Can categorical classes in $L_{\omega_1,\omega}$ be bounded in size? UIC Seminar: 2025

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February 23, 2025

Formal and Formalism-Free Mathematics

Formal

1. Explicit vocabulary, syntax and notion of truth
   e.g. $L_{\omega,\omega}$, *admissiblefragments*, $L_{\lambda,\omega}(Q)$, L_{2nd} etc.
   'elementary' submodel.
2. Natural axiomatizations of much of mathematics: axiom of Archimedes around structures up to the continuum.
3. Lindenbaum algebra and Stone space: Shelah for anything beyond first order [She75]

Formalism-Free [Ken21]

1. standard mathematics including
   AEC: fixed vocabulary; abstract properties of 'elementary' submodel
2. 'Galois' types

ABSTRACT ELEMENTARY CLASSES defined

Definition

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism and satisfy the following conditions.

- **A1**. If $M \prec_{\boldsymbol{K}} N$ then $M \subseteq N$.
- **A2**. $\prec_{\boldsymbol{K}}$ is a partial order on $\boldsymbol{K}$.
- **A3**. If $\langle A_i : i < \delta \rangle$ is $\prec_{\boldsymbol{K}}$-increasing chain:
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. for each $j < \delta$, $A_j \prec_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$
  3. if each $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ then $\bigcup_{i<\delta} A_i \prec_{\boldsymbol{K}} M$.

- **A4**. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.
- **A5**. There is a Löwenheim-Skolem number $\mathrm{LS}(\boldsymbol{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \boldsymbol{K}$ with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and
  $|A'| < \mathrm{LS}(\boldsymbol{K}) + |\mathrm{A}|$.

Historical Background

When do $\aleph_1$-categorical theories (AEC) have a bounded size of models?

In the mid-70's Shelah answered my question as to whether a sentence of $L_{\omega_1,\omega}(Q)$ could be *categorical in the philosophers sense*, have only one model. In different papers he proved in different ways that $\aleph_1$-categorical such sentence has a model in $\aleph_2$.

Two questions: Under what conditions does a sentence of $L_{\omega_1,\omega}$ (with LN $\aleph_0$) that is $\aleph_1$-categorical have models in $\aleph_2$, $2^{\aleph_0}$, or even larger?

More generally, *Grossberg's question* Must an aec categorical in λ with $I(\boldsymbol{K}, \lambda^+) < 2^{\lambda^+}$ have a model in λ^{++}?

We already know the second is independent of ZFC. As follows.

One Completely General Result

WGCH(λ): $2^\lambda < 2^{\lambda^+}$
Let $\boldsymbol{K}$ be an abstract elementary class (AEC).

Theorem

[WGCH (λ)] Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. If amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality $\kappa = \lambda^+$.

Uses $[\hat{\Theta}_{\lambda^+}(S)]$ (weak diamond) for many S.

λ-categoricity plays a fundamental role.

No really specific model theoretic hypothesis but a set-theoretic one?
Definitely not provable in ZFC for AEC (even for $L_{\omega_1,\omega}(Q_1)$ maybe for $L_{\omega_1,\omega}$).

THE counterexample: Φ

$\boldsymbol{K}$ is the models in a vocabulary with two unary relations P, Q and two binary relations E, R which satisfy:
For any model $M \in \boldsymbol{K}$,

1. P and Q partition M.
2. E is an equivalence relation on Q.
3. P and each equivalence class of E is denumerably infinite.
4. R is a relation on $P \times Q$ so that each element of Q codes a subset of P.
5. R induces the independence property on $P \cup Q$.

This class is axiomatized by a sentence Φ in $L_{\omega_1,\omega}(Q_1)$.

Properties of models of Φ

amalgamation, ω-stablilty, and arbitrarily large models FAIL

Under MA Φ is $\aleph_1$-categorical but is not ω-stable, fails amalgamation in $\aleph_0$, and has no models beyond the continuum.

Shelah suggested a variant, axiomatized in $L_{\omega_1,\omega}$ with the same properties in $\aleph_0$. Laskowski showed that sentence had at least $2^{\aleph_0}$ models in $\aleph_1$.

The AEC attached to Φ ([She, 6.3]) is the $\boldsymbol{K}_3$.
$M, N \in \boldsymbol{K}$, $M \leq_{\boldsymbol{K}_3} N$ if $M \subseteq N$ and $[a]^M = [b]^N$.
[She87, She83, She],[Bal09, §17]

The $L_{\omega_1,\omega}$ case
Syntax and Semantics

The class of models

$\boldsymbol{K}_T$ is the class of atomic models of the countable first order theory T.

