# Strongly minimal Steiner Systems: Model Theory, Universal Algebra, Combinatorics
# University of Chicago logic seminar

John T. Baldwin
University of Illinois at Chicago

Nov 15, 2021

# Two Goals

1. Introduce some of the wide variety of strongly minimal ‘Hrushovski structures’, especially Steiner systems.
   1. Summarise connections with universal algebra and ‘coordinatization’.
   2. Describe applications of variants of concepts from the study of finite Steiner systems to the infinite.
2. Gesture at the proof that many (most??) strongly minimal sets given by an *ab initio* Hrushovski construction do not admit elimination of imaginaries and have very limited definable closure

1. Strongly Minimal Theories
2. Quasi-groups and Steiner systems
3. Coordinatization by varieties of algebras
4. Groups, definable closure, and elimination of imaginaries
5. The General Construction
6. The structure of $\mathrm{acl}(X)$
7. Applications to Combinatorics
8. Diversity and Classification

Joint work with Vitkor Verbovskiy
Thanks to Joel Berman, Gianluca Paolini,Omer Mermelstein, and Viktor Verbovskiy.

# Strongly Minimal Theories

# STRONGLY MINIMAL

**Definition**

$T$ is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

# STRONGLY MINIMAL

**Definition**

$T$ is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

**Definition**

$a$ is in the algebraic closure of $B$ ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

**Theorem**

If $T$ is strongly minimal algebraic closure defines matroid/combinatorial geometry.

# Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

## Definition

A closure system is a set $G$ together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

**A1.** $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
**A2.** $X \subseteq cl(X)$
**A3.** $cl(cl(X)) = cl(X)$

$(G, \mathrm{cl})$ is pregeometry if in addition:
**A4.** If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Usually this $\mathrm{acl}$ pre-geometry is not definable.

# The trichotomy

## Zilber Conjecture

The $\mathrm{acl}$-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. 'bi-interpretable' with an algebraically closed field (non-locally modular)

Zilber: geometries $\leftrightarrow$ canonical structures

Hrushovski gave a method of constructing strongly minimal sets that have flat geometries and admit no associative binary function.

There is no apparent canonical structure - only a (very flexible) method.
Zariski Geometries aim at canonical structures with more restrictions.

# The diversity of flat strongly minimal sets

The ‘Hrushovski construction’ actually has 5 parameters:

## Describing Hrushovski constructions

1. $\sigma$: vocabulary
2. $\boldsymbol{L}_0$: A $\forall\exists$ axiomatized collection of finite $\sigma$-structures.
3. $\epsilon$: A flat (hence submodular) function from $\boldsymbol{L}_0^*$ to $\mathbb{Z}$.
4. $\boldsymbol{L}_0$: $\boldsymbol{L}_0^*$ defined using $\epsilon$.
5. $\mu$: a function bounding the number of 0-primitive extensions of an $A \in \boldsymbol{L}_0$ are in $L_\mu$.

To organize the classification of the theories each choice of a class **U** of $\mu$ yields a collection of $T_\mu$ with similar properties.

# Quasi-groups and Steiner systems

# Definitions

A Steiner system with parameters $t, k, n$ written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$ and allow infinite $n$.

# Some History

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

# Some History

For which n's does an $S(2, k, n)$ exist?
for $k = 3$

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939) Skolem ( $6n + 1$, 1958)

# Linear Spaces

## Definition: linear space

The vocabulary contains a single ternary predicate $R$, interpreted as collinearity. A linear space satisfies

1. $R$ is a predicate of sets (hypergraph)
2. Two points determine a line

$\alpha$ is the iso type of $(\{a, b\}, \{c\})$ where $R(a, b, c)$.

## Groupoids and quasi-groups

1. A groupoid (magma) is a set $A$ with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancelation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

# existentially closed 3-Steiner Systems

## Barbina-Casanovas

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures $(A, R)$ which are the graphs of a Steiner quasigroup.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory $T^*_{sq}$.
2. $T^*_{sq}$ has
    1. quantifier elimination
    2. $2^{\aleph_0}$ 3-types;
    3. the generic model is prime and locally finite;
    4. $T^*_{sq}$ has $TP_2$ and $NSOP_1$.

