Generalized Quantifiers, Infinitary Logics, and Abstract Elementary Classes

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December 14, 2007

In this paper we discuss extensions of first order logic both to infinitary logics and by adding some generalized quantifiers. In a certain sense this topic is orthogonal to the usual study of generalized quantifiers. That program considers the basic properties: compactness, interpolation, etc. of various logics. An abstract elementary class is a generalization of the class of models of a specific theory in first order logic. The goal of the study is to find the properties of such classes that make sense for 'elementary classes' in many logics. Thus our goal in this paper is threefold 1) to expound the definition of abstract elementary class and some key concepts of their study 2) to examine and solicit specific examples, especially those defined with generalized quantifiers and 3) to investigate the distinction between those AEC defined via infinitary logic and those using generalized quantifiers.

In contrast to the usual goals of abstract model theory, our study is motivated by considering classes of structures of mathematical interest that can be axiomatized in a suitable logic. Thus, a great deal of algebraic geometry can be viewed as the model theory of the first order theory of algebraically closed fields (albeit with an emphasis on positive formulas). In contrast, Zilber's study [24, 2] of the complex field with exponentiation requires axiomatization in $L_{\omega_1,\omega}(Q)$ thus providing a clear example of the need for both infinitary logic and generalized quantifiers. A second major innovation is that an AEC is a pair of a class of models and a notion of 'elementary submodel'. Since $\prec_{\boldsymbol{K}}$ is given axiomatically the first reaction is to expect some examples could be very different from the basic examples defined by logics. At least ostensibly this is the case for ${}^{\perp}N$, a certain family of modules that we discuss in Section 4; $A \prec_{\boldsymbol{K}} B$ is defined by a property of the quotient B/A. This expectation has not yet been confirmed. In particular, consider the current state of ${}^{\perp}N$ as discussed in Section 4.

Question 0.1 *Are there further examples of classes of structures of mathematical interest, along with a notion of elementary submodel, that can be axiomatized only by extending first order logic by infinitary and/or generalized quantifiers?*

*Partially supported by NSF-0500841.

As Vaananen [23] suggested at the conference there is a natural splitting of extensions of first order logic into three families. An extension has 'first order like-quantification' if quantification is still restricted to individuals; one may allow quantification over infinite sequences of elements, or infinite Boolean combinations of formulas in the matrix. A second alternative is the logics with generalized quantifiers as introduced by Mostowski. Finally there are higher order logics. We restrict here to the first two families.

A Lindström theorem provides syntactic/semantic conditions on a logic that pin down certain logics. We discuss a similar phenomena. But we work not with entire logics but with class of structures along with a notion of 'elementary submodel' on that class and consider in Section 2 what semantic properties of the pair are reflected by syntactic conditions.

Shelah has a massive program to study categoricity in general AEC see his work on 'frames') but the most complete categoricity transfer results [17, 18] are only for $L_{\omega_1,\omega}$. In section 3, we explore some of the reasons for that restriction by studying the more simple question of the existence of models in $\aleph_2$ [21]. We expound the work of Andrew Coppola who found a more natural framework for the result.

1 Abstract Elementary Classes

The basic context of AEC is the same as for studying generalized quantifiers. We are looking at a collection of models for a fixed vocabulary. But rather than develop a syntax and semantics for a logic, we try to isolate the properties of a class of defined by a sentence; these models are connected by some notion of 'elementary submodel', $\prec_{\boldsymbol{K}}$. For background on AEC see e.g. [6, 1, 21, 19]

Definition 1.1 *A class of τ–structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an* abstract elementary class (AEC) *if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ on $\boldsymbol{K}$ are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If $M \prec_{\boldsymbol{K}} N$ then $M \subseteq N$.*
- **A2**. *$\prec_{\boldsymbol{K}}$ is a partial order on $\boldsymbol{K}$.*
- **A3**. *If $\langle A_i : i < \delta \rangle$ is a continuous $\prec_{\boldsymbol{K}}$–increasing chain:*
    1. *$\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;*
    2. *for each $j < \delta$, $A_j \prec_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$;*
    3. *if each $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ then $\bigcup_{i<\delta} A_i \prec_{\boldsymbol{K}} M$.*
- **A4**. *If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.*
- **A5**. *There is a Löwenheim–Skolem number* $\mathrm{LS}(\boldsymbol{K})$ *such that if $A \subseteq B \in \boldsymbol{K}$ there is a $A' \in \boldsymbol{K}$ with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and $|A'| \leq |A| + \mathrm{LS}(\boldsymbol{K})$.*

Here, $\langle A_i : i < \delta \rangle$ is a *continuous* $\prec_{\boldsymbol{K}}$*–increasing chain* provided that $A_i \in \boldsymbol{K}$ and $A_i \prec_{\boldsymbol{K}} A_{i+1}$ for all $i < \delta$, and $A_i = \bigcup_{j<i} A_j$ for all limit ordinals $i < \delta$.

