A \subset N_\alpha$  with  $A \in \mathbf{K}_0$  and every finite extension  $B \in \mathbf{K}_0$  of  $A$ , at some stage  $\beta > \alpha$ ,  $B$  is amalgamated with  $N_\beta$  over  $A$ .  $N_{\kappa^+}$  is rich and so is model of  $\phi_M$ .  $\square$

Theorem 2.6 and Corollary 2.11 complete the first two parts of the proof of Theorem 1.5. The models of  $\phi_M$  of size  $\kappa$  produced by Corollary 2.11 are of type  $(\kappa, \kappa)$ , i.e. both  $|X|, |Y|$  have the same cardinality  $\kappa$ . We want to prove the analogue of Corollary 2.7 for cardinals greater than  $\aleph_1$ . The argument does not directly generalize because, unlike in  $\aleph_0$ , the  $\phi_M$  is not uncountably categorical. We need a little more work to overcome this difficulty.

**Definition 2.12.** Let  $C$  be a model of  $\phi_M$  of size  $\kappa$ . Then  $C$  admits a filtering if there exists a strictly increasing sequence  $(C_\alpha)_{\alpha < \kappa}$  with  $\bigcup_{\alpha < \kappa} C_\alpha = C$  that satisfies

- (1) for all  $\alpha$ ,  $C_\alpha \in \hat{\mathbf{K}}_0$ ;
- (2) for all successors  $\alpha + 1$ ,  $C_{\alpha+1}$  is the disjoint amalgam of  $C_\alpha$  and some  $B_\alpha \in \mathbf{K}_0$  over a substructure  $A_\alpha \subset C_\alpha$  with  $A_\alpha \in \mathbf{K}_0$ ;
- (3) for all  $\alpha$ , the pair  $(B_\alpha, A_\alpha)$  is a good pair (cf. Definition 2.4); and
- (4) for each limit  $\alpha$ ,  $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ .

If in addition the sequence  $(C_\alpha)_{\alpha < \kappa}$  satisfies the following property, then  $C$  admits an ample filtering:

- (5) For each good pair  $(A, B)$  and each  $\hat{A} \subset C_\alpha$  with  $\hat{A} \cong A$ , there are  $\kappa$ -many stages  $\alpha$  such that  $A_\alpha = \hat{A}$  and  $B_\alpha \cong B$ .

All models of  $\phi_M$  produced by Lemma 2.11 admit a filtering and it is not hard to modify the proof of the same lemma to produce models of  $\phi_M$  of type  $(\kappa, \kappa)$  that admit ample filtering. In fact a stronger statement is true.

**Lemma 2.13.** Let  $C$  be a model of  $\phi_M$  of size  $\kappa$ . If  $C$  admits a filtering, then there is some model  $C'$  of  $\phi_M$  such that  $C \subset C'$  and  $C'$  admits an ample filtering.

*Proof.* Let  $(C_\alpha)_{\alpha < \kappa}$  be a filtering of  $C$ . If it is an ample filtering, then take  $C' = C$ . If not, then proceed for another  $\kappa$  more stages to construct an extension of  $(C_\alpha)_{\alpha < \kappa}$  to  $(C_\alpha)_{\alpha < \kappa \cdot 2}$  and take  $C' = \bigcup_{\alpha < \kappa \cdot 2} C_\alpha$ . Iterate the argument of 2.11, requiring now for every  $A \subset C_\alpha$ ,  $A \in \mathbf{K}_0$ ,  $\alpha < \kappa \cdot 2$ , and every finite extension  $B \in \mathbf{K}_0$  of  $A$ , there are  $\kappa$ -many stages  $\beta_i^{(\alpha)} > \alpha$ ,  $i < \kappa$ , and  $\kappa$ -many disjoint isomorphic copies of  $B$ ,  $B_i$ , such that  $B_i$  is amalgamated with  $C_{\beta_i^{(\alpha)}}$  over  $A$  at stage  $\beta_i^{(\alpha)}$ . The construction can be carried through in  $\kappa$  stages. It follows that  $C'$  is rich, and therefore, a model of  $\phi_M$ .  $\square$

The next theorem proves that under the assumption of ample filtering, a model  $C$  of  $\phi_M$  of type  $(\kappa, \kappa)$  can be properly extended to a model  $D$  of  $\phi_M$ , while  $X^C = X^D$ .

