Interactions of Set Theory, $L_{\omega_1,\omega}$, and ACE INFINITY Workshop

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Using Extensions of ZFC in Model Theory

Three Strategies

1. Force a model for a result known to be absolute.
2. Consistency implies truth.

   Find a model $\mathcal{N}$ of set theory where the result is true and then an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ where the result is absolute with V.
3. Use an intervening model of set theory to prove the result in V.

Today's Topics

Acknowledgements/References

1. Harrington's theorem: New proof (with S. Friedman, C.Laskowski, M. Koerwien) (Discussions with Marker) Also a recent proof by Knight, Montalban, and Schweber.
2. Few models in $\aleph_1$ of a PC_δ AEC implies
   a. few types over the empty set. (With P. Larson)
   b. almost Galois ω-stable implies
      Galois ω-stable and (if ap) absoluteness of $\aleph_1$-categoricity. (with P. Larson and S. Shelah)
3. Few models in $\aleph_1$ of an $L_{\omega_1,\omega}$-sentence implies the density of pseudominimal types. (With S. Shelah, C.Laskowski)

Harrington's theorem

Theorem: Harrington 70's unpublished

If ϕ is a counterexample to Vaught's conjecture then ϕ has models in $\aleph_1$ with arbitrarily high Scott rank.

Definition

$\varphi \in L_{\omega_1,\omega}$ is scattered if for every countable fragment F of $L_{\omega_1\omega}$ only countably many F-types are realized in a model of φ.

The Morley tree: $\mathcal{T}$

Inductive definition of the Morley tree $\mathcal{T}$

Suppose that φ is scattered.

1. Choose a countable fragment F_0 containing φ and let $\mathcal{T}_0$, consists all complete F_0-theories containing φ.
2. Define level $\alpha + 1$ of $\mathcal{T}$ by
   1. enlarge the fragment F_α to the least fragment $F_{\alpha+1}$ containing F_α and the conjunctions of the F_α-types realized in models of φ
   2. extend each theory T in $\mathcal{T}_\alpha$ which is not $\aleph_0$-categorical to the complete $F_{\alpha+1}$ theories containing T.
3. For limit δ:
   1. F_δ is the union of the fragments F_α, $\alpha < \delta$ and
   2. $\mathcal{T}_\delta$, is the collection of unions along paths cofinal through $\mathcal{T}_{<\delta}$.

Properties of the Morley tree: $\mathcal{T}$

(a) Each theory appearing in the Morley tree is an atomic theory, i.e. if T lies in the fragment F then each F-formula which is T-consistent is implied by a formula which is T-complete.

(b) If T lies on level α of the Morley tree and α is a limit ordinal, then any model of T has Scott rank at least α.

(c) Every countable model M of φ is the unique model of some theory on a terminal node of the Morley tree of φ.

(d) φ is a counterexample to the (absolute) Vaught conjecture iff $\mathcal{T}$ has uncountable height.

A counterexample to Vaught's conjecture has $\aleph_1$ countable models because each level of the Morley tree is countable.

The Generic Morley tree: $\mathcal{T}^*$

Fix $\mathbb{P}$ as the set of all finite partial functions from ω to ω_1, ordered by reverse inclusion.

Definition: The Generic Morley tree

Enlarge the universe V by making the ω_1 of V countable, with a standard Lévy collapse to a forcing extension $V^* = V[G]$ with G generic for $\mathbb{P}$.
The generic Morley tree $\mathcal{T}^*$ is the Morley tree for φ built in V^*.

If φ is a counterexample to VC, the generic Morley tree will have height $\omega_1^{V^*}$, the ω_2 of V.

Scott rank in $L_{\omega_2,\omega}$

As usual for $\alpha < \omega_2$, define by induction α-equivalence of finite tuples from a model M of cardinality $\aleph_1$.