If $M \prec_{\boldsymbol{K}} N$ we say that M is a *strong submodel* of N. If $f : M \mapsto N$ is 1-1 and $fM \prec_{\boldsymbol{K}} N$, we call f a *strong embedding.* Note that **A3** in toto says that $\boldsymbol{K}$ is closed under well–ordered direct limits of *strong embeddings*.

This notion generalizes the framework in which Jónnson proved the existence and uniqueness of universal-homogeneous models for classes defined by universal sentences (i.e. closed under substructure). The main innovation is to introduce a variable relation $\prec_{\boldsymbol{K}}$ for submodel. The most natural examples are elementary classes in $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$. $\prec_{\boldsymbol{K}}$ becomes elementary submodel in an appropriate fragment of $L_{\omega_1,\omega}$.

Note that classes defined in logics $L_{\kappa,\lambda}$ with $\lambda > \aleph_0$ and taking elementary submodel in that logic are *not* AEC. Nor does the class of models of a theory in $L_{\infty,\omega}$ with the natural notion of submodel since there may be no Löwenheim-Skolem number. Let n be finite and L^n be logic with n-variables. Then any L^n-definable class with L^n-elementary submodel is an AEC [15, 4].

2 From Semantics to Syntax

In this section, we note several results of the following form: A class of models satisfying certain semantic conditions can be defined as class of models in a certain logic. The first theorem of this sort is Tarski's proof that a class $\boldsymbol{K}$ of relational structures that is closed under substructure and satisfies the condition: if every finite substructure of A is in $\boldsymbol{K}$ then A is in $\boldsymbol{K}$ is defined by set of first order universal sentences. This kind of result can be seen as a kind of Lindström's theorem but for elementary classes rather than logics.

We generalize by allowing both infinitary sentences and reducts. Recall that a *PC* (*pseudoelementary*) class is the collection of reducts to a vocabulary τ of models of a theory T' in an expanded vocabulary τ'. We extend this notion to allow the omission of types as part of the defining condition.

Definition 2.1 *Let Γ be a collection of types in finitely many variables over the empty set in a vocabulary τ'. A $PC(T, \Gamma)$* class *is the class of reducts to $\tau \subset \tau'$ of models of a first order theory τ'-theory which omit all members of the specified collection Γ of partial types.*

We write $PC\Gamma$ to denote such a class without specifying either T or Γ. And we write $\boldsymbol{K}$ is $PC(\lambda, \mu)$ if $\boldsymbol{K}$ can be presented as $PC(T, \Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \mu$. (We sometimes write $PC\Gamma(\lambda, \mu)$ to emphasize the type omission. In the simplest case, we say $\boldsymbol{K}$ is λ-presented if $\boldsymbol{K}$ is $PC(\lambda, \lambda)$.)

First we note that with these definitions we can provide a syntactic definition for each AEC. On the one hand this characterization is hopeless as it requires an expansion to a larger language and an extremely arbitrary construction in the expansion; thus the 'algebraic' information about the original class is lost. But the presentation theorem does allow one (at least for classes with arbitrarily large models) to use Ehrenfeucht-Mostowski models. This proves an extremely valuable tool that has been exploited by Shelah in many situations; one of the many examples is in [1, 19].