This means that we extend only the  $Y$ -sort. To motivate the argument we note that the conclusion of the next theorem in fact implies  $Y^D$  is an end-extension of  $Y^C$ . Indeed, assume that there exist points  $y_0 \in Y^C$  and  $y_1 \in Y^D \setminus Y^C$  such that  $y_0 > y_1$ . By Theorem 2.6, there must be some  $x \in X^D$  such that  $g^D(x, y_0) = y_1$ . Since we assumed that  $X^D = X^C$ ,  $x$  must also belong to  $X^C$ . But then  $g^C(x, y_0)$  is defined and belongs to  $Y^C$ , which means that it can not equal  $y_1$ . Because  $C$  is a substructure of  $D$ ,  $g^D(x, y_0) = g^C(x, y_0)$  and we get a contradiction. Thus, under the assumption  $X^D = X^C$ , it is necessary for  $Y^D$  to be an end-extension of  $Y^C$ . Furthermore, we also prove that the resulting model  $D$  admits ample filtering, which enables us to apply the theorem inductively.

**Theorem 2.14.** Assume  $\kappa > \aleph_0$  and  $C$  is a model of  $\phi_M$  of type  $(\kappa, \kappa)$  that admits an ample filtering. Then there exists a model  $D$  of  $\phi_M$  such that  $C \subset D$ ,  $X^C = X^D$ , and  $Y^D$  is an end-extension of  $Y^C$ . Furthermore,  $D$  admits an ample filtering.

*Proof.* Fix a sequence  $\langle C_\alpha, A_\alpha, B_\alpha : \alpha < \kappa \rangle$  witnessing that  $C$  admits an ample filtering. In particular, at each stage  $\alpha + 1$ ,  $C_{\alpha+1}$  is the disjoint amalgam of  $C_\alpha$  and  $B_\alpha$ , over  $A_\alpha \subset C_\alpha$  with  $A_\alpha, B_\alpha \in \mathbf{K}_0$ .

As in Corollary 2.7, extend the vocabulary  $\tau$  to  $\tau'$  by adding a new predicate  $Q$ , and let  $\mathbf{K}'_0$  be the class of finite  $\tau'$ -structures  $N$  such that

- (i)  $N \upharpoonright \tau \in \mathbf{K}_0$ ;
- (ii)  $X^N \subset Q^N$ ; and

(iii)  $Y^N \cap Q^N$  is an initial segment of  $Y^N$ .

Similarly to Definition 2.8, define that a finite  $\tau'$ -structure  $N$  is in  $\hat{\mathbf{K}}'_0$  if

- (1) for every finite  $A \subset N$  there exists some finite substructure  $A'$  of  $N$  with  $A \subset A'$  and  $A' \in \mathbf{K}'_0$ .
- (2)  $(Y^N, <)$  is a linear order without endpoints.

We now create a new sequence  $\langle D_\alpha, A'_\alpha, B'_\alpha : \alpha < \kappa \rangle$  of  $\tau'$ -structures, so that the restriction of this new sequence to the  $Q$ -sort, is the original sequence  $\langle C_\alpha, A_\alpha, B_\alpha : \alpha < \kappa \rangle$ . More precisely,  $Q^{D_\alpha} \upharpoonright \tau = C_\alpha$ ,  $Q^{A'_\alpha} \upharpoonright \tau = A_\alpha$  and  $Q^{B'_\alpha} \upharpoonright \tau = B_\alpha$ . The relationship we will require between the pairs  $(A_\alpha, B_\alpha)$  and  $(A'_\alpha, B'_\alpha)$  is captured by the following definitions.

Recall Definition 2.4 of a good pair.

**Definition 2.15.** We call  $A', B' \in \mathbf{K}'_0$  a good pair of  $\tau'$ -structures if  $(A' \upharpoonright \tau, B' \upharpoonright \tau)$  is a good pair of  $\tau$ -structures.

If  $A', B' \in \mathbf{K}'_0$  is a good pair of  $\tau'$ -structures then  $(Q(A') \upharpoonright \tau, Q(B') \upharpoonright \tau)$  is a good pair of  $\tau$  structures since all of  $X^{A'}$  satisfies  $Q$ .

**Definition 2.16.** Let  $A, B \in \mathbf{K}_0$  be a good pair of  $\tau$ -structures and let  $A', B' \in \mathbf{K}'_0$  be a good pair of  $\tau'$ -structures. We call  $(A', B')$  a good extension of  $(A, B)$ , if  $A', B'$  are extensions of  $A, B$  respectively, and  $Q^{A'} \upharpoonright \tau = A$  and  $Q^{B'} \upharpoonright \tau = B$ . Call  $(A', B')$  a good extension, if it is a good extension of some  $(A, B)$ .

Note that if  $(A', B')$  is a good extension, then  $|(X^{B'} \cap Q^{B'}) \setminus (X^{A'} \cap Q^{A'})| \geq |(Y^{B'} \cap Q^{B'}) \setminus (Y^{A'} \cap Q^{A'})|$ .

