Reflections on the DeZolt axiom: Mathematical, Philosophical and Historical

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Hartshorne's question

> *We avoid the use of the words 'greater' and 'lesser' because these imply the existence of an order relation among figures which we have not yet established. In fact, the existence of an order relation for content depends on (Z) (Exercise 22.7). [We will also see that 'if squares are equal then their sides are equal' follows from (Z) (Exercise 22.6).]*
>
> *I do not know of any purely geometric proof of axiom(Z)* from the definition of content [area] we have given. *. . . (Z) holds however, whenever there is a measure of area function defined in the geometry.*
> *([Har00, p.202])*

He repeats the same sentiment in more detail on page 210.
'The proof [of De Zolt and of area function] is analytic in that it makes use of the field of segment arithmetic and similar triangles.' ([Har00, p.210]).

Questions

These passages raises several questions.

1. What is De Zolt's axiom (Z)?
2. What is De Zolt good for?
3. What is a geometric proof?

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Hartshorne's version of De Zolt (Z)

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De Zolt's version

If a polygon is divided into parts in a given way, it is not possible, when one of these parts is omitted to recompose the remaining parts in such a way that they cover entirely the polygon.
(De Zolt 1881 p. 12)

One distinction is that De Zolt is referring to scissors congruence (dissection), which we call *equidecomposable*.

Hartshorne uses *equal content*, which we call equicomplementable.

In this case subtraction is allowed as in Euclid I.35.

What is De Zolt good for?

Underlying problems

It is quite clear that congruence classes of line segments are linearly ordered by $AB < CD$ if AB is congruent with a subsegment of CD.
How do we show such an order for polygons in two dimensions?
De Zolt used the postulate above to argue for such an ordering [Gio21, §3.2].

The *De Zolt order* with respect to an equivalence relation on figures with non-empty interior is defined by
$[P] < (\leq?)[Q]$ if $(\exists R)[Q] = [P + R]$.
But there are two issues:

1. Is the ordering strict?
2. Is it linear? Does it satisfy the trichotomy property?

Fundamental Dilemma

De Zolt's axiom is a true statement in models of $HP5$; the extant proof uses a measure of area function. Is this technique necessary?

In any plane satisfying $HP5$: DeZolt HOLDS and a measure of area function exists.

How can we possibly show they are independent (or dependent)?

Frege

Frege [Fre84] faced the same problem in his dialog with Hilbert. He felt that the various axioms of geometry all were inherent in the thought of geometry but nevertheless separately needed to expound geometry. But since they were all true, one could not apply Hilbert's 'formal method' [Bla07] to show independence. We are not in the same boat but a very similar one.
Our approach is (I hope) in the spirit that Tappenden described yesterday.
We will introduce some further geometrical notions that have distinct interpretations (concerning the calculation of magnitudes) and
We aimed to:

1. Either give a uniform argument that forces a measure of area function, vindicating Hartshorne's intuition.
2. or establish a linear order of polygons without invoking such a function.

Confronting the Dilemma

We investigate ‘same magnitude’ in several contexts.
Properties of such ‘equal magnitude’ equivalence relations depend on a number of factors:

- background theory
- definition of figures
- precise description of the equivalence relation
- dimension of the space

Background Theories

Axiom systems for geometry

1. First-order axioms:

   HP, HP5 We write HP (Hilbert plane) for Hilbert's incidence, betweenness, and congruence axioms. We write HP5 for HP plus the parallel postulate. HP is often known as absolute or neutral geometry.

   EG The *axioms for Euclidean geometry*, *HP*5 + circle-circle intersection.

2. Hilbert's group continuity axioms, must be formalized in infinitary and second-order logic

   Archimedes The sentence in the logic $L_{\omega_1,\omega}$ expressing that any segment is contained in a finite number of copies of any other.

   Dedekind Every Dedekind cut is realized.

‘Equal Magnitude’ equivalence relations

Definition

1. Two figures P, P' are *equidecomposable* if they each can be written as a non-overlapping union of the same number of pairwise congruent atoms.
2. Two figures P, P' are *equicomplementable* if there are other figures Q, Q' such that:
   a. P and Q are non-overlapping;
   b. P' and Q' are non-overlapping;
   c. Q and Q' are equidecomposable
   d. $P \cup Q$ and $P' \cup Q'$ are equidecomposable.
3. Two figures P, P' are *equimeasured* (by α) if there is a measure of content function α such that $\alpha(P) = \alpha(P')$
4. For a subgroup G of the group of rigid motions of the space, two figures P, P' are *G-equivalent* if there is $g \in G$ with $g(P) = P'$.