Theorem 2.2 (Shelah's Presentation Theorem) *If K is an AEC with Löwenheim-number* $\mathrm{LS}(\boldsymbol{K})$ *(in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$), there is a vocabulary $\tau' \supseteq \tau$ with cardinality $|\mathrm{LS}(\boldsymbol{K})|$, a first order τ'-theory T' and a set Γ of at most $2^{\mathrm{LS}(\boldsymbol{K})}$ partial types such that:*

$$\boldsymbol{K} = \{M' \restriction \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, the $\prec_{\boldsymbol{K}}$ relation satisfies the following conditions:

1. *if M' is a τ'-substructure of N' where M', N' satisfy T' and omit Γ then $M' \restriction \tau \prec_{\boldsymbol{K}} N' \restriction \tau$;*
2. *if $M \prec_{\boldsymbol{K}} N$ there is an expansion of N to a τ'-structure such that M is the universe of a τ'-substructure of N;*
3. *Finally, the class of pairs (M, N) with $M \prec_{\boldsymbol{K}} N$ forms a $PC\Gamma(\aleph_0, 2^{\aleph_0})$-class in the sense of Definition 2.1.*

   Without loss of generality we can guarantee that T' has Skolem functions.

The proof of this result is wildly non-constructive. The connection between the expanded language and the original is only what is demanded by the theorem. The proof has two stages. By adding $\mathrm{LS}(\boldsymbol{K})$ function symbols to form a language τ' we can regard each model of cardinality at most $\mathrm{LS}(\boldsymbol{K})$ as being finitely generated. If we look at finitely generated τ'-structures, the question of whether the structure is in $\boldsymbol{K}$ is a property of the quantifier free type of the generators. Similarly the question of whether one τ' finitely generated structure is strong in another is a property of the τ' type of the generators of the larger model. Thus, we can determine membership in $\boldsymbol{K}$ and strong submodel for finitely generated (and so all models of cardinality $\mathrm{LS}(\boldsymbol{K})$) by omitting types. But every model is a direct limit of finitely generated models so using the AEC axioms on unions of chains (and coherence) we can extend this representation to models of all cardinalities. For detailed proofs see [1] or [21].

A second example is Kirby's [13] treatment of quasi-minimal excellent classes. In [25], Zilber defines a generalization of the notion of a strongly minimal set designed to capture the structure of complex exponentiation. Key to this study is the generalization of the first order notion of algebraic closure by defining $a \in \mathrm{cl}(B)$ if for some $L_{\omega_1,\omega}$ formula $\phi(x, \mathbf{b})$ we have both $\phi(a, \mathbf{b})$ and there are only countably many solutions for ϕ. Zilber's classes can be defined in $L(Q)$; but there is a negative occurrence of Q. However, the class of models is still closed under unions of chains with respect to the notion of closed submodel [13, 1]. M is a closed submodel of N if for every $X \subset M$, $\mathrm{cl}_M(X) = \mathrm{cl}_N(X)$. Zilber demands eventually that his 'quasiminimal excellent' class be definable in $L_{\omega_1,\omega}(Q)$. Then he proves that classes which satisfies all of these conditions are categorical in all uncountable powers. (The term quasiminality arises because the conditions imply that every definable subset is countable or co-countable.) Kirby takes a different tack. He axiomatizes the situation entirely by properties of models and concludes that the class can be axiomatized in $L_{\omega_1,\omega}(Q)$. He defines the notion of strong submodel in terms of closure in the underlying combinatorial geometry.

For our third example we look at the notion of finite character introuduced by Hyttinen and Kesala [9] and various results of Kueker [14]. Using his method of countable approximations and game quantifiers he has proved a number of definability results for Abstract Elementary Classes. We use Kueker's definition which is equivalent to the earlier definition of Hyttinen and Kesala in classes with amalgamation.

Definition 2.3 *$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has* finite character *if and only if for $M, N \in \boldsymbol{K}$ we have $M \prec_{\boldsymbol{K}} N$ if $M \subseteq N$ and for every finite $\boldsymbol{a} \in M$ there is some $\boldsymbol{K}$-embedding $f_{\boldsymbol{a}}$ of M into N fixing $\boldsymbol{a}$.*

Clearly if $\boldsymbol{K}$ is defined by a sentence in $L_{\infty,\omega}$ and $\prec_{\boldsymbol{K}}$ is elementary submodel with respect to that fragment (or even with respect to a smaller nicely closed class of formulas), $\boldsymbol{K}$ has finite character. Equally clearly, elementary submodel with respect to cardinality quantifiers does not have finite character.