Definition

The atomic class $\boldsymbol{K}_T$ is extendible if there is a pair $M \preceq N$ of countable, atomic models, with $N \neq M$.

Equivalently, $\boldsymbol{K}_T$ is extendible if and only if there is an uncountable, atomic model of T.

We assume throughout that $\boldsymbol{K}_T$ is extendible. We work in the monster model of T, which is usually not atomic.

A complete sentence of has such a representation by Chang's trick: Expanding the language by introducing predicates for countable conjunctions (theory T^*) and making them correct by omitting types.

The $L_{\omega_1,\omega}$-class is the reducts of atomic models of T^*.

ω-stability in Atomic Classes

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.
$\boldsymbol{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

But, there may be $A \subseteq M$, $p \in S_{at}(A)$ that has no extension to $S_{at}(M)$.
Note also ϕ may be κ-stable in this sense while the associated AEC is not κ-stable (for Galois types) [BK09].

First order absoluteness

Theorem (Morley-Baldwin-Lachlan)

A first order theory T in a countable language is $\aleph_1$ categorical iff

1. T has no 2-cardinal models and
2. T is ω-stable.

1) is arithmetic and 2) is Π^1_1.

Fact

A first order theory T in a countable language whose class of atomic models satisfies 1) and 2) is $\aleph_1$-categorical.

I emphasize Morley because it is his direction:
'$\aleph_1$-categorical implies ω-stable' that is problematic for $L_{\omega_1,\omega}$.

Getting ω-stability: I

Theorem: Keisler/Shelah

$\boldsymbol{K} = \mathrm{mod}(\psi)$, $\psi \in L_{\omega_1,\omega}$

1. *(Keisler) ZFC* If some uncountable model in $\boldsymbol{K}$ realizes uncountably many types (in a countable fragment) over $\emptyset$ then $\boldsymbol{K}$ has $2^{\aleph_1}$ models in $\aleph_1$.
2. *(Shelah)* $(2^{\aleph_0} < 2^{\aleph_1})$ If $\boldsymbol{K}$ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then $\boldsymbol{K}$ is ω-stable.

Two uses of WCH

A. WCH implies AP in $\aleph_0$. Thus, if $\boldsymbol{K}$ is not ω-stable there is a countable model M and an uncountable $N \in \boldsymbol{K}$ which realizes uncountably many types over M.
B. By Keisler, $\mathrm{Th}_M(M)$ has $2^{\aleph_1}$ models. From WCH we conclude $\mathrm{Th}(M)$ has $2^{\aleph_1}$ models.

Getting ω-stability: II

Morley's original *first order* proof using Hanf number for omitting types, EM-models, and the Skolem hull gives:

Theorem

If a complete first order theory has arbitrarily large models and is $\aleph_1$-categorical then it is ω-stable.

More generally,

Theorem

An $\aleph_1$-categorical atomic class $\boldsymbol{K}$ that has arbitrarily large models and amalgamation in $\aleph_0$ is ω-stable.

Tradeoff: $\beth_{\omega_1}$ for weak CH

A new notion of closure

Definition

An atomic tuple **c** is in the pseudo-algebraic closure of the finite, atomic set B ($\mathbf{c} \in \mathrm{pcl}(B)$) if
for every atomic model M such that $B \subseteq M$, and $M\mathbf{c}$ is atomic, $\mathbf{c} \subseteq M$.

When this occurs, and **b** is any enumeration of B and $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, we say that $p(\mathbf{x}, \mathbf{b})$ *is pseudo-algebraic.*

Example I

Our notion, pcl of *algebraic* differs from the classical first-order notion of algebraic as the following examples show:

Example

Suppose that an atomic model M consists of two sorts. The U-part is countable, but non-extendible (e.g., U infinite, and has a successor function S on it, in which every element has a unique predecessor). On the other sort, V is an infinite set with no structure (hence arbitrarily large atomic models). Then, an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

Example II

Example

Let $L = A, B, \pi, S$ and T say that A and B partition the universe with B infinite, $\pi : A \to B$ is a total surjective function and S is a successor function on A such that every π-fiber is the union of S-components. K_T is the class of $M \models T$ such that every π-fiber contains exactly one S-component. Now choose elements $a, b \in M$ for such an M such that $a \in A$ and $b \in B$ and $\pi(a) = b$. Clearly, a is not algebraic over b in the classical sense, but $a \in \mathrm{pcl}(b)$.