We construct a strictly increasing sequence  $(D_\alpha)_{\alpha < \kappa}$  of  $\mathbf{K}'_0$ -structures with the following properties:

- (1) for all  $\alpha$ ,  $D_\alpha \in \hat{\mathbf{K}}'_0$ ;
- (2) for all  $\alpha$ ,  $Q^{D_\alpha} \upharpoonright \tau = C_\alpha$ ;
- (3) for all successors  $\alpha + 1$ ,  $D_{\alpha+1}$  is a disjoint amalgam of  $D_\alpha$  and  $B'_\alpha$  over  $A'_\alpha$ , where  $(A'_\alpha, B'_\alpha)$  is a good extension of  $(A_\alpha, B_\alpha)$ ;
- (4) for each limit  $\alpha$ ,  $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$ ;
- (5) for each  $\alpha$ , if  $A' \subset D_\alpha, B'$  and  $A', B' \in \mathbf{K}'_0$ , there are  $\kappa$ -many stages  $\alpha$  such that  $A'_\alpha = A'$  and  $B'_\alpha \cong B'$ .

Let  $A \subset C$ ,  $A \in \mathbf{K}_0$ , and  $B \in \mathbf{K}_0$  is an extension of  $A$ . By assumption on the sequence  $(C_\alpha)$ , there are  $\kappa$ -many stages such that  $A_\alpha = A$  and  $B_\alpha \cong B$ . Consider all possible  $\tau'$ -structures  $A' \subset B'$  such that  $B = Q^{B'} \upharpoonright \tau$  and  $A = Q^{A'} \upharpoonright \tau$ . Organize the induction so that there are  $\kappa$ -many stages such that  $A'_\alpha = A'$  and  $B'_\alpha \cong B'$ . Then use Theorem 2.10 to disjointly amalgamate  $D_\alpha$  and  $B'_\alpha$  over  $A'_\alpha$  to  $D_{\alpha+1} \in \mathbf{K}'_0$ , making sure that (a) in the amalgam  $Y \cap Q$  is an initial segment<sup>7</sup> of  $Y$  and (b) the amalgamation agrees with the amalgamation of  $C_\alpha$  and  $B_\alpha$  over  $A_\alpha$ .

Let  $D = \bigcup_{\alpha < \kappa} D_\alpha \upharpoonright \tau$ . Then  $D$  is rich and it satisfies the conclusion of Theorem 2.14.  $\square$

**Theorem 2.17.** For each  $\kappa$ , there exist a model of  $\phi_M$  of type  $(\kappa^+, \kappa)$ .

*Proof.* If  $\kappa = \aleph_0$ , the result holds true by Corollary 2.7. Assume  $\kappa$  is uncountable and let  $C_0$  be a model of type  $(\kappa, \kappa)$  given by Corollary 2.11. By Lemma 2.13, assume that  $C_0$  admits ample filtering.

<sup>7</sup>This is easy to obtain as we can place  $Y^B \setminus Y^C$  at the end of  $Y^C$ .

Construct a sequence  $(C_\alpha)_{\alpha < \kappa^+}$  such that each  $C_\alpha$  is a model of  $\phi_M$  that admits ample filtering,  $X^{C_\alpha} = X^{C_0}$  and  $Y^{C_{\alpha+1}}$  is an end-extension of  $Y^{C_\alpha}$ . Use Theorem 2.14 for the successor stages. At limit stages take unions. The construction continues past the limit stages, because the union of models of the same cardinality that admit ample filtering, also admits ample filtering.  $\square$

This completes the proof of Theorem 1.5. We can actually do a little better. We prove Theorem 2.19 which is an extension of Theorem 1.5. We first state the useful corollary Theorem 2.18 of Theorem 2.19.

**Theorem 2.18.** *If  $\kappa$  is homogeneously characterized by a complete  $\mathcal{L}_{\omega_1, \omega}(\tau)$ -sentence, there is another complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\chi$  in a vocabulary  $\tau'$  extending  $\tau$  with a predicate  $U$  such that:*

- (1)  $\chi$  characterizes  $\kappa^+$ ;
- (2)  $U$  is a set of absolute indiscernibles for the countable model of  $\chi$ ; and
- (3)  $U$  has size  $\leq \kappa$  in all models of  $\chi$  with maximum attained.

The proof of this result is straightforward from the following theorem.

