M is not Mary: Variables from grade 3 to 13

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March 15, 2008

Outline

Collaborators

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The preparation of this talk was supported in part by a grant from the National Science Foundation to the Center for the Mathematics Education of Latinos (No. ESI-0424983). The findings and opinions expressed here are those of the author(s) and do not necessarily reflect the views of the funding agency.

What is a variable?

Substitutional approach

A variable is a symbol that can be replaced by a number.

Something that varies

A variable is the argument of a function

Which explanation is right?

Mathematically, the many notions of variable can be explained using the first approach.

Psychologically, there are many different perceptions.

The Pizza Problem

Problem

I went to the Pompeii restaurant and bought the same number of salads and small pizzas. Salads cost two dollars each and pizzas cost six dollars each. I spent \$40 all together. Assume that the equation $2S + 6P = 40$ is correct. Then,

$$2S + 6P = 40.$$

Since $S = P$, I can write

$$2P + 6P = 40.$$

So

$$8P = 40.$$

The last equation says 8 pizzas is equal to \$40 so each pizza costs \$5. Is this reasoning right?

What is P?

What is wrong with the above reasoning? Be as detailed as possible. How would you try to help a student who made this mistake?

(taken from Lockhead and Mestre, From Words to Algebra, Mending Misconceptions, Algebraic Thinking: grades K-12, NCTM [LM99])

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P is the number of pizzas I bought.

Note that after applying the distributive law P is number of 'meals' bought.

One advantage of algebra is that we do not keep an assigned and constant meaning for each variable during the course of computation.

Watching the Video

I have handed out the problem the children are working on. We will listen to about three minutes of tape. This tape is of 3rd graders in the CEMELA after school program. The investigation of this incident was undertaken with Alexander Radosavljevic.

What do you observe in the video?

Some Observations

How many chickens are there?

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How many chickens are there?

1. Three, it says so at the top.
2. Six, two are weighed in each of three combinations.
3. Nine, count them.

Moral I

This group of third graders don't understand the conventions for reading cartoons.
How could the teacher address that problem?

Moral II

The question is not, "How many chickens there are?"

The question is, "What is the weight of the three chickens, of each chicken? "

Questions

How, if at all, can such problems be used in elementary school to develop students understanding of algebra?

It may be very important that these are third graders; the materials were taken from a 6th grade curriculum but have been used with all ages.

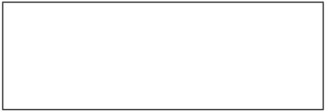
Fence problem

Hillary and Barack can paint a fence in one hour.
So can Barack and John.
But Hillary and John take two hours.

How long does it take Hillary, Barack and John?

Fraction strip method

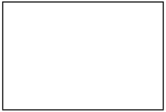
Hillary and Barack paint one fence in one hour.



John and Barack paint one fence in one hour.

Hillary and John paint one fence in two hours.

So, Hillary and John paint 1/2 fence in one hour.



Fraction strip method continued

So if we had two each of Hillary, Barack and John they would paint 2 1/2 fences in one hour.

Thus, the actual three can paint 5/4 of a fence in an hour.

Fraction strip method continued

So if we had two each of Hillary, Barack and John they would paint 2 1/2 fences in one hour.

Thus, the actual three can paint 5/4 of a fence in an hour.

And so it takes them 4/5 of an hour to paint the fence.

Another attempt

This is a concrete method of solution. A pre-service elementary teacher was given the problem in the 'a job of work form'. She attempted the problem by writing the following system of equations.

$$A + B = 1 \tag{1}$$
$$B + C = 1 \tag{2}$$
$$A + C = 2 \tag{3}$$

What is the difficulty?

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What is the difficulty?

The variables seem to represent the amount of time taken by each person. But this is contradictory.

From fraction strips to equations

The fraction strips lead naturally to the equations:

$$A + B = 1 \tag{4}$$
$$B + C = 1 \tag{5}$$
$$A + C = 1/2 \tag{6}$$

What do the variables represent?

From fraction strips to equations

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What do the variables represent?