Definability of pseudo-algebraic closure

Strong ω-homogeneity of the monster model of T yields:

Fact

If $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, then

$$\mathbf{c} \in \mathrm{pcl}(\mathbf{b}) \quad \text{if and only if} \quad \mathbf{c}' \in \mathrm{pcl}(\mathbf{b}')$$

for any $\mathbf{c}'\mathbf{b}'$ realizing $p(\mathbf{x}, \mathbf{y})$.
In particular, the truth of $c \in \mathrm{pcl}(\mathbf{b})$ does not depend on an ambient atomic model.

Further, since a model which atomic over the empty set is also atomic over any finite subset, moving M to N we have:

Fact

If $\mathbf{c} \not\in \mathrm{pcl}(B)$, witnessed by M then for every countable, atomic $N \supset B$, there is a realization $\mathbf{c}'$ of $p(\mathbf{x}, B)$ such that $\mathbf{c}' \not\subseteq N$.

Pseudo-minimal sets

Definition

1. A *possibly incomplete* type q over $\mathbf{b}$ is *pseudominimal* if for any finite, $\mathbf{b}^* \supseteq \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that $\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, if $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, and $\mathbf{c} \notin \mathrm{pcl}(\mathbf{b}^*)$, then $\boldsymbol{a} \in \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.
2. M is pseudominimal if $x = x$ is pseudominimal in M.

I.e, pcl satisfies exchange (and more); we have a geometry.

'Density'

Definition

$\boldsymbol{K}_T$ satisfies *'density' of pseudominimal types* if for every atomic $\mathbf{e}$ and atomic type $p(\mathbf{e}, \mathbf{x})$ there is a $\mathbf{b}$ with $\mathbf{eb}$ atomic and $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending p such that q is pseudominimal.

So density fails if there is a single type $p(\mathbf{e}, \mathbf{x})$ over which exchange fails.

Method: 'Consistency implies Truth':I

[BL16]
Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let A be the countable ω-model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct B, an uncountable model of set theory, which is an elementary extension of A, such that B is correct about uncountability. Then the model of ϕ in B is actually an uncountable model of ϕ.

Main Theorem

Goal Theorem [BLS16]

If $\boldsymbol{K}_T$ fails 'density of pseudominimal types' then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

We prove this in two steps

1. Force to construct a model (M, E) of set theory in which a model of T codes model theoretic and combinatorial information sufficient to guarantee the non-isomorphism of its image in the different ultralimits.
2. Apply Skolem ultralimits of the models of set theory from 1) to construct $2^{\aleph_1}$ atomic models of T with cardinality $\aleph_1$ in V.

Getting Models in the continuum

Getting models in $2^{\aleph_0}$: Method

In the novel *White Light* [Ruc80], Rudy Rucker proposes a metaphor for the continuum hypothesis. One can reach $\aleph_1$ by a laborious climb up the side of Mt. ON, pausing at ϵ_0.

Or one can take
Cantor's elevator An instantaneous trip up a shaft at the center of the mountain.

For atomic models we take the slightly slower
Shelah's elevator The elevator is a bit slower but has only countably many floors. After building finitely many rooms at each step we reach an object of cardinality $2^{\aleph_0}$.

Asymptotic similarity

Definition

Fix an L-structure M. A subset of M, indexed by $\{a_\eta : \eta \in 2^\omega\}$, is *asymptotically similar* if, for every k-ary L-formula θ, there is an integer N_θ such that for every $\ell \geq N_\theta$,

$$M \models \theta(a_{\eta_0}, \ldots, a_{\eta_{k-1}}) \leftrightarrow \theta(a_{\tau_0}, \ldots, a_{\tau_{k-1}})$$

whenever $(\eta_0, \ldots, \eta_{k-1})$ and $(\tau_0, \ldots, \tau_{k-1})$ satisfy: $\eta_i \restriction \ell = \tau_i \restriction \ell$ and the $\eta_i \restriction \ell$ are distinct.

Remark

Asymptotic similarity is a type of indiscernibility; the indiscernibility is only formula by formula. Let $M = (2^\omega, \{U_i : i < \omega\}$ where the U_i are independent unary predicates. The entire universe is asymptotically similar, although no two elements have the same 1-type.

Getting models in $2^{\aleph_0}$

Theorem [BL19]

If a countable first order theory T has an atomic pseudominimal model M of cardinality $\aleph_1$ then there is an atomic pseudominimal model N of T which a contains a set of *asymptotically similar* elements with cardinality $2^{\aleph_0}$.
Equivalently, if the models of a complete sentence Φ in $L_{\omega_1,\omega}$ are pseudominimal and Φ has an uncountable model, it has a model in the continuum.

A simple application of the method gives Borel models in the continuum of any theory with trivial definable closure.