**Theorem 2.19.** *There is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi_{M'}$  in a vocabulary with unary predicates  $X, Y$  and a binary predicate  $<$  such that any model of  $\phi_{M'}$  satisfies the first 3 conditions of Theorem 1.5*

- (1)  $\phi_{M'}$  has arbitrarily large models,
- (2) if  $M \models \phi_{M'}$ ,  $|M| \leq |X^M|^+$ ,
- (3) for each  $\kappa$ , there exist a model of  $\phi_{M'}$  of type  $(\kappa^+, \kappa)$ ;

and also there is a unary predicate  $U$  such that

- (1) if  $M'$  is the countable model of  $\phi_{M'}$ ,  $U^{M'}$  is a set of absolute indiscernibles,
- (2) if  $M \models \phi_{M'}$ ,  $|U^M| \leq |X^M|$ ,
- (3) for each  $\kappa$ , there is a model  $N$  of  $\phi_{M'}$  with  $|N| = \kappa^+$  and  $|U^{M'}| = \kappa$ .

The proof of Theorem 2.19 mirrors the proof of Theorem 1.5 starting with Construction 2.2 up to Lemma 2.17. Most of these proofs work with minor modifications. We only outline the key differences here and leave the details to the reader.

**Construction 2.20.** *Let  $\tau'$  be union of  $\tau \cup \{V, U, P\}$ , where  $\tau$  is the vocabulary from Construction 2.2. Let  $\mathbf{K}_1$  be the collection of finite  $(\tau')$ -structures  $A'$  such that*

- (1)  $X, Y, U$  are disjoint;
- (2)  $P$  defines a projection function from  $X$  into  $U$ , and  $P$  is void everywhere else;
- (3)  $(A' \setminus U) \upharpoonright \tau$  is a structure  $A$  in  $\mathbf{K}_0$ .

Because  $\mathbf{K}_0$  satisfies disjoint amalgamation, the proof of Lemma 2.5 can be used to prove that the same is true for  $\mathbf{K}_1$ ; to amalgamate  $B'$  and  $C'$  over  $A'$  first amalgamate the associated  $B$  and  $C$  over  $A$  in  $\mathbf{K}_0$  by 2.5, then add the disjoint union of  $U^{B'}$  and  $U^{C'}$  over  $U^{A'}$  with the same values for  $P$ . This guarantees the following analogue of Theorem 2.6; as usual the generic model construction guarantees that  $P$  is onto.

**Theorem 2.21.** *There is a countable  $\mathbf{K}_1$ -generic (and atomic) model  $M'$  (with Scott sentence  $\phi_{M'}$ ) which is an expansion of the model  $M$  from Theorem 1.5 by a set of absolute indiscernibles  $U^{M'}$ , and  $P^{M'}$  defines a projection function from  $X^{M'}$  onto  $U^{M'}$  with the set  $\{x \in X^{M'} \mid P^{M'}(x, u)\}$  being infinite for every  $u \in U^{M'}$ .*

Thus, if  $N \models \phi_{M'}$ ,  $|U^N| \leq |X^N|$  and  $|Y^N| \leq |X^N|^+$ .

To complete the proof of Theorem 2.19, we can repeat the proofs of Theorem 2.10 through Theorem 2.17 with  $\phi_{M'}$  replacing  $\phi_M$ . In the proof of the revised Theorem 2.10, to amalgamate  $B'$  and  $C'$  over  $A'$  first amalgamate the associated  $B$  and  $C$  over  $A$  in  $\mathbf{K}_0$  by 2.10, then then add the disjoint union of  $U^{B'}$  and  $U^{C'}$  over  $U^{A'}$  with the same values for  $P$ . If the amalgam contains new  $X$ -elements that are not in  $X^{B'} \cup X^{C'}$ , e.g. elements introduced by Lemma 2.3, or elements in  $X_{new}$ , then expand the  $U$ -sort by adding a new set  $U_{new}$  of  $U$ -elements disjoint from  $U^{B'} \cup U^{C'}$ , and assign  $P$ -values for the new  $X$ -elements in  $U_{new}$ .

The proofs of Corollary 2.11, Lemma 2.13 and Theorem 2.17 require no changes. In the proof of the revised Theorem 2.14, add to the requirement (ii) that  $X \subset Q$  that also  $U \subset Q$  and then the same argument works.

This concludes the outline of the proof of Theorem 2.19.