Theorem 2.4 (Kueker) *1. If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is an AEC and $\mathrm{LS}(\boldsymbol{K}) = \kappa$ then $\boldsymbol{K}$ is closed under L_{∞,κ^+}-elementary equivalence.*

2. Assume that $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has finite character. Let $M \in \boldsymbol{K}$ and assume that $M \equiv_{L_{\infty,\omega}} N$. Then $N \in \boldsymbol{K}$.

Very little more has appeared beyond this result although there are a number of examples showing limits to the most immediate conjectures. We discuss some other examples in Section 4. An AEC with finite character is called finitary if also has arbitrarily large models, the amalgamation property and Löwenheim number $\aleph_0$. The work of Hyttinen, Kesala and Kueker raised the question.

Question 2.5 *Suppose a finitary AEC is categorical in all uncountable models. Must it be $L_{\omega_1,\omega}$ definable?*

A crucial point in the study of AEC is that compactness fails in a very strong way; in general the class does not have the upward Löwenheim-Skolem property. This raises the question:

Question 2.6 *Can the 'compactness aspect' of various Lindström theorems characterizing logics be weakened to 'upward Löwenheim-Skolem property'?*

There have been some recent results on Lindström theorems for logics that allow the study of analysis, specifically Banach Spaces. Bradd Hart has announced a new Lindström theorem for continuous logic.

Theorem 2.7 (Hart) *First order continuous logic is the maximal logic for continuous structures such that satisfies:*

1. *closure under ultraproducts*
2. *the DLS property*
3. *closure under unions of elementary chains of substructures*

Iovino [10] proved a related result earlier. He studies a logic $\mathbb{H}$ of positive bounded formulas under approximate satisfaction. This framework stems from Henson [8].

Theorem 2.8 (Iovino) *There is no logic for analytic structures that extends* $\mathbb{H}$ *properly and satisfies both the compactness and the elementary chain property*

Both of these theorems are formulated for notions of 'structure' that are ostensibly different from those in Lindström's context. Specifically for both 'continuous' and 'analytic' structures, there is a specified sort for the real numbers that is held standard in all models and certain uniform continuity conditions are imposed on basic functions into this sort. Note however, that any class defined in such a logic is a candidate for being an AEC in the normal interpretation of 'structure'. Thus there is a possibility of reformulating these results (or proving related results) as Lindström theorems for AEC.

3 AEC and generalized quantifiers

A natural question is to try to extend this framework to study classes defined with the adjunction of the Q quantifier introduced by Mostowski. We focus here on the $\aleph_1$ interpretation but there is a natural question of extending the analysis to other interpretations. We close the section by discussing some similar logics.

The models of an arbitrary sentence of $L(Q)$ with the associated notion of elementary submodel as $\prec_{\boldsymbol{K}}$ does *not* give an AEC; it easy for the interpretation of a formula $\phi(x)$ to have countably many solutions in each model of an elementary chain but not in their union. As Caiceido pointed out at the conference, this problem does not arise if each occurrence of Q is 'positive'.

Question 3.1 *Are there positive sentences in $L(Q)$ that describe mathematically interesting situations?*

Keisler and others (e.g. [11, 1]) described some other notions of strong submodel for the $L(Q)$ setting.

Definition 3.2 *Let ψ be a sentence in $L_{\omega_1,\omega}(Q)$ in a countable vocabulary and let L^* be the smallest countable fragment of $L_{\omega_1,\omega}(Q)$ containing ψ. Define a class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ by letting $\boldsymbol{K}$ be the class of models of ψ in the standard interpretation. We consider several notions of strong submodel.*

1. $M \prec^* N$ *if*

   (a) $M \prec_{L^*} N$ *and*

   (b) $M \models \neg(Qx)\theta(x, \boldsymbol{a})$ *then* $\{b \in N : N \models \theta(b, \boldsymbol{a}) = \{b \in M : N \models \theta(b, \boldsymbol{a})$.

2. $M \prec^{**} N$ *if*

   (a) $M \prec_{L^*} N$,

   (b) $M \models \neg(Qx)\theta(x, \boldsymbol{a})$ *then* $\{b \in N : N \models \theta(b, \boldsymbol{a}) = \{b \in M : N \models \theta(b, \boldsymbol{a})$, *and*

   (c) $M \models (Qx)\theta(x, \boldsymbol{a})$ *implies* $\{b \in N : N \models \theta(b, \boldsymbol{a})$ *properly contains* $\{b \in M : N \models \theta(b, \boldsymbol{a})$.