# Omitting classes of Steiner quasigroups

## Horsley- Webb

Consider the class $\tilde{\boldsymbol{K}}$ of finite structures $(A, *)$ which are Steiner quasigroups that omit a family $\boldsymbol{F}$ of finite quasigroups (such that there exists an $A$ which neither extends nor embeds in any member of $\boldsymbol{F}$).

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus
2. $\tilde{\boldsymbol{K}}$ has a countable locally finite generic model.

They have a rather complex notion of subsystem but for quasigroups $(Q, *)$ it is just subalgebra.

# Hrushovki's basic construction vs Steiner

## Example

1. $\sigma$ has a single ternary relation $R$;
2. $\boldsymbol{L}_0$: All finite $\sigma$-structures
   finite linear spaces
3. $\epsilon(A)$ is $|A| - r(A)$, where $r(A)$ is the number of tuples realizing $R$.
   $\delta(A) = |A| - \sum_{\ell \in L(A)}(|\ell| - 2)$.
4. $A \in \boldsymbol{L}_0^*$ if $\epsilon(B) \geq 0$ for all $B \subseteq A$.
   Replace $\epsilon$ by $\delta$.
5. **U** is those $\mu$ with $\mu(A/B) \geq \epsilon(B)$.
   $\mu(\boldsymbol{\alpha}) = q - 2$ gives line length 2.

# Strongly minimal linear spaces I

## Fact

Suppose $(M, R)$ is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

## Corollary

If $(M, R)$ is a strongly minimal linear system, for some $k$, all lines have length at most $k$.

# Specific Strongly minimal Steiner Systems

## Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with $v$ points such that each line has $k$ points.

## Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family $T_\mu$ of strongly minimal $(2, k, \infty)$ Steiner-systems.

There is no infinite group definable in any $T_\mu$. More strongly, Associativity is forbidden.

# Coordinatization by varieties of algebras

# Coordinatizing Steiner Systems

## Weakly coordinatized

A collection of algebras $V$ '(weakly) coordinatizes' a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in $V$ definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in $V$.

## Coordinatized

A collection of algebras $V$ definably coordinatizes a class $\mathcal{S}$ of $k$-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

# Coordinatizing Steiner triple systems

## Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$$

# Coordinatizing Steiner triple systems

## Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$$

Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct $a, b, c$:

$$R(a, b, c) \text{ if and only if } a * b = c$$

## Theorem

Every strongly minimal Steiner (2,3)-system given by $T_\mu$ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroup definable in the system.

# 2 VARIABLE IDENTITIES

## Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

## Definition

Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on $F$ by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

# Coordinatizing Steiner Systems

## Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from $V$ then so does $\mathcal{S}$.

## Definition [Pad72]

An $(r, k)$ variety is one in which every $r$-generated algebra has cardinality $k$ and is freely generated by every $n$-elements.

## Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality $k$, each $A \in V$ determines a Steiner $k$-system.
(The 2-generated subalgebras.)
And each Steiner $k$-system admits a weak coordinatization.

Can this coordinatization be definable in the strongly minimal $(M, R)$?

# Forcing a prime power

## Fact (Ganter-Werner et al)

1. [Ś61, Grä63] The only $(r, k)$ varieties are those where $r = 0$, $k = 0$; $r = k$; $r = 2$, $k = q = p^n$, for a prime $p$ and a natural number $n$; $r = 3$, $k = 4$.
2. [GW75, GW80] For each $q$, the class of $q$-Steiner systems is coordinatized by a $(2, q)$-variety of block algebras

Proof: As, if an algebra $A$ is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

# Definability

## Definability Theorem

Suppose $q$ is a prime power and $\mu(\alpha) = q - 2$. Then

1. Each $(M, R) \models T_\mu$ is *coordinatizable* by an algebra $(Q_M, *)$ in $V$.
2. $R(x, y, z)$ is definable in $(Q_M, *)$ by the formula $\theta_F(x, y, z)$ that is the disjunction of the terms $z = f_i(x, y)$ where the $f_i(x, y)$ list the terms generating $F = F_2(V)$. Thus, $(M, R)$ is definable in $(Q_M, *)$.
3. There is an (incomplete) first order theory $\check{T}_\mu$ in the vocabulary $\{*\}$ such that each model of $T_\mu$ is coordinatized by a model of $\check{T}_\mu$.