Area Assumptions to be formalized

Hartshorne lists Euclid's implicit assumptions about area; they include

Two Assumptions

1. The whole is greater than the part. (CN5)
2. 'equal' squares have equal sides.

Hartshorne proves

For equicomplementation, in models of *EG*, de Zolt implies the *square property*: 'equal' squares have equal sides.

We prove the converse

For equicomplementation, in models of *EG*, the square property implies de Zolt for equicomplementation;

The same argument works in HP5, replacing the square by the rectangle property: 'rectangles with the same height and 'equal' areas have equal bases'.

General framework

Definition: Admissible equivalence relations

Let $n = 2$ or 3.

1. An *atom* is an n-dimensional convex hull of $n+1$ points in n-space.
2. A *figure* in n-space is a non-overlapping (intersection cannot have an interior) union of atoms.
3. An equivalence relation E on figures is *admissible* if
   1. Congruent atoms are E-equivalent;
   2. For disjoint figures P, P' and Q, Q', if $E(P, P')$ and $E(Q, Q')$, then $E(P \cup P', Q \cup Q')$.
4. The *De Zolt order* with respect to an admissible equivalence relation on figures with non-empty interior is defined by $[P] < [Q]$ if $(\exists R)[Q] = [P + R]$.

How can one measure area?

Global Method: Fix a unit; say, a square; tile the plane with congruent squares. Then to measure a figure, continually refines the measure by cutting the squares in quarters and counting only those (possibly fractional) squares which are contained in the figure.

Local Method: (Hilbert) Triangulate a figure with finitely many triangles, which are each assigned area

- Euclidean Geometry: $\frac{bh}{2}$
- Hyperbolic Geometry: $(0, \delta)$ or $(1, \delta)$ depending on the size of the defect δ

and the area of the figure is the sum of the areas of the triangles.

Representative Method: Fix a representative of each equivalence class.

The first two examples are described in [Bol78]; the third in [Har00, §36]; we introduce the method here.

Measure of Area function

Hartshorne provides a general definition that applies to all three methods.

Definition

A *measure of area function* on figures in *n*-space is a function α with values in *an ordered semigroup* $(G, +, <)$ satisfying:

1. Congruent *atoms* have the same value.
2. For disjoint figures P, Q, $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.

Hartshorne requires a group (needed for hyperbolic case).

An ordered semigroup is a structure $(G, +, <)$ such that

1. $+$ is associative and satisfies $(\forall x, y, z) x < y \rightarrow (xz < yz \wedge zx < yz)$.
2. $<$ is a strict linear order.

Scales

Definition

1. A *figure type* is a first order formula $\phi(\overline{x})$ such that if $\phi(\boldsymbol{a})$ then $\boldsymbol{a}$ are the points of a figure P.
   Suppose P and P' realize ϕ; fix an enumeration of a_i of P and b_i of P'.
2. A figure type ϕ is a *scale* if $\phi(\boldsymbol{a})$ and $\phi(\mathbf{b})$ describe figures P and P'
   1. for some i, $a_i a_{i+1} \approx b_i b_{i+1}$
      then P and P' are congruent.
      Example: for some n, ϕ describes the collection of regular n-gons.
   2. Suppose there is a fixed segment AB such that $AB \approx x_0 x_1$ is subformula of ϕ so $AB \approx a_0 a_1 \approx b_0 b_1$ and a fixed i such that for any P, P' satisfying ϕ if $x_i x_{i+1} \approx y_i y_{i+1}$ then $P \approx P'$.
      Example: Rectangle of fixed height.

Scaled and well-scaled

Definition

1. An equivalence relation on figures is *scaled* (by a scale ϕ) if every equivalence class contains at least one instance of ϕ.
2. An equivalence relation E on figures is *well-scaled* (by a scale ϕ) if P, P' satisfy ϕ and $E(P, P')$ implies $P \approx P'$. (Here and below $P \approx P'$ means P and P' are congruent.)