Now, $(\boldsymbol{K}, \prec^*)$ is an AEC with Löwenheim Number $\aleph_1$. But, in general, $(\boldsymbol{K}, \prec^{**})$ is *not* an AEC. (Hint: Consider the second union axiom **A3.3** in Definition 1.1 and a model with a definable uncountable set.)

I asked in the early 70's whether there was a sentence of $L(Q)$ that had a unique model and that model had cardinality $\aleph_1$. Shelah replied with the following result. He gave two proofs of this proposition. In the first [16], he assumed $V = L$ and developed a large amount of stability theory for $L_{\omega_1,\omega}(Q)$. It developed that this argument really uses only $2^{\aleph_0} < 2^{\aleph_1}$ and the set theory is used to reduce to a complete sentence in $L_{\omega_1,\omega}$ that is ω-stable. (See the Chapter, Independence in ω-stable theories in [1].) The use of $2^{\aleph_0} < 2^{\aleph_1}$ to show an $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ is ω-stable is essential. The crucial example is described in [20]; Baldwin introduces a new set of forcing conditions to verify that Martin's axiom implies $\aleph_1$-categoricity of the class in [1].

In [21] Shelah proved the result in ZFC from basic principles without stability theory; the argument goes in two stages. (See also [1].)

Theorem 3.3 (Shelah) *If* $\boldsymbol{K}$ *is a* $\aleph_0$*-categorical* $PC\Gamma(\aleph_0, \aleph_0)$ *class that is also an AEC and has a unique model of power* $\aleph_1$*, then there is a model of power* $\aleph_2$*.*

Corollary 3.4 (Shelah) *An* $\aleph_1$*-categorical sentence* ψ *in* $L_{\omega_1,\omega}$ *has a model of power* $\aleph_2$*.*

But what about $L_{\omega_1,\omega}(Q)$? Can we derive the result for $L_{\omega_1,\omega}(Q)$ from Theorem 3.3? Let $\boldsymbol{K}$ be the class of models of $L(Q)$-sentences and $\prec$ denote $L(Q)$-elementary submodel. We are asking, "is $(\boldsymbol{K}, \prec)$ an AEC?" And answers vary.

1. In the $\aleph_0$ interpretation, yes.
2. In the $\aleph_1, \aleph_2$, equi-cardinal interpretations, no.

So the extension to $L(Q)$ requires some further effort. Shelah does this in [21]. But the extension is completely *ad hoc.* We sketch Shelah's argument and then point out another formalism.

Definition 3.5 *A sentence ψ in $L_{\omega_1,\omega}(Q)$ is called* complete *for $L_{\omega_1,\omega}(Q)$ if for every sentence ϕ in $L_{\omega_1,\omega}(Q)$, either $\psi \models \phi$ or $\psi \models \neg\phi$.*

The details of the following argument [1] are due to David Kueker; the assertion is implicit in [16].

Theorem 3.6 *Suppose the τ-structure M realizes only countably many $L_{\omega_1,\omega}(Q)$ types. Then there is a* complete *sentence σ_M of $L_{\omega_1,\omega}(Q)$ such that $M \models \sigma_M$.*

We call a model M that realizes only countably many $L_{\omega_1,\omega}(Q)$ types, $L_{\omega_1,\omega}(Q)$-small. Now Keisler [12] showed (see treatment in [1]:

Theorem 3.7 *If an $L_{\omega_1,\omega}(Q)$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete* small *$L_{\omega_1,\omega}(Q)$-sentence ψ_0 that implies ψ and has a model of cardinality $\aleph_1$ and such that every model of ψ_0 is small.*

Shelah [17] proved by a nice application of the undefinability of well-order in $L_{\omega_1,\omega}(Q)$:

Theorem 3.8 *If the $L_{\omega_1,\omega}(Q)$-τ-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}(Q)$, then ψ has a $L_{\omega_1,\omega}(Q)$-small model of cardinality $\aleph_1$.*

Combining these results, we have:

Theorem 3.9 *If an $L_{\omega_1,\omega}$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete $L_{\omega_1,\omega}$-sentence ψ_0 that implies ψ and has a model of cardinality $\aleph_1$.*

This gets close to the situation of a complete sentence of $L_{\omega_1,\omega}$ that allows Shelah to prove categoricity transfer for $L_{\omega_1,\omega}$. But the transfer for $L(Q)$ is dodgy at best and the transfer of categoricity result for $L_{\omega_1,\omega}(Q)$ remains open. Even, as mentioned above, the nice proof that a complete $\aleph_1$-categorical sentence $L_{\omega_1,\omega}(Q)$ has a model in $\aleph_2$ requires *ad hoc* methods that seem unlikely to generalize.