Proof of Theorem 2.18. If  $\theta$  homogeneously characterizes  $\kappa$  with the receptive pair  $(\theta, Q)$ , consider the merger  $\chi_{\theta, Q, \phi_{M'}, X}$ . This will restrict the size of  $X$  by the size of  $Q$ . The merger characterizes  $\kappa^+$  and has a set of absolute indiscernibles  $U$  of size  $\leq \kappa$  with maximum attained.  $\square_{2.18}$

Lastly, notice that if  $\mathcal{N}' \models \phi_{M'}$ ,  $|X^{\mathcal{N}'}| = \kappa$  and  $|Y^{\mathcal{N}'}| = \kappa^+$ , the size of the predicate  $U(\mathcal{N}')$  can not be of size  $\kappa^+$ . This makes it impossible to use this technique to homogeneously characterize  $\kappa^+$ .

### 3. MAXIMAL MODELS IN $\kappa$ AND $\kappa^\omega$

Working similarly to Section 1 we construct a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence that admits maximal models in  $\kappa$  and  $\kappa^\omega$ , and has no larger models. But we must define a sentence that transfers from  $\kappa$  to  $\kappa^\omega$  rather than  $\kappa^+$ .

**Theorem 3.1.** *Let  $\phi$  be a complete  $\mathcal{L}_{\omega_1, \omega}(\tau)$ -sentence (in vocabulary  $\tau$ ) with a set of absolute indiscernibles  $U$ . Then there is a complete  $\mathcal{L}_{\omega_1, \omega}(\tau')$ -sentence  $\phi^*$  (in vocabulary  $\tau' \supset \tau$ ) with the property:*

*If  $\phi$  homogeneously characterizes  $\kappa$ , then  $\phi^*$  homogeneously characterizes  $\kappa^\omega$ .*

Proof sketch: Here is the basic idea of the construction.

Fix an infinite set  $X$  and consider the structure  $N$  with universe the disjoint union of  $\omega$ ,  $X^{<\omega}$  and a subset of  $X^\omega$ . Fix a vocabulary  $\tau_1$  with unary predicates  $K, V, F$  denoting these sets, binary predicates  $H, R$  and a ternary predicate  $E$ . Interpret the ‘height’ predicate  $H(\cdot, \cdot)$  on  $K \times V$  so that  $H(n, v)$  holds if and only if  $v \in X^n$ , and the predicate  $R(u, v)$  on  $V^2$  which holds if  $v$  is an immediate successor of  $u$ , and the restriction function  $E$  on  $K \times F \times V$  such that  $E(k, f, v)$  holds if and only if  $f \upharpoonright k = v$ . Of course, by a sentence in  $\mathcal{L}_{\omega_1, \omega}(\tau_1)$  we can require that  $K$  is the standard  $\omega$  and each of  $V$  and  $F$  are sets of functions.

If  $X$  is countable and  $F$  is also countable, the resulting structure has a Scott sentence  $\mu$ . Furthermore, if for every  $v \in X^n$  there exist infinitely many  $f \in F$  with  $f \upharpoonright n = v$ , by Theorem 3.4 of [Sou14] this structure is back and forth equivalent with the model where  $X$  is countable and  $F$  is the set of eventually constant sequences.

Now using the  $\tau$ -sentence  $\phi$  we are able (in an expanded model  $N_*$ ) to bound  $|X| = |K^{N_*}|$  by  $\kappa$  and  $|V^{N_*}|$  by  $\kappa^\omega$ . Form  $\tau'$  by adding a binary symbol  $M(\cdot, \cdot)$  to  $\tau_1 \cup \tau_2$  and predicates  $S(x, \cdot)$  for each  $\tau$ -relation  $S(\cdot)$ .

Assert that the sets  $M(u, \cdot)$  for  $u \in V$  are disjoint and require that for each  $u \in V$ , the set  $M(u, \cdot)$  (under the relations  $S(u, \cdot)$ ) is a model of  $\phi$ . Require further that the set  $R(u, \cdot)$  of the immediate successors of  $u$  is also the set  $U(u, \cdot)$  of absolute indiscernibles of the model  $M(u, \cdot)$  of  $\phi$ . Since  $\phi$  homogeneously characterizes  $\kappa$ , if  $N_* \models \phi^*$ ,  $R^{N_*}(u, \cdot)$  cannot be larger

than  $\kappa$  so the number of immediate successors of  $u$  can not be more than  $\kappa$ . The resulting tree has height  $\omega$  and is  $\leq \kappa$ -splitting. To prevent the first level  $V(0, \cdot)$  from growing arbitrarily we require that  $V(0, \cdot)$  has only one element (the root).

The detailed axiomatization of this structure by a complete sentence of  $\mathcal{L}_{\omega_1, \omega}$  and the proof that it characterizes  $\kappa^\omega$  appear in [Sou14].