## Proof

1. 1) and 2)are immediate from the general coordinatization theorem.
2. 3) Let $\Delta_F(x, y, f_1(x, y), \dots f_k(x, y))$ be the quantifier-free diagram of $F$. By 2-transitivity of $F_2$ , any $x, y$ does. Axiomatize $\check{T}_\mu$ by:

$$Eq(V) \cup \{(\forall x, y)\Delta_F(x, y, f_1(x, y), \dots f_k(x, y))\} \cup \{\phi\restriction(R/\theta_F) : \phi \in T_\mu\}$$

# Non-definability

**Theorem: (B) Non-definability in $(M, R)$**

If $\mu(\alpha) = k > 1$ this coordinatization is not definable in $(M, R)$.

**Proof**

Without loss of generality, let $(M, R)$ be the countable generic and suppose it is coordinatized by $(Q_M, *)$.
Let $\{a, b\}$ be a strong substructure of $(M, R)$ (i.e. $d(\{a, b\} = 2)$ and let $c_1, \ldots c_k$ fill out the line through $a, b$ to a structure $A$. By genericity there is a strong embedding of $A$ into $M$.

Then all triples $a, b, c_i$ realize the same quantifier free $R$-type and $A \leq M$ implies for any permutation $\nu$ of $k$ fixing $0, 1$, for $2 \leq i < k$, there is an automorphism of $(M, R)$ fixing $a, b$ and taking $c_i$ to $c_{\nu(i)}$.
Thus, $a * b$ cannot be definable in $(M, R)$.

# Groups, definable closure, and elimination of imaginaries

This section is about arbitrary strongly minimal theories not just Hrushovski constructions.

# Baizhanov's Question

**Question (1990's)**

Does every strongly minimal set that admits elimination of imaginaries interpret an algebraically closed field?

**Partial Answer**

1. Infinite language: No! Verbovskiy
2. finite language:
   1. Yes! for one ternary relation: constructions of [Hru93, BP20].
   2. A program for other flat geometries

# Group Action and Definable Closure

Fix $I$, a finite set of independent points in the model $M \models T$.

## 2 groups

Let $G_{\{I\}}$ be the set of automorphisms of $M$ that fix $I$ setwise and $G_I$ be the set of automorphisms of $M$ that fix $I$ pointwise.

## Definition

1. $\mathrm{dcl}^*(I)$ consists of those elements that are fixed by $G_I$ but not by $G_X$ for any $X \subsetneq I$.
2. The *symmetric definable closure* of $I$, $\mathrm{sdcl}^*(\mathrm{I})$, consists of those elements that are fixed by $G_{\{I\}}$ but not by $G_{\{X\}}$ for any $X \subsetneq I$.

$\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ implies $T$ does not admit elimination of imaginaries.

# Finite Coding

## Definition

A finite set $F = \{\overline{a}_1, \ldots, \overline{a}_k\}$ of tuples from $M$ is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over $A$ if

$$\sigma(F) = F \Leftrightarrow \sigma|S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples $F$ is coded by some set $S$ over $\emptyset$.

If there exists $I$ with $\mathrm{dcl}^*(I) = \emptyset$, $T$ does not have the finite set property.

# $\mathrm{dcl}^*$ and elimination of imaginaries

**Fact: Elimination of imaginaries**

A theory $T$ admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no

**Fact**

*If $T$ admits weak elimination of imaginaries then $T$ satisfies the finite set property if and only $T$ admits elimination of imaginaries.*

Since every strongly minimal theory with $\mathrm{acl}(\emptyset)$ infinite has weak elimination of imaginaries. [Pil99], we have

If such a strongly minimal $T$ has only essentially unary definable binary functions it does not admit elimination of imaginaries.

# No definable binary function/elimination of imaginaries: Sufficient

## Lemma

*Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and $M$ is a generic model of a strongly minimal theory.*

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then $I$ is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then $I$ is not finitely coded and there is no parameter free definable binary function.*

# ‘Non-trivial definable functions’

## Definition

Let $T$ be a strongly minimal theory. function $f(x_0 \ldots x_{n-1})$ is called *essentially unary* if there is an $\emptyset$-definable function $g(u)$ such that for some $i$, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.

## Lemma

For a strongly minimal $T$ the following conditions are equivalent:

1. for any $n > 1$ and any independent set $I = \{a_1, a_2, \ldots a_n\}$, $\mathrm{dcl}^*(I) = \emptyset$;
2. every $\emptyset$-definable $n$-ary function ($n > 0$) is essentially unary;
3. for each $n > 1$ there is no $\emptyset$-definable truly $n$-ary function in any $M \models T$.

