What is the right type-space? Canadian Mathematical Society June 1, 2007

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Goals

The fundamental notion of a Stone space is delicate for infinitary logic.
I will describe several possibilities.

There will be a quiz.

Infinitary Logic and Omitting Types

Key Insight

(Chang, Lopez-Escobar) Any sentence ϕ of $L_{\omega_1,\omega}$ can be coded by omitting a set of types in a countable first order theory.

Reduction

(Shelah) Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory.

A model $A \subset M \models T$ is atomic if every finite sequence from A realizes a principal (isolated) type over the empty set.

Our Context

T is a countable first order theory that admits quantifier elimination.

$\mathbf{K}$ is the class of atomic models of T.

$\prec_{\mathbf{K}}$ is elementary submodel.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:

$$\begin{tikzcd} & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

there exists an N_1 such that

$$\begin{tikzcd} & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$$

Assumption

K has the amalgamation property over models and the joint embeddding property.

This is a highly nontrivial assumption; it follows from categoricity up to $\aleph_\omega$ and the WGCH.

Amalgamation over arbitrary subsets is a still stronger hypothesis which misses mathematically important examples.

The Monster Model

If **K** has the amalgamation property and the joint embedding property then we can work inside a monster model (universal domain) $\mathcal{M}$ that is $|\mathcal{M}|$-model homegeneous. That is,

If $N \prec_{\mathbf{K}} \mathcal{M}$ and $N \prec_{\mathbf{K}} M \in \mathbf{K}$ and $|M| < |\mathcal{M}|$ there is embedding of M into $\mathcal{M}$ over N.

Stone Space I

Ordinary $S(A)$

Let $A \subset M \in \mathbf{K}$. A syntactic 1-type is an ultrafilter in the Boolean algebra of 1-ary formulas with parameters from A.

$S(A)$ is the set of types over A.

Since T eliminates quantifiers there is no dependence on the choice of M.

Stone Space II

$S^*(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S^*(A)$ means there is an atomic model M of T with $A \subseteq M$ such that p is realized in M.

Stone Space III

$S_{at}(A)$

Let $p \in S(A)$ where A is atomic.

Definition

$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

Simple Example

Wake up!

T is the theory of an infinite set under equality. $M \models T$. p asserts $x \neq m$ for every $m \in M$. Then $p \in S_{at}(M)$.

Complicated Example

Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

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What is the monster model for $\mathbf{K}$?

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What is the monster model for **K**?
Every $p \in S_{at}(M)$ is realized in M.

Complicated Example

Fact

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.

Consequences

What is the monster model for **K**?
Every $p \in S_{at}(M)$ is realized in M.

This does not mean:
For any $A \subset M$, every $p \in S_{at}(A)$ is realized in M.

Stone Space IV

$S_{\mathbb{M}}(A)$

Fix a monster model $\mathbb{M}$ of $\mathbf{K}$.
Let $p \in S(A)$ where $A \subset \mathbb{M}$.

Definition

$p \in S_{\mathbb{M}}(A)$ means that p is realized in $\mathbb{M}$.

Question

$S_{\mathbb{M}}(A)$

Fix a monster model $\mathbb{M}$ of $\mathbf{K}$.

What are the relations among
$S_{\mathbb{M}}(A)$,
$S^*(A)$,
$S_{at}(A)$,
$S(A)$?

Does the cardinality of A matter?

Observations

$\subset$ means proper subset.

Always

$S_{\mathbb{M}}(A) \subseteq S^*(A) \subseteq S_{at}(A) \subseteq S(A)$

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Marcus

$$S_{\mathbb{M}}(A) \subset S^*(A)$$

***A*-countable**

$$S^*(A) = S_{at}(A)$$

Stone Space V: Galois Types

$\mathbb{S}(A)$

Fix a monster model $\mathbb{M}$ of $\mathbf{K}$.

Definition

Let $A \subseteq M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The *Galois type* of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of $\mathbb{M}$ which fix A.

The set of Galois types over A is denoted $\mathbb{S}(A)$.

Stone Space V: Galois Types: Subtleties

Over sets

Clearly, for fixed $\mathbb{M}$, there exists $\pi : \mathbb{S}(A) \twoheadrightarrow S_{\mathbb{M}}(A)$.
But there is a choice of $\mathbb{M}$.
In general the relation between $\mathbb{S}(A)$ and $S_{at}(A)$ is unclear.

Over models

But for models: $\pi : \mathbb{S}(M) \twoheadrightarrow S_{at}(M)$.
There is only one 'monster model'.
And Shelah's notion corresponds to the 'monster model' version.

Very Complicated Example

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
4. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
5. ϕ_k is not $\aleph_{k-2}$-Galois stable;
6. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;
7. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Some further consequences

In this example:

1. $\pi : \mathbb{S}(M) \twoheadrightarrow S_{\mathbb{M}}(A)$ is not 1-1.
2. There exists an independent pair of countable models M_1, M_2 over M_2 so that $|S_{at}(M_1 M_2)| = 2^{\aleph_0}$

The last example exhibits the failure of 'automorphism stationarity'.
If $M_1 \approx M_1'$ over M_0 and both are independent from M_2 over M_0 then they are isomorphic over M_0.
But in 2) they need not be automorphic over M_0.

One more inequality

The Marcus example gives an uncountable atomic set A which cannot be embedded in an atomic model.
In fact, $S^*(A) \neq S_{\mathrm{at}}(A)$.

From the Baldwin-Kolesnikov analysis there is a model M with

$$S^*(M) \neq S_{\mathrm{at}}(M).$$

Conclusion

- $\mathbb{S}(M)$ is the natural tool for Abstract elementary classes.
- $S_{\mathrm{at}}(M)$ is Shelah's tool for $L_{\omega_1,\omega}$.
  Excellence implies they agree.
- But Hytinnen and Kesala have interesting results with $S_{\mathbb{M}}(A)$ for finitary AEC.