**Theorem 3.2.** *Assume  $\lambda \leq \lambda^\omega < \kappa < \kappa^\omega$  and  $\phi_\kappa$  homogeneously characterizes  $\kappa$ . Then there is a complete sentence  $\phi_\kappa^*$  that has maximal models in  $\kappa$  and  $\kappa^\omega$ , no maximal models in any other cardinality, and no models larger than  $\kappa^\omega$ .*

*Proof.* By Theorem 1.6, we can assume  $\phi_\kappa$  has maximal models of type  $(\kappa, \lambda)$  for all  $\lambda \leq \kappa$ . Let  $\phi_\kappa^*$  be the sentence from Theorem 3.1. If for every  $u \in V$ , the set  $M(u, \cdot)$  is a maximal model of  $\phi_\kappa$  of type  $(\kappa, \lambda)$ , then the resulting tree is  $\lambda$ -splitting and the associate model is a maximal model of  $\phi_\kappa^*$  of size  $\max\{\kappa, \lambda^\omega\} = \kappa$ .

Further, if for every  $u \in V$ , the set  $M(u, \cdot)$  is a maximal model of  $\phi_\kappa$  of type  $(\kappa, \kappa)$ , then the resulting tree is  $\kappa$ -splitting and yields a maximal model of  $\phi_\kappa^*$  of size  $\max\{\kappa, \kappa^\omega\} = \kappa^\omega$ .

Finally notice that for all  $\mu < \kappa$ , if  $\mu^\omega > \kappa$ , then  $\mu^\omega = \kappa^\omega$ . So, we get maximal models only in cardinality  $\kappa$  and in cardinality  $\kappa^\omega$ .  $\square$

Replacing the construction that characterized  $\kappa^\omega$  from [Sou14] with the construction that characterized  $\kappa^{\aleph_\alpha}$ ,  $\alpha < \omega_1$ , from [Sou12] (cf. Theorem 3) one can prove the following theorem.

**Theorem 3.3.** *Assume  $\alpha < \omega_1$ ,  $\lambda \leq \lambda^{\aleph_\alpha} < \kappa < \kappa^{\aleph_\alpha}$  and there is a sentence  $\phi_\kappa$  that homogeneously characterizes  $\kappa$ . Then there is a complete sentence  $\phi_\kappa^*$  that has maximal models in  $\kappa$  and  $\kappa^{\aleph_\alpha}$ , no maximal models in any other cardinality, and no models larger than  $\kappa^{\aleph_\alpha}$ .*

The next theorem describes where JEP holds/fails in the examples of this section. Note that the notion of strong embedding  $\prec_{\mathbf{K}}$  specified below, maybe different than elementary substructure in the fragment generated by  $\phi_\kappa^*$ .

Let  $\phi_\kappa$  be a sentence that homogeneously characterizes  $\kappa$  with  $P$  a set of absolute indiscernibles and let  $\phi_\kappa^*$  be as in Theorem 3.1. Let  $\mathbf{K}$  be the collection of models that satisfy  $\phi_\kappa^*$  and let  $\mathcal{N}_0 \prec_{\mathbf{K}} \mathcal{N}_1$  if  $\mathcal{N}_0 \subset \mathcal{N}_1$  and for each  $u \in V$ ,  $M(u, \cdot)^{\mathcal{N}_0} \cup R(u, \cdot)^{\mathcal{N}_0} \prec M(u, \cdot)^{\mathcal{N}_1} \cup R(u, \cdot)^{\mathcal{N}_1}$ , where  $\prec$  is understood as elementary substructure in the fragment generated by  $\phi_\kappa$ .

**Theorem 3.4.** *If the models of  $\phi_\kappa$  satisfy JEP( $< \kappa$ ), then the same is true for  $(\mathbf{K}, \prec_{\mathbf{K}})$ .*

*Proof.* Let  $\mathcal{N}_0, \mathcal{N}_1 \in \mathbf{K}$  and  $|\mathcal{N}_0|, |\mathcal{N}_1| < \kappa$ . It follows that the tree contained in either  $\mathcal{N}_0$  or  $\mathcal{N}_1$  must satisfy  $|V(0, \cdot)| = 1$  and for each  $u \in V(n, \cdot)$ ,  $|M(u, \cdot) \cup R(u, \cdot)| < \kappa$ . The goal is to embed  $\mathcal{N}_0, \mathcal{N}_1$  into a common  $\mathcal{N} \in \mathbf{K}$ .