## The General Construction

This section applies to the original Hrushovski context (one ternary function) and to $q$-Steiner systems.

# Amalgamation and Generic model

We study classes $\boldsymbol{K}_0$ of finite structures $A$
with $\delta(A') \geq 0$, for every $A' \subset A$.
$d_M(A/B) = \min\{\delta(A'/B) : A \subseteq A' \subset M\}$.

$A \leq M$ if $\delta(A) = d(A)$.
When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

# The main result: Classifying $\mathrm{dcl}$ [BV21]

## Theorem

Let $T_\mu$ be a strongly minimal theory as in Hrushovski's original paper. I.e. $\mu \in \mathcal{U} = \{\mu : \mu(A/B) \geq \delta(B)\}$). Let $I = \{a_1, \ldots, a_v\}$ be a tuple of independent points with $v \geq 2$.

$G_I$ If $T_\mu$ triples

$$\mathcal{U} \supseteq \mathcal{T} = \{\mu : \mu(A/B) \geq 3\}$$

then $\mathrm{dcl}^*(I) = \emptyset$
$\mathrm{dcl}(I) = \bigcup_{a \in I} \mathrm{dcl}(a)$
and every definable function is essentially unary (Definition 28).

$G_{\{I\}}$ In any case $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$
$\mathrm{sdcl}(\mathrm{I}) = \bigcup_{\mathrm{a} \in \mathrm{I}} \mathrm{sdcl}(\mathrm{a})$
and there are no $\emptyset$-definable symmetric (value does not depend on order of the arguments) truly $v$-ary function.

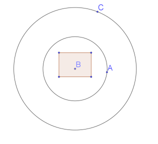
In both cases $T_\mu$ does not admit elimination of imaginaries and the algebraic closure geometry is not disintegrated.

# Primitive Extensions and Good Pairs

## Definition

Let $A, B, C \in \mathbf{K}_0$.

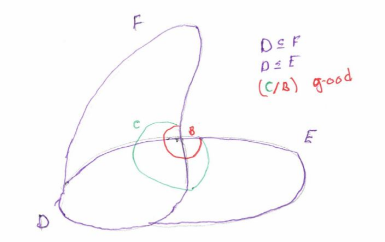
1. $C$ is a 0-*primitive extension* of $A$ if $C$ is minimal with $\delta(C/A) = 0$.



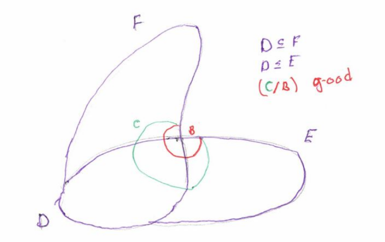

2. $C$ is good over $B \subseteq A$ if $B$ is minimal contained in $A$ such that $C$ is a 0-*primitive extension* of $B$. We call such a $B$ a *base*.

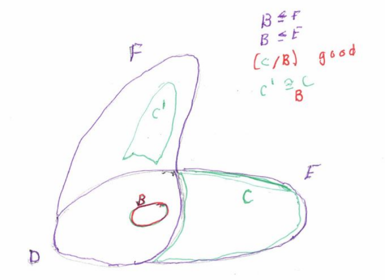
$\alpha$ is the isomorphism type of $(\{a, b\}, \{c\})$,

# The Amalgamation



# The Amalgamation



# Overview of the analysis of $\mathrm{acl}(I)$

## Realization of good pairs

1. A good pair $C/B$ *well-placed* by $A$ in a model $M$, if $B \subseteq A \leq M$ and $C$ is 0-primitive over $X$.
2. For any good pair $(C/B)$, $\chi_M(B, C)$ is the maximal number of disjoint copies of $C$ over $B$ appearing in $M$.
3. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair $(A, B)$.