Note that a model M in V with Scott sentence $\phi \in L_{\omega_2,\omega}$ has the same Scott rank β in V^*, though in V^*, $\phi \in L_{\omega_1,\omega}$. and β is uncountable in V, but countable in V^*.

The Extended Morley tree: $\widetilde{F}$

Definition: Extended Morley Tree

We define simultaneously fragments $\widetilde{F}_\alpha \subset L_{\omega_2,\omega}$ and collections $\widetilde{\mathcal{T}}_\alpha$ of $\widetilde{F}_\alpha$-theories by induction over $\alpha < \omega_2$: Just do the construction of the Morley tree for ω_2 steps. But

- Given $\widetilde{F}_\alpha$, let $\widetilde{\mathcal{T}}_\alpha$ be the collection of all sets $A \subset \widetilde{F}_\alpha$ such that
  - $\phi \in A$
  - there is some $p \in \mathbb{P}$ with $p \Vdash$ "A is a satisfiable, $\widetilde{F}_\alpha$-complete theory and no $A \restriction \widetilde{F}_\beta$ is $\aleph_0$-categorical for $\beta < \alpha$"

Generic atomicity

Definition

Let F be an $L_{\omega_2,\omega}$-fragment of size at most $\aleph_1$ and T a collection of F-sentences. T is generically F-atomic if in V^*, T is a satisfiable F-atomic $L_{\omega_1,\omega}$-theory

$\mathcal{T}^* = \widetilde{\mathcal{T}}$

Theorem

The extended Morley tree $\widetilde{\mathcal{T}}$ equals the generic Morley tree $\mathcal{T}^*$. In particular, $\mathcal{T}^*$ is an element of V.
Moreover, $\widetilde{\mathcal{T}}$ contains $\mathcal{T}$ (the standard Morley tree in V) as an initial segment.

The identification of $\widetilde{\mathcal{T}}$ and $\mathcal{T}^*$ yields:

Corollary

In V, for any $\alpha < \omega_2$, any theory $T \in \widetilde{T}_\alpha$ is generically F_α-atomic.

Computing Scott rank

Lemma

Suppose that T lies on level α of the extended Morley tree and α is a limit ordinal. Then any model of T has Scott rank at least α.

Suppose $M \models T$ and $sr(M) = \beta < \alpha$.

In V^*, we contradict:
Suppose that T lies on level α of the Morley tree and α is a limit ordinal. Then any model of T has Scott rank at least α.

Key Lemma

Model Existence theorem

If T is a theory on $\widetilde{\mathcal{T}} = \mathcal{T}^*$ then T has a model.

Corollary

[Harrington] If ϕ is counterexample to Vaught's conjecture then ϕ has models of Scott rank α for arbitrarily large $\alpha < \omega_2$.

Proof. If $T \in T^*_\alpha$, T has a model by the model existence theorem and it has Scott rank at least α by the previous slide.

Model Existence Theorem: the guts

Theorem

Let F be a fragment of $L_{\omega_2,\omega}$ with cardinality $\aleph_1$ and suppose the F-complete theory T is generically atomic. Then there is a directed system $(F_i, T_i, \pi_{ij}) : i < \omega_1)$ where T_i is a theory in the fragment F_i such that the direct limit of $(F_i, T_i, \pi_{ij}) : i < \omega_1)$ is (F, T).

Further, for each i, T_i is an atomic theory so has an atomic model M_i and an embedding σ_{ij} into M_j so $(F_i, \pi_{ij}, M_i, \sigma_{ij} : i < \beta)$ is an atomic directed system and the limit of $(M_i, \sigma_{ij} : i < \omega_1)$ is a model of T of cardinality $\aleph_1$.

The intuition

Let $T_i = p_i(T) \in \overline{A}_i$. Since T is a generically atomic F-theory; by the definability of forcing this property is preserved by elementary equivalence (in set theory) so for each i, T_i is generically atomic in $\overline{A}_i$.