First embed the root  $a_0$  of the tree in  $\mathcal{N}_0$  and the root  $a_1$  of the tree in  $\mathcal{N}_1$  into the root of the tree in  $\mathcal{N}$ , call it  $a$ . Since the models of  $\phi_\kappa$  satisfy JEP( $< \kappa$ ), joint embed  $M(a_0, \cdot)^{\mathcal{N}_0} \cup R(a_0, \cdot)^{\mathcal{N}_0}$  and  $M(a_1, \cdot)^{\mathcal{N}_1} \cup R(a_1, \cdot)^{\mathcal{N}_1}$  to a common model  $M(a, \cdot)^{\mathcal{N}} \cup R(a, \cdot)^{\mathcal{N}}$ , say through embeddings  $f_0, f_1$ . If  $v \in R(a, \cdot)^{\mathcal{N}}$  and  $v \notin \text{range}(f_0) \cup \text{range}(f_1)$ , then attach a copy of  $(\aleph_0)^\omega$  with root  $v$  into  $\mathcal{N}$ . If  $v \in \text{range}(f_0) \setminus \text{range}(f_1)$ , then embed  $M(f_0^{-1}(v), \cdot)^{\mathcal{N}_0} \cup R(f_0^{-1}(v), \cdot)^{\mathcal{N}_0}$  into  $M(v, \cdot)^{\mathcal{N}} \cup R(v, \cdot)^{\mathcal{N}}$ . Similarly work if  $v \in \text{range}(f_1) \setminus \text{range}(f_0)$ . If  $v \in \text{range}(f_0) \cap \text{range}(f_1)$ , then joint embed the models  $M(f_0^{-1}(v), \cdot)^{\mathcal{N}_0} \cup R(f_0^{-1}(v), \cdot)^{\mathcal{N}_0}$  and  $M(f_1^{-1}(v), \cdot)^{\mathcal{N}_1} \cup R(f_1^{-1}(v), \cdot)^{\mathcal{N}_1}$  into  $M(v, \cdot)^{\mathcal{N}} \cup R(v, \cdot)^{\mathcal{N}}$ . Then repeat by induction the same process for each  $u \in V(n, \cdot)$ , and finally embed each  $f \in F^{\mathcal{N}_0}$  and  $f \in F^{\mathcal{N}_1}$  into the corresponding branch in  $F^{\mathcal{N}}$ . The reader can verify that  $\mathcal{N}_0, \mathcal{N}_1 \prec_{\mathbf{K}} \mathcal{N}$ .  $\square$

Notice that disjoint-JEP fails for  $(\mathbf{K}, \prec_{\mathbf{K}})$ . One reason is that the root of the tree in  $\mathcal{N}_0$  and the root of the tree in  $\mathcal{N}_1$  must embed into the same element, namely the root of the tree in  $\mathcal{N}$ .

#### 4. AN EXAMPLE WITH MAXIMAL MODELS IN COUNTABLY MANY CARDINALITIES

In this section we construct a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence that admits maximal models in countably many cardinalities.

We adapt ideas from [Hjo02] and [Sou13] to characterize pairs of cardinals  $\kappa, 2^\kappa$ . Combining this result with our basic technique we are able to construct sentences with maximal models in cardinalities  $2^\lambda$ , for all  $\lambda \leq \kappa$ , where  $\kappa$  is a homogeneously characterizable cardinal; this is interesting when  $\kappa < 2^\lambda$ . The following version of Theorem 4.29 of [Sou13] establishes the necessary transfer result.

**Theorem 4.1.** *There is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi(X, Y)$  such that if  $M \models \phi(X, Y)$  and  $|X| = \kappa$  then  $|Y| \leq 2^\kappa$ ; the maximum is attained.*

*Proof Sketch:* Fix a set  $X$  and a binary relation on  $X$  which defines a dense linear order. As in Section 4 of [Sou13], the key idea is the following condition crystallizes the property of the meet function on  $2^X$ . Construct (via a generalized Fraïssé construction) a map  $f$  from  $Y^2$  to  $X$  such that for distinct<sup>8</sup>  $a_0, a_1, a_2 \in Y$ , if  $f(a_0, a_1) \neq f(a_0, a_2)$  then  $f(a_1, a_2) = \min\{f(a_0, a_1), f(a_0, a_2)\}$ . But if  $f(a_0, a_1) = f(a_0, a_2)$  then  $f(a_1, a_2) > f(a_0, a_1) = f(a_0, a_2)$ . The resulting structure has a Scott sentence  $\phi'$  in  $\mathcal{L}_{\omega_1, \omega}$  such that if  $\mathcal{M} \models \phi'$ ,  $|Y^{\mathcal{M}}| \leq 2^{|X^{\mathcal{M}}|}$  and  $2^{|X^{\mathcal{M}}|}$  is possible. Namely, the maximum is achieved if  $(X, <)$  has a  $<$ -cofinal sequence of length  $\kappa$ .