If $C/B$ is well-placed by $A \leq M$, $\chi_M(B, C) = \mu(B/C)$

# The structure of $\mathrm{acl}(X)$

# $G$-decomposable sets

## Definition

Let $G$ be $G_I$ or $G_{\{I\}}$, $I \leq M$ independent.
$\mathcal{A} \subseteq M$ is $G$-decomposable if

1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is $G$-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

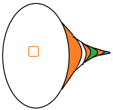
## Fact

There are $G$-decomposable sets.
Namely for any finite $U$ with $d(U/I) = 0$,

$$\mathcal{A} = \mathrm{icl}(\mathrm{I} \cup \mathrm{G}(\mathrm{U}))$$

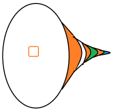
# Constructing a $G$-decomposition

Linear Decomposition: Each annulus is primitive over its predecessors.

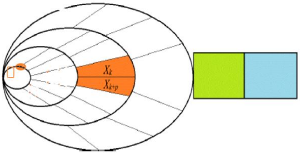


# Constructing a $G$-decomposition

Linear Decomposition: Each annulus is primitive over its predecessors.



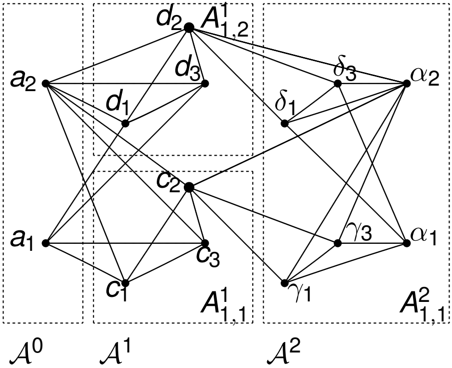
Tree Decomposition: Each pie piece is primitive over previous level.




Prove by induction on levels that $\mathrm{dcl}^*(I) = \emptyset$. $(\mathrm{sdcl}^*(\mathrm{I}) = \emptyset)$

# A non-trivial definable binary function

In the diagrams, we represent a triple satisfying $R$ by a triangle.



# Pasch Configuration

## Definition

Let $X$ be finite partial Steiner system. A Steiner system $(M, R)$ is *anti-$X$* if there no embedding of $X$ into $M$.

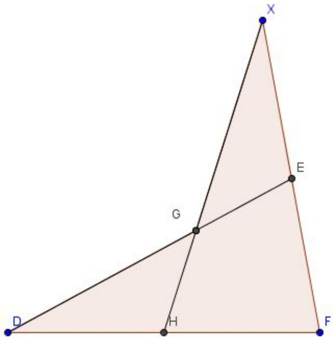



Figure: Pasch configuration: $\mathcal{P}$

# 'few' Pasch and no Pasch

## Fact

If there is an infinite group (associative binary operation) interpretable in a structure $M$ then there is an instance of the Pasch diagram where a point $c$ is on a line through $ab$ if $c \in \mathrm{acl}(a, b)$. No such diagram with points having positive dimension is possible in a strongly minimal set with a flat geometry.

But since $\delta(\mathcal{P}) = 2$ there will be copies of $\mathcal{P}$ in our Steiner systems as constructed. But,

## Theorem

The subclass of $\boldsymbol{K}_0^P$ of those finite structures with 3-element lines that omit the Pasch configuration satisfies amalgamation. Thus, there are strongly minimal anti-Pasch Steiner triple systems.

Without the strongly minimal, this is proved in [HW21].

# $\infty$-sparse configurations

[CGGW10, page 116] prove there is are continuum many countable $\infty$-sparse configurations.

## Definition

A Steiner triple system $(M, R)$ is $\infty$-sparse if there is no $A \subseteq M$ with $|A| \geq 6$ and $\delta(A) = 2$.

## Definition

Let $\boldsymbol{K}_0^{sp}$ be the subclass of $\boldsymbol{K}^*$ (linear spaces) such that for every $B \subseteq A$:

$$(\#)\ |B| > 1 \ \rightarrow\ \delta(B) > 1\ \&\ |B| > 3 \rightarrow \delta(B) > 2.$$

## Theorem

The system $(\boldsymbol{K}_0^{sp}, \leq)$ has $\leq$-amalgamation. And so for any $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu^{sp}$ has $\leq$-amalgamation. So there are $2^{\aleph_0}$ strongly minimal sparse 3-Steiner systems of every infinite cardinality.

