

What is the
right
type-space?
Canadian
Mathematical
Society
June 1, 2007

John T.
Baldwin

What is the right type-space? Canadian Mathematical Society June 1, 2007

John T. Baldwin

June 30, 2007

Goals

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The fundamental notion of a Stone space is delicate for
infinitary logic.

I will describe several possibilities.

There will be a quiz.

Infinitary Logic and Omitting Types

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Key Insight

(Chang, Lopez-Escobar) Any sentence ϕ of $L_{\omega_1, \omega}$ can be coded by omitting a set of types in a countable first order theory.

Reduction

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(Shelah) Any κ -categorical sentence of $L_{\omega_1, \omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory.

A model $A \subset M \models T$ is atomic if every finite sequence from A realizes a principal (isolated) type over the empty set.

Our Context

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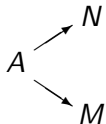
T is a countable first order theory that admits quantifier elimination.

\mathbf{K} is the class of **atomic models** of T .

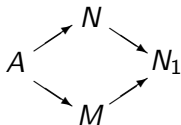
$\prec_{\mathbf{K}}$ is elementary submodel.

AMALGAMATION PROPERTY

The class \mathbf{K} satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



Assumption

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K has the amalgamation property **over models** and the joint embedding property.

This is a highly nontrivial assumption; it follows from categoricity up to \aleph_ω and the WGCH.

Amalgamation over arbitrary subsets is a still stronger hypothesis which misses mathematically important examples.

The Monster Model

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If \mathbf{K} has the amalgamation property and the joint embedding property then we can work inside a monster model (universal domain) \mathcal{M} that is $|\mathcal{M}|$ -model homogeneous. That is,

If $N \prec_{\mathbf{K}} \mathcal{M}$ and $N \prec_{\mathbf{K}} M \in \mathbf{K}$ and $|M| < |\mathcal{M}|$ there is embedding of M into \mathcal{M} over N .

Stone Space I

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Ordinary $S(A)$

Let $A \subset M \in \mathbf{K}$. A syntactic 1-type is an ultrafilter in the Boolean algebra of 1-ary formulas with parameters from A .
 $S(A)$ is the set of types over A .

Since T eliminates quantifiers there is no dependence on the choice of M .

Stone Space II

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$S^*(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S^*(A)$ means there is an atomic model M of T with $A \subseteq M$ such that p is realized in M .

Stone Space III

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$S_{at}(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S_{at}(A)$ means Aa is atomic if a realizes p .

Simple Example

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Wake up!

T is the theory of an infinite set under equality. $M \models T$. p asserts $x \neq m$ for every $m \in M$. Then $p \in S_{at}(M)$.

Complicated Example

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Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Complicated Example

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Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Consequences

What is the monster model for \mathbf{K} ?

Complicated Example

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Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Consequences

What is the monster model for \mathbf{K} ?
Every $p \in S_{at}(M)$ is realized in M .

Complicated Example

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Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Consequences

What is the monster model for \mathbf{K} ?

Every $p \in S_{at}(M)$ is realized in M .

This does not mean:

For any $A \subset M$, every $p \in S_{at}(A)$ is realized in M .

Stone Space IV

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$$S_{\mathbb{M}}(A)$$

Fix a monster model \mathbb{M} of \mathbf{K} .
Let $p \in S(A)$ where $A \subset \mathbb{M}$.

Definition

$p \in S_{\mathbb{M}}(A)$ means that p is realized in \mathbb{M} .

Question

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$$S_{\mathbb{M}}(A)$$

Fix a monster model \mathbb{M} of \mathbf{K} .

What are the relations among

$$S_{\mathbb{M}}(A),$$
$$S^*(A),$$
$$S_{at}(A),$$
$$S(A)?$$

Does the cardinality of A matter?

Observations

\subset means **proper** subset.

Always

$$S_{\mathbb{M}}(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$$

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Observations

\subset means **proper** subset.

Always

$$S_{\mathbb{M}}(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$$

Usually

$$S_{at}(A) \subset S(A)$$

Observations

\subset means **proper** subset.

Always

$$S_M(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$$

Usually

$$S_{at}(A) \subset S(A)$$

Marcus

$$S_M(A) \subset S^*(A)$$

Observations

\subset means **proper** subset.

Always

$$S_M(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$$

Usually

$$S_{at}(A) \subset S(A)$$

Marcus

$$S_M(A) \subset S^*(A)$$

A -countable

$$S^*(A) = S_{at}(A)$$

Stone Space V: Galois Types

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$\mathbb{S}(A)$

Fix a monster model \mathbb{M} of \mathbf{K} .

Definition

Let $A \subseteq M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The *Galois type* of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of \mathbb{M} which fix A .

The set of Galois types over A is denoted $\mathbb{S}(A)$.

Stone Space V: Galois Types: Subtleties

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Over sets

Clearly, for fixed \mathbb{M} , there exists $\pi : \mathbb{S}(A) \rightarrow S_{\mathbb{M}}(A)$.

But there is a choice of \mathbb{M} .

In general the relation between $\mathbb{S}(A)$ and $S_{at}(A)$ is unclear.

Over models

But for models: $\pi : \mathbb{S}(M) \rightarrow S_{at}(M)$.

There is only one 'monster model'.

And Shelah's notion corresponds to the 'monster model' version.

Very Complicated Example

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1, \omega}$ sentence ϕ_k such that:

- 1 ϕ_k has the disjoint amalgamation property;
- 2 Syntactic types determine Galois types over models of cardinality at most \aleph_{k-3} ;
- 3 But there are syntactic types over models of size \aleph_{k-3} that split into $2^{\aleph_{k-3}}$ -Galois types.
- 4 ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
- 5 ϕ_k is not \aleph_{k-2} -Galois stable;
- 6 But for $m \leq k - 3$, ϕ_k is \aleph_m -Galois stable;
- 7 ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Some further consequences

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In this example:

- 1 $\pi : \mathbb{S}(M) \rightarrow S_{\mathbb{M}}(A)$ is **not** 1-1.
- 2 There exists an independent pair of countable models M_1, M_2 over M_0 so that $|S_{at}(M_1 M_2)| = 2^{\aleph_0}$

The last example exhibits the failure of 'automorphism stationarity'.

If $M_1 \approx M'_1$ over M_0 and both are independent from M_2 over M_0 then they are isomorphic over M_0 .

But in 2) they need not be automorphic over M_0 .

One more inequality

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The Marcus example gives an uncountable atomic set A which cannot be embedded in an atomic model.

In fact, $S^*(A) \neq S_{\text{at}}(A)$.

From the Baldwin-Kolesnikov analysis there is a model M with

$$S^*(M) \neq S_{\text{at}}(M).$$

Conclusion

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- $\mathbb{S}(M)$ is the natural tool for Abstract elementary classes.
- $S_{\text{at}}(M)$ is Shelah's tool for $L_{\omega_1, \omega}$.
Excellence implies they agree.
- But Hyttinen and Kesala have interesting results with $S_{\mathbb{M}}(A)$ for finitary AEC.