Since $\overline{A}_i$ is countable we can build (in V) an $\overline{A}_i$-generic G for $\mathbb{P}^{\overline{A}_i}$. In $\overline{A}_i[G]$, T_i is an atomic theory with an atomic model M_i.

We have triples (M_i, F_i, T_i) for $i < \omega_1$.

IDEA: Take the union of the M_i to get the model in $\aleph_1$.

The complication

What happens to an F-formula $\bigwedge_{x\in X}\chi_x$ where each $\chi_x \in F$ and $|X| = \aleph_1$.

First note that each χ_x is in some A_i. But some χ_x may themselves be uncountable conjunctions and then some of the conjuncts will be missing from A_i (and so from $\overline{A}_i$).

So while each π_{ij} is the identity on $L_{\omega,\omega}(\tau)$ an infinite conjunction (disjunction) will gain elements as we pass from $\overline{A}_i$ to $\overline{A}_j$.

A suitable direct limit solves this problem.

Direct systems of fragments

Definition

A directed system of fragments is a continuous directed system (F_i, π_{ij}) where for $i < \omega_1$ each F_i is a countable fragment of $L_{\infty,\omega}(\tau_i)$ and the maps π_{ij} satisfy the following for each $i < j < \omega_1$:

- π_{ij} is the identity on atomic formulas;
- π_{ij} commutes with each of $\neg, \wedge, \vee, \exists$; and
- for each $\theta(\mathbf{x}) \in F_i$,
  - θ and $\pi_{ij}(\theta)$ have the same free variables;
  - θ is a disjunction (conjunction) if and only if $\pi_{ij}(\theta)$ is a disjunction (conjunction); and
  - ϕ is a disjunct (conjunct) of θ if and only if $\pi_{ij}(\phi)$ is a disjunct (conjunct) of $\pi_{ij}(\theta)$.

Directed systems of fragments and models

Definition

Suppose that $(F_i, \pi_{ij} : i < \beta)$ is a continuous directed system of countable fragments of length ω_1 and that for each i, M_i is an τ_i-structure.

1. A mapping $\sigma_{ij} : M_i \to M_j$ is $\underline{\pi_{ij}\text{-elementary}}$ if, for all $\theta(\mathbf{x}) \in F_i$ and all $\boldsymbol{a} \in M_i^{lg(\mathbf{x})}$,

$$M_i \models \theta(\boldsymbol{a}) \qquad \text{if and only if} \qquad M_j \models \pi_{ij}(\theta)(\sigma_{ij}(\boldsymbol{a})).$$

2. A directed system $(F_i, \pi_{ij}, M_i, \sigma_{ij})$ of fragments AND models is a pair of a directed system of fragments (F_i, π_{ij}) and a directed system of τ_i-structures (M_{ij}, σ_{ij}) such that for each $i < j$, σ_{ij} is π_{ij}-elementary.

Method: ‘Making more things absolute

Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$
or $L_{\omega_1,\omega}(aa)$
such that it is consistent that ϕ has a model.
Let $\mathcal{A}$ be the countable ω-model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct $\mathcal{B}$, an uncountable model of set theory, which is an elementary extension of $\mathcal{A}$
such that $\mathcal{B}$ is correct about uncountability (stationarity).
Then the model of ϕ in $\mathcal{B}$ is actually an uncountable model of ϕ.

How to build $\mathcal{B}$

MT Iterate a theorem of Keisler and Morley (refined by Hutchinson).

ST Iterations of 'special' ultrapowers. (stationary tower forcing)

Crucial points

1. Each $\mathcal{B}_\alpha$ is countable.
2. $\mathcal{B}_{\alpha+1}$ increases exactly the sets that

$$\mathcal{B}_\alpha$$

thinks are uncountable.

ZFC° denotes a sufficient subtheory of ZFC for our purposes.

Really distinct interations

Theorem (Larson)

If M is a countable model of $\mathrm{ZFC}^\circ + \mathrm{MA}_{\aleph