The amount of fence each person does in one hour.

Work problem Critique

I quote at length from Zal Usiskin [Usi80, Usi07].

'What should not be in the algebra curriculum ...!'

Usiskin– Deletion 1: the traditional word problems

The traditional word problems (coin, age, mixture, distance-rate-time, and digit) are in the curriculum because of a very valuable goal, the goal of translating from the real world into mathematics. But except for mixture problems, they do not help achieve that goal. In fact, they convince students that are no real applications of algebra because they are so ridiculous.

Usiskin confronted

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giving easily accessible examples of the use of variable

Usiskin continued

Who can row upstream 5 hours at a time? Anyone who can belongs in the Guiness Book of World Records, not an algebra book.

I was appalled to see the following problem in a 1977 algebra book. ...

John can shovel snow from a walk in four minutes.
Mary can shovel the same walk in three minutes.
How long will it take them to do it together?

Usiskin continued

The only homage to reality in this problem is that people do shovel snow. However, the problem must have been written by someone from well-below the Mason-Dixon line. Anyone from the North knows that if a walk can be shoveled in less than five minutes it is not worth shoveling at all. Also two people would not shovel independently as the method in the book suggests. They would talk, or kiss, or help each other, or something.

Realism and Representation

The next series of problems are from a co-taught course in basic chemistry and intermediate algebra. They attempt to put this kind of problem in a coherent framework through the use of functions. These problems followed a lot of work on linear functions and then on linear equations.

A rate problem

Filling Sinks I

The hot water tap delivers 3 quarts per minute; the cold water tap delivers 4 quarts per minute. If both taps are turned on how long does it take to fill a sink that holds 12 quarts?

In the this case we are given the two rates:
cold water: 3 quarts per minute
hot water: 4 quarts per minute
So in any t minutes, the cold water delivers 3t quarts:

$$C(t) = 3t$$

and the hot water delivers 4t quarts:

$$H(t) = 4t.$$

How long does it take?

We are asked how long it takes for them together to fill one sink which holds 12 quarts.

$$C(t) + H(t) = 12$$

So, we must solve:

$$3t + 4t = 12.$$

Easily,
$t = 12/7$ minutes

A work problem

Filling Sinks II

The hot water tap can fill the sink in 3 minutes; the cold water tap can fill it in 4 minutes. If both taps are turned on how long does it take to fill the sink.

In the second case we are given the two rates:
cold water: $1/4$ sink per minute
hot water: $1/3$ sink per minute
So in any t minutes, the hot water fills $t/3$ sinks and the cold water fills $t/4$ sinks.

How long does it take?

We are asked how long it takes for the two taps together to fill one sink.

$$C(t) + H(t) = 1$$

$$t/3 + t/4 = 1$$

$$t(1/3 + 1/4) = 1$$

$$\frac{7}{12}t = 1$$

$t = 12/7$ minutes.

How long does it take to paint the fence

Let A, B and C be the rates at which Hilary, Barack, and John respectively paints in fences per hour.
Now

$$A \cdot 1 + B \cdot 1 = 1 \tag{7}$$
$$B \cdot 1 + C \cdot 1 = 1 \tag{8}$$
$$A \cdot 2 + C \cdot 2 = 1 \tag{9}$$

This of course yields an equivalent system to the fraction strip approach; but the method is uniform.

Distinctions of last approach

1. Aimed at more advanced students (grade 11 instead of grade 9);
2. example is more realistic as parties can cooperate;
3. variable is clearly identified;
4. the time is doubled–not the person.

What is the variable?

In the fraction strip approach the variable was:
the amount done in a unit time.

In the functions approach the variable was:
the amount done per unit time.

By identifying the variable as a rate the problems are made part of a general pattern – rather than one more isolated technique.

More General questions

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1. Are all solutions equal?
2. Are different solution schema more appropriate at different stages of a student's development?
3. More precisely, can one coordinate different approaches to develop the students conceptions?
4. Does such coordination make any sense over a 10-year span? Is it a realistic element of curriculum design?

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