**Theorem 4.2.** *Suppose that  $\psi$  is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence that homogeneously characterizes  $\kappa$  with absolute indiscernibles in the predicate  $P$ . Then there is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\psi'$  that characterizes  $2^\kappa$ .*

*Furthermore for every  $\lambda \leq \kappa$ , there exists a maximal model of  $\psi'$  of size  $\max\{\kappa, 2^\lambda\}$ .*

*Proof.* By Theorem 1.6, we can assume  $\psi$  has maximal models of type  $(\kappa, \lambda)$ , for all  $\lambda \leq \kappa$ .

Merge  $\psi$  with the complete sentence  $\phi$  from Theorem 4.1 identifying  $X$  with  $P$ . Let  $\psi' = \chi_{\psi, P, \phi, X}$ . By Fact 1.3 (1),  $\psi'$  is a complete sentence.

Now suppose  $M$  is a maximal model of  $\psi$  of type  $(\kappa, \lambda)$ . Linearly order  $P^M$  so that  $(P^M, <)$  is a dense linear order with a  $\lambda$ -cofinal sequence. Then by Theorem 4.1 we get a maximal model of  $\psi'$  with cardinality  $\max\{\kappa, 2^\lambda\}$ .  $\square$

Exactly what this says about the cardinality of maximal models depends on the cardinal arithmetic. We just give some sample applications of Theorem 4.2 with various choices of the  $\lambda$ .

**Theorem 4.3.** *Assume  $\kappa$  is a homogeneously characterizable cardinal and for simplicity let  $2^{\aleph_0} \geq \kappa$ . Then there is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi_\kappa$  with maximal models in cardinalities  $2^\lambda$ , for all  $\lambda \leq \kappa$ .*

**Theorem 4.4.** *Assume for simplicity that  $2^{\aleph_0} > \aleph_\omega$ . For each  $n \in \omega$ , there is a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi'_n$  with maximal models in cardinalities  $2^{\aleph_0}, 2^{\aleph_1}, \dots, 2^{\aleph_n}$ .*

**Theorem 4.5.** *Let  $(\kappa_i | i \leq n)$  be an increasing sequence of cardinals such that  $\text{cf}(\kappa_i) > \aleph_i$ . It is consistent that there is a complete sentence with maximal models in cardinalities  $\kappa_0, \kappa_1, \dots, \kappa_n$ .*

<sup>8</sup>For further restrictions see Definition 4.6 of [Sou13].

*Proof.* By Easton's Theorem, there is model of ZFC where  $2^{\aleph_i} = \aleph_i$ . Then apply Theorem 4.4.  $\square$

## 5. CONCLUSION

The examples from [Sou12] and [Sou14] have the maximal number of models in the cardinals they characterize, namely  $\aleph^\omega$  and  $\aleph^{\aleph_\alpha}$  respectively. As a consequence, the sentence  $\phi_\aleph^*$  in Theorem 3.2 has the maximal number of maximal models in  $\aleph$  and  $\aleph^\omega$ , and the sentence  $\phi_\aleph^*$  in Theorem 3.3 has the maximal number of maximal models in  $\aleph$  and  $\aleph^{\aleph_\alpha}$ .

This motivates the following question.

**Open Question 5.1.** *Is there a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence  $\phi$  which has at least one maximal model in an uncountable cardinal  $\aleph$ , but less than  $2^\aleph$  many models?*

In particular, a negative answer to Open Question 5.1 implies a negative answer to the following Open Question 5.2, which was asked in [BKL14] and which in return relates to old conjectures of S. Shelah.

**Open Question 5.2** ([BKL14]). *Is there a complete  $\mathcal{L}_{\omega_1, \omega}$ -sentence which characterizes an uncountable cardinal  $\aleph$  and it has less than  $2^\aleph$  many models in cardinality  $\aleph$ ?*

Finally, we want to stress the differences in techniques of this paper from [BKS15]. The main idea behind [BKS15] is certain combinatorial properties of bipartite graphs. Here the main construction is a refinement of the construction from [Kni77] combined with repeated use of sets of absolute indiscernibles. All the examples presented here are *complete* sentences with maximal models in more than one cardinality, which *do not* have arbitrarily large models. In [BKS15] the examples are *incomplete* sentences with maximal models in more than one cardinality, which *do* have arbitrarily large models. The following question arises naturally: Find examples of complete  $\mathcal{L}_{\omega_1, \omega}$ -sentences with maximal models in more than one cardinality, which also have arbitrarily large models. Also, in [BKS15], the JEP- and AP-spectra of the sentences presented there are precisely calculated. The JEP- and AP-spectra of our examples seem harder to calculate and the question remains open.

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