$K_{(2^{\aleph_0})^+} \neq \emptyset$ implies ω-stability

Goal Theorem [BLS24]

Theorem

If an atomic class At *is* $\aleph_1$*-categorical and has a model of size* $(2^{\aleph_0})^+$ *then* At *is* ω*-stable.*

Old and new definitions:

Definition

1. *A type* $p \in S_{at}(M)$ splits over $F \subseteq M$ *if there are tuples* $\mathbf{b}, \mathbf{b}' \subseteq M$ *and a formula* $\phi(\mathbf{x}, \mathbf{y})$ *such that* $\mathrm{tp}(\mathbf{b}/F) = \mathrm{tp}(\mathbf{b}'/F)$, *but* $\phi(\mathbf{x}, \mathbf{b}) \wedge \neg\phi(\mathbf{x}, \mathbf{b}') \in p$.
2. *We call* $p \in S_{at}(M)$ constrained *if* p *does not split over some finite* $F \subseteq M$ *and* unconstrained *if* p *splits over every finite subset of* M.
3. $C_M := \{p \in S_{at}(M) : p \text{ is constrained}\}$, *for an atomic model* M, . *We say* At *has* **only constrained types** *if* $S_{at}(N) = C_N$ *for every atomic model* N.

Basic Properties

Lemma

1. If M is a countable atomic model and $p \in S_{at}(M)$ then p is realized in an atomic extension of M.
2. For any atomic models $M \preceq N$ and finite $A \subseteq M$, then for any $q \in S_{at}(N)$ that does not split over A, the restriction $q \restriction M$ does not split over A; and any $p \in S_{at}(M)$ that does not split over A has a unique non-splitting extension $q \in S_{at}(N)$.
3. If some atomic N has an unconstrained $p \in S_{at}(N)$, then for every countable $A \subseteq N$, there is a countable $M \preceq N$ with $A \subseteq M$ for which the restriction $p \restriction M$ is unconstrained.
4. At has only constrained types if and only if $S_{at}(M) = C_M$ for every/some countable atomic model M.

Limit types

Definition

For $|N| = \aleph_1$, a type $p \in S_{at}(N)$ is a $\mathrm{limit\ type}$ if the restriction $p \restriction M$ is realized in N for every countable $M \preceq N$.

Trivially, for every N, every type in $S_{at}(N)$ realized in N is a limit type. Since we allow $M = N$ in the definition of a limit type, if M is countable, then the only limit types in $S_{at}(M)$ are those realized in M.

'Consistency implies Truth': II

Note that there are no additional assumptions on At, other than the existence of an uncountable, atomic model.

KEY Theorem:

If At admits an uncountable, atomic model, then there is some $N \in \mathrm{At}$ with $|N| = \aleph_1$ for which every limit type in $S_{at}(N)$ is constrained.

So if $\aleph_1$ -categorical: limit $=$ constrained on the model in $\aleph_1$.

Linking a largish model with ω-stability

Theorem

If an atomic class At is $\aleph_1$-categorical and has a model of size $(2^{\aleph_0})^+$ with a relatively $\aleph_1$-saturated submodel of cardinality continuum, then $S_{At}(M)$ has only constrained types

Pf. Choose a c realizing an unconstrained type and use relative $\aleph_1$-saturation to build an unconstrained limit type. This contra the KEY.

Similarly argue that if there is a unconstrained type over a countable model then there is a model in $\aleph_1$ with an unconstrained limit type. Apply KEY again [BLS24, Theorem 2.4.4]

Relatively saturated models exist!

Proof.

Let M^{**} be an atomic model of size $(2^{\aleph_0})^+$. We construct a *relatively* $\aleph_1$-saturated elementary substructure $M^* \preceq M^{**}$ of size $2^{\aleph_0}$ as the union of a continuous chain $(N_\alpha : \alpha \in \omega_1)$ of elementary substructures of M^{**}, each of size $2^{\aleph_0}$, where, for each $\alpha < \omega_1$ and each of the $2^{\aleph_0}$ countable $M \preceq N_\alpha$, $N_{\alpha+1}$ realizes each of the at most $2^{\aleph_0}$ $p \in S(M)$ that is realized in M^{**}. □

Note there is no reason to think any of these models are even $\aleph_1$-saturated, until we conclude ω-stability. [BLS24, Theorem 2.4.5]

Further Context

Covers of Algebraic Varieties

$$\exp : (\mathcal{C}, +) \twoheadrightarrow (\mathcal{C}, \times).$$

$$j : H \twoheadrightarrow . C$$

$$p : C \twoheadrightarrow S(\mathcal{C}).$$

Zilber conjecture that the most complicated (Shimura Varieties) were (almost) quasiminimal exellent and so uncountably categorical. The many partial results/methods are represented in the next chart.