# Further Combinatorial Applications

Unlike many construction in infinite combinatorics these methods give a family of infinite structures with similar properties [Bal21a, Bal21b]. Among the properties investigated are:

1. cycle graphs in 3-Steiner systems [CW12] generalized to paths in Steiner $k$-system;
2. preventing or demanding 2-transitivity

# Diversity and Classification

# Examples

## Strongly minimal theories with non-locally modular algebraic closure

1. the Hrushovski (Steiner) examples $2^{\aleph_0}$ theories of strongly minimal Steiner systems $(M, R)$ with
   1. no $\emptyset$-definable binary function. (i.e. triplable)
   2. Some definable functions (examples in [BV21])
2. $2^{\aleph_0}$ theories of strongly minimal quasigroups $(M, R, *)$ + a 3-Steiner example of Hrushovski
3. strongly minimal Steiner systems with combinatorial interesting properties
4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
5. 2-ample but not 3-ample sm sets (not flat) [MT19]
6. strongly minimal eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)

# Classifying the examples

1. discrete
2. non-trivial but no binary function
3. non-trivial but no commutative binary function
4. Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]

# References I

John T. Baldwin.
Some projective planes of Lenz Barlotti class I.
*Proceedings of the A.M.S.*, 123:251–256, 1995.

John T. Baldwin.
Strongly minimal Steiner Systems II: Coordinatizaton and Strongly Minimal Quasigroups.
Math arXiv:2106.13704, 2021.

John T. Baldwin.
Strongly minimal Steiner Systems III: Path Graphs and Sparse configurations.
in preparation, 2021.

# References II

John T. Baldwin and G. Paolini.
Strongly Minimal Steiner Systems I.
*Journal of Symbolic Logic*, pages 1–15, 2020.
published online oct 22, 2020 `arXiv:1903.03541`.

W. Burnside.
*Groups of Finite Order*.
Cambridge, 1897.

John T. Baldwin and V. Verbovskiy.
Towards a finer classification of strongly minimal sets.
preprint: Math Arxiv:2106.15567, 2021.

K. M. Chicot, M. J. Grannell, T. S. Griggs, and B. S. Webb.
On sparse countably infinite Steiner triple systems.
*J. Combin. Des.*, 18(2):115–122, 2010.

# References III

P. J. Cameron and B. S. Webb.
Perfect countably infinite Steiner triple systems.
*Australas. J. Combin.*, 54:273–278, 2012.

Trevor Evans.
Universal Algebra and Euler's Officer Problem.
*The American Mathematical Monthly,*, 86(6):466–473, 1976.

Trevor Evans.
Finite representations of two-variable identities or why are finite fields important in combinatorics?
In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 135–141. North-Holland, Amsterdam, 1982.

# References IV


G. Grätzer.
A theorem on two transitive permutation groups with application to universal algebras.
*Fundamenta Mathematica*, 53, 1963.

Bernhard Ganter and Heinrich Werner.
Equational classes of Steiner systems.
*Algebra Universalis*, 5:125–140, 1975.

Bernhard Ganter and Heinrich Werner.
Co-ordinatizing Steiner systems.
In C.C. Lindner and A. Rosa, editors, *Topics on Steiner Systems*, pages 3–24. North Holland, 1980.

E. Hrushovski.
A new strongly minimal set.
*Annals of Pure and Applied Logic*, 62:147–166, 1993.

# References V


D. Horsley and B. Webb.
Countable homogeneous steiner triple systems avoiding specified subsystems.
*Journal of Combinatorial Theory, Series A*, 180, 2021.
https://www.sciencedirect.com/science/article/pii/S0097316521000339.

I. Muller and K. Tent.
Building-like geometries of finite morley rank.
*J. Eur. Math. Soc.*, 21:3739–3757, 2019.
DOI: 10.4171/JEMS/912.

R. Padmanabhan.
Characterization of a class of groupoids.
*Algebra Universalis*, 1:374–382, 1971/72.

# References VI


Anand Pillay.
Model theory of algebraically closed fields.
In E. Bouscaren, editor, *Model Theory and Algebraic Geometry : An Introduction to E. Hrushovski's Proof of the Geometric Mordell-Lang Conjecture*, pages 61–834. Springer-Verlag, 1999.

D.J.S. Robinson.
*A Course in the Theory of Groups*.
Springer-Verlag, 1982.

S. Świerczkowski.
Algebras which are independently generated by every $n$ elements.
*Fund. Math.*, 49:93–104, 1960/1961.

Sherman K Stein.
Homogeneous quasigroups.
*Pacific Journal of Mathematics*, 14:1091–1102, 1964.