Main Theorem

The Well-Scaled Theorem

Work in neutral geometry. Suppose E is scaled by a scale ϕ.

1. If E is well-scaled by ϕ then there is a measure function for E, the equivalence classes are linearly ordered, and E satisfies De Zolt.
2. Suppose E is scaled by a scale ϕ and E satisfies De Zolt, then E is well-scaled.

Corollary

If the plane π satisfies either $HP5$ and the rectangle property for equicomplementation
or EG and the squares property for equicomplementation
then it satisfies De Zolt
and there is a measure of area function on π.

Proof. By II.14 of [Euc56], equicomplementation is scaled by squares and well-scaled by the square property.

Why is De Zolt a worry?

Euclid implicitly gives a formula for the area of a triangle.

But he uses the method of exhaustion to show 'triangular pyramids of the same angle are to each other as to their bases.'

Removing such limit processes is one of the goals of 19th century rigorizing.

fact

[Wallace-Bolyai-Gerwien Theorem] Two polygons *in an Archimedean plane* are equidecomposable (scissors congruent) if and only if they have the same measure of area.

Fact (Dehn-Snyder Theorem)

Two polyhedra in $\Re^3$ *are equidecomposable iff they have the same volume and the same Dehn invariant.*

Non-Archimedean planes

A crucial contribution of the Grundlagen is to show the geometrical results of Euclid do NOT depend on Archimedes.

Fact

There is a non-Archimedean plane satisfying HP5.

Proof. Fix points A and C on a plane satisfying $HP5$. Consider the set of sentences $(\exists x)\phi_n(A, x, B_1 \ldots B_n, C)$ which asserts the B_i are decreasing, each B_{i+1}, B_i is half the length of B_i, B_{i-1} and x is between A and B_n. Since each $(\exists x)\phi_n$ is true for an appropriate choice of a B^* to witness x, the compactness theorem for first order logic implies there is a B_∞ such that $\phi_n(A, B_\infty, B_1 \ldots B_n, C)$ for every n. Then B_∞ is an infinitessimal. $\square$

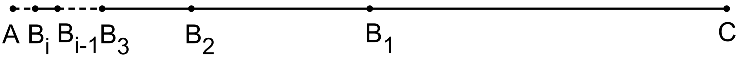


Figure: Euclid I.35

Archimedes matters

Lemma

There is a model of HP5 where Equidecomposition is not scaled by squares.

Proof. We show there is a model of $HP5$ with a parallelogram $EBCF$ that is not equidecomposable with a square. Consider Hilbert's example in a Cartesian plane π over a non-archimedean elementary extension F of the reals $\Re$. We say a point A in π is standard or finite, if both coordinates of A are in $\Re$.

Proof continued

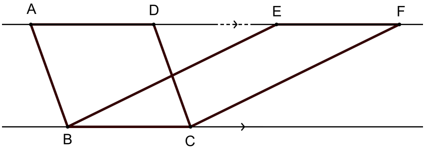


Figure: Euclid I.35

Using the diagram for Euclid I.35, suppose *A*, *B*, *C*, *D* are finite (standard) but E and F are not; although the length of EF must be standard since $EF \approx BC$. Now we know *ABCD* is equidecomposable with a (standard) square, since all its sides are in the real field and the WBG theorem applies. Since $\pi \models HP5$, Hilbert's measure of area function gives the same finite area to both *ABCD* and *EBCF*. But *EBCF* is not equidecomposable with any finite square, as it has a side of infinite length. But then *EBCF* is not equidecomposable with any square. Since de Zolt implies that at most one congruence class of squares appears in an equicomplementability equivalence class.

Geometric Proof

Three topics

1. area by equicomplementation
2. multiplication
3. proportionality

Euclid's path is from 1) to 3) (using Archimedes) and 2) is a corollary (for Descartes).
19th century worries about the rigor of 1).
Hilbert's path is from 2) to 3) to 1).
But the only use of 3) in Hilbert's area theory is

Theorem

Any of the three choices of base for a triangle give the same value for the product of the base and the height.

So Hilbert is actually going from 2) to 1).

Algebra of segments

Hilbert (following Von Staudt) define an algebra of segments which is a linearly ordered semigroup.
In order to define the area function he must fix a unit length.
He asserts that this is a geometric argument.
In the definition of the area function he

1. fixes a unit length
2. uses proportionality (side-splitter)
3. rigid motions are used to show the transitivity of equicomplementability [Har00, Lemma 22.4] (and justified [Har00, §17]).

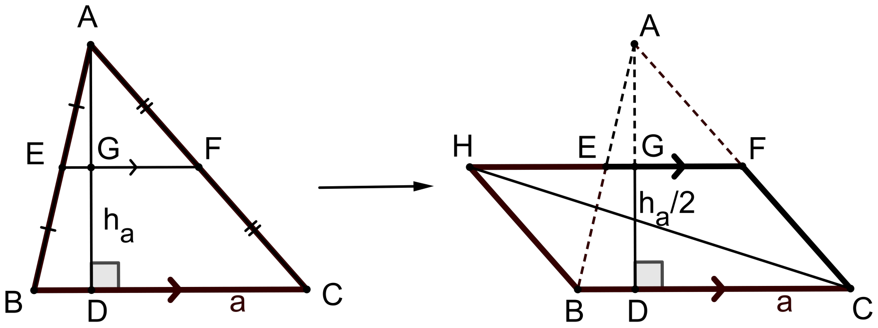
Proportionality is used only to show the area of a triangle does not depend on the choice of base. But this use can be avoided.

Independence of base

We prove the following without use of proportionality.

Theorem

Any of the three choices of base for a triangle give the same value for the product of the base and the height.



Proof

Take triangle ABC. Choose an arbitrary side, say a. Draw the midline EF parallel to side a. The height h_a perpendicular to a is bisected by the midline at G (or by its extension).
Rotate triangle AEF about point E to get parallelogram $BCFH$ with base a and height $\frac{1}{2}\ h_a$.
HC divides the parallelogram into two congruent triangles (BCH and FHC) with base a and height $\frac{1}{2}\ h_a$ and therefore area $\frac{1}{2}\ (\frac{1}{2}\ a\ h_a)$, so the area of the parallelogram $BCFH$ is $\frac{1}{2}\ a\ h_a$.
Since that is done without preference of base, the argument is valid for the other two sides and corresponding heights of the triangle, leading the second and third area formulas $\frac{1}{2}\ b\ h_b$ and $\frac{1}{2}\ c\ h_c$, all three measures of areas of parallelograms with equal content as triangle ABC.
This shows that the triangle area formula $\frac{1}{2}(base)(height)$ is independent of choice of base.

Conclusion

Hartshorne asks about the significance of the existence of a measure function in establishing the theory of area.

Equidecomposablity and thus equicomplementability are described by infinite *disjunctions* of first order formulas. Thus, (as grasped by De Zolt), non-equidecomposablity is a treacherous notion; to establish it requires checking infinitely many possibilities. Moreover, these possibilities are too wild to support an induction.

The independence of Hilbert's measure of area on triangulation shows that a figure with area g with respect to some triangulation is equicomplementable with a triangle of height 1 and base g.

Thus, two triangles are equicomplementable if and only if some/ any calculation of their areas give the same value.

This immediately yields either the rectangle property or De Zolt.

Any well-scaled equivalence relation will work. But, like Hartshorne, we see a complete proof only using Hilbert's function.

References I

Patricia Blanchette.
Frege on consistency and conceptual analysis.
Philosophia Mathematica, 15:321–346, 2007.

Vladimir Boltyanskii.
Hilbert's third problem.
V.H. Winston and Sons, 1978.

Euclid.
Euclid's elements.
Dover, New York, New York, 1956.
In 3 volumes, translated by T.L. Heath; first edition 1908; online at `http://aleph0.clarku.edu/~djoyce/java/elements/`.

References II

Gottlob Frege.
On the foundations of geometry: Second series.
In Brian McGuinnes, editor, *Collected papers on Mathematics, Logic, and Philosophy*. Oxford: Basil Blackwell, 1984.

E. Giovannini.
David Hilbert and the foundations of the theory of plane area.
Archive for History of Exact Sciences, 2021.
`https://doi.org/10.1007/s00407-021-00278-z`.

Robin Hartshorne.
Geometry: Euclid and Beyond.
Springer-Verlag, 2000.