Approaches to categoricity of covers

	topic	paper	method/context	sect
1	Complex exponentiation	[Zil05]	quasiminimality	§?
2	cov mult group	[Zil06]	quasiminimality	§?
3		[BZ11]	quasiminimality	
4	*j*-function	[Har14]	background	§?
5	Modular/Shimura Curves	[DH17]	quasiminimality	§?
6	Modular/Shimura Curves	[DZ22]	quasiminimality	
7	Abelian Varieties	[BGH14]	finite Morley rank groups	§?
8	Abelian Varieties	[BHP20]	fmr & notop	§?
9	Shimura *varieties*	[Ete22]	notop	§?
10	Smooth varieties	[Zil22]	o-quasiminimality	§?

Almost quasiminimal excellence: I

Definition (Quasiminimal structure)

A structure M is *quasiminimal* if every first order ($L_{\omega_1,\omega}$) definable subset of M is countable or cocountable. Algebraic closure is generalized by saying $b \in \mathrm{acl}'(X)$ if there is a first order formula with **countably many** solutions over X which is satisfied by b.

Almost quasiminimal excellence: II

Definition (Quasiminimal excellent geometry)

Let $\boldsymbol{K}$ be a class of L-structures such that $M \in \boldsymbol{K}$ admits a closure relation cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ that satisfies the following properties.

1. **Basic Conditions**
   1. Each cl_M defines a pregeometry on M.
   2. For each $X \subseteq M$, $\mathrm{cl}_M(X) \in \boldsymbol{K}$.
   3. countable closure property (ccp): If $|X| \leq \aleph_0$ then $|\mathrm{cl}(X)| \leq \aleph_0$.

Almost quasiminimal excellence: II

4. **Homogeneity**
   1. A class $\boldsymbol{K}$ of models has $\aleph_0$-**homogeneity over** $\emptyset$ (Definition 5) if the models of $\boldsymbol{K}$ are pairwise qf-back and forth equivalent
   2. A class $\boldsymbol{K}$ of models has $\aleph_0$-**homogeneity over models** if for any $G \in \boldsymbol{K}$ with G empty or a countable member of $\boldsymbol{K}$, any H, H' with $G \leq H, G \leq H'$, H is qf-back and forth equivalent with H' over G.
5. $\boldsymbol{K}$ is an *almost quasiminimal excellent geometry* if the universe of any model $H \in \boldsymbol{K}$ is in $\mathrm{cl}(X)$ for any maximal cl-independent set $X \subseteq H$.
6. We call a class which satisfies these conditions an *almost quasiminimal excellent geometry* [BHH+14].

Terminology

The **almost** is essential because the structures are two-sorted. But in the quasi-minimal covers: Galois type are quantifier-free first order types (in a suitably Morleyized theory).

Arbitrarily large models, bounded categoricity

Galois may properly refine Syntactic

Theorem Hart-Shelah/B-Kolesnikov

For each $2 \leq k < \omega$ there is an $L_{\omega_1,\omega}$-sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not $\aleph_{k-2}$-Galois stable;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the disjoint amalgamation property in every κ;
5. For $k > 2$,
    1. ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic first-order types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
    2. ϕ_k is $\aleph_m$-Galois stable for $m \leq k-3$;
    3. ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

[Bal09, BK09] refining an example of [HS90].

Galois may properly refine Syntactic: II

1. ϕ_2 is $\aleph_0$-categorical but not ω-Galois stable nor categorical in any power. (ω-synactic stability unclear in paper.)
2. ϕ_3 is categorial in $\aleph_1, \aleph_2$ and never again:
   In $\aleph_0$, syntactic $=$ Galois and ω-stable.

Thus, Morley's result that ω-stable implies κ-stable for all κ is gone (for Galois and likely? syntactic).

Questions/Problems

Let ϕ be a complete sentence of $L_{\omega_1,\omega}$.

1. Give a definition of a 'complete' that eliminates uninformative counterexamples [BKS16, BHK13].
2. Do the BLS results on $L_{\omega_1,\omega}$ generalize at all? E.g. to analytic classes? [BL16]
3. If ϕ characterizes $\kappa > \aleph_0$, must ϕ have 2^κ models in κ?
4. For $\kappa < \aleph_{\omega_1}$, describe an explicit sentence that characterizes κ. [BKL17]

Theorem

[BS24] There is a complete sentence ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$.

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