The significance of the transfer can be seen even in $L_{\omega_1,\omega}$. It is not hard (Marker) to construct a sentence in $L_{\omega_1,\omega}$ that is $\aleph_1$-categorical but has $2^{\aleph_0}$ countable models. Choosing ϕ_0 is throwing away all but one of those countable models.

It is pointed out in [11] that the omitting types result of $L(Q)$ extend to $L(aa)$. So it should not be hard to give a variant of Theorem 3.7 for $L(aa)$. But are there any interesting AEC defined by sentences in $L(aa)$? Since the language has second order variables, it is not clear that one will be able to find AEC defined in the logic. The connections of AEC with the Whitehead problem that lead to the discussion of ${}^{\perp}N$ in Section 4 raise the question of whether using $L(aa)$ to describe filtrations of models of cardinality $\aleph_1$ might provide some examples.

Definition 3.10 *Let ψ_0 be a small $L_{\omega_1,\omega}(Q)$-complete sentence with vocabulary τ in the countable fragment L^* of $L_{\omega_1,\omega}(Q)$. Form τ' by adding predicates for formulas as in Morley's procedure for first order theories; but also add for each formula $(Qx)\phi(x,\mathbf{y})$ a predicate $R_{(Qx)\phi(x,\mathbf{y})}$ and add the axiom*

$$(\forall x)[(Qx)\phi(x,\mathbf{y}) \leftrightarrow R_{(Qx)\phi(x,\mathbf{y})}].$$

Let ψ' be the conjunction of ψ_0 with the $L_{\omega_1,\omega}(Q)$-τ'-axioms encoding this expansion. Let $\boldsymbol{K}_1$ be the class of atomic τ'-models of $T(\psi)$, the first order τ' theory containing all first order consequences of ψ'.

The two roles of the union axiom conflict. $(\boldsymbol{K}_1, \leq^*)$ is an AEC with Löwenheim number $\aleph_0$. But, to get Löwenheim number $\aleph_0$, we allow models of $\boldsymbol{K}_1$ that are not models of ψ. Unfortunately, we may also have gained uncountable models of $\boldsymbol{K}_1$ that are not models of ψ. Working with $(\boldsymbol{K}_1, \leq^*)$, one cannot show that many models for $\boldsymbol{K}_1$ implies many models of ψ. $(\boldsymbol{K}_1, \leq^{**})$ solves this problem. But, $(\boldsymbol{K}_1, \leq^{**})$, does not satisfy A.3.3.

Coppola [5] introduced the notion of a Q-AEC, which has two notions of submodel. This provides a completely axiomatic proof of the existence of a model in $\aleph_2$ for $L_{\omega_1,\omega}(Q)$.

Definition 3.11 (Coppola) *A* Q-Abstract Elementary Class *is a collection of τ-structures K equipped with a notion of submodel $\prec_{\boldsymbol{K}}$, a refined notion of submodel to build chains $\prec^{\cup}_{\boldsymbol{K}}$, $\mathcal{K} = (K, \prec_{\boldsymbol{K}}, \prec^{\cup}_{\boldsymbol{K}})$ such that*

- **A0** *$K, \prec_{\boldsymbol{K}}, \prec^{\cup}_{\boldsymbol{K}}$ are closed under isomorphism, i.e.*
  - **a.** *If $M \in K$ and $M \approx M'$ then $M' \in K$;*
  - **b.** *If $M \prec_{\boldsymbol{K}} N$ and $f : N \tilde{\hookrightarrow} N'$, then $f(M) = M' \prec_{\boldsymbol{K}} N'$;*
  - **c.** *If $M \prec^{\cup}_{\boldsymbol{K}} N$ and $f : N \tilde{\hookrightarrow} N'$, then $f(M) = M' \prec^{\cup}_{\boldsymbol{K}} N'$;*
- **A1** *$\prec_{\boldsymbol{K}}$ is a partial order, and $\prec^{\cup}_{\boldsymbol{K}}$ is transitive on K;*
- **A2** *$\prec_{\boldsymbol{K}}$ refines the notion of substructure, $\prec^{\cup}_{\boldsymbol{K}}$ refines $\prec_{\boldsymbol{K}}$;*
- **A3** *If $M_0 \subset M_1$ and $M_0, M_1 \prec_{\boldsymbol{K}} N$ then $M_0 \prec_{\boldsymbol{K}} M_1$ (coherence for $\prec_{\boldsymbol{K}}$);*
- **A4** *There is a* Löwenheim -Skolem number, *$LS(\boldsymbol{K})$ such that for all $N \in \mathcal{K}$ and $A \subseteq N$ there is $M \prec^{\cup}_{\boldsymbol{K}} N$ containing A of size at most $|A| + \mathrm{LS}(\boldsymbol{K})$;*
- **A5** *If $(M_i : i < \lambda)$ is $\prec^{\cup}_{\boldsymbol{K}}$-increasing, continuous, then*
  - **a** *$M = \bigcup_{i<\lambda} M_i \in \mathcal{K}$;*
  - **b** *$M_i \prec^{\cup}_{\boldsymbol{K}} \bigcup_{i<\lambda} M_i$, for each $i < \lambda$;*
  - **c** *If $M_i \prec_{\boldsymbol{K}} N$ for each $i < \lambda$, then $\bigcup_{i<\lambda} M_i \prec_{\boldsymbol{K}} N$.*
- **A6** *$\mathcal{K}$ satisfies Assumptions I,II,III (below).*

With this result we have a clean argument in ZFC without invoking stability theoretic arguments that a complete sentence in $L_{\omega_1,\omega}(Q)$ has a model in $\aleph_2$.

4 AEC of modules

The notion of an AEC provides a way to describe certain classes of modules. Recall that an abelian group A is a *Whitehead group* if $Ext(A, \mathbb{Z}) = 0$. That is, every short exact sequence

$$0 \to \mathbb{Z} \to H \to A \to 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathbb{Z}$. This notion is generalized by defining for a group N, Z in the case of Whitehead groups, the class ${}^\perp N$ to be those groups A such that any short exact sequence with kernel N and image A splits. These classes have been widely studied by module theorists; a recent summary is [7]. Baldwin, Eklof and Trlifaj discovered that the properties of such classes could be neatly summarized when they form an AEC $\boldsymbol{K}$ with the following notion of strong submodel. $A \prec_N B$ if $A \subset B$, both are in $\boldsymbol{K}$ and $B/A \in \boldsymbol{K}$. (To simplify some of the statements, we replace $Ext(A, \mathbb{Z}) = 0$ by for each i, $Ext^i(A, \mathbb{Z}) = 0$.) We, [3] have the following result.

Theorem 4.1 (Baldwin, Eklof, Trlifaj) *1. For any module N, if the class $({}^\perp N, \prec_N)$ is an abstract elementary class then N is a cotorsion module.*

2. For any R-module N, over a ring R, if N is a pure-injective module then the class $({}^\perp N, \prec_N)$ is an abstract elementary class.

3. For an abelian group N, (module over a Dedekind domain R), the class $({}^\perp N, \prec_N)$ is an abstract elementary class if and only if N is a cotorsion module.

Thus the question whether $({}^\perp N, \prec_N)$ imposes natural restrictions on N. For e.g. abelian groups, the condition is exactly that N is cotorsion. But we show in [3] that although if N is a pure-injective module then the class $({}^\perp N, \prec_N)$ is an abstract elementary class, there are such N which cotorsion but not pure-injective. In particular, $({}^\perp N, \prec_N)$ is an abstract elementary class, then ${}^\perp N$ is closed under arbitrary direct limits (of homomorphisms) not just direct limits of strong embeddings in the sense of $({}^\perp N, \prec_N)$. Recently Trlifaj [22] has proved that a large class of classes $({}^\perp N, \prec_N)$ which are AEC have finite character. It remains open whether there are AEC of ${}^\perp N$ that do not have finite character. This is a particularly intriguing question since the notion of defining $M_0 \prec_N M_1$ by making a requirement on the quotient M_1/M_0 seemed at first to be a radically new notion of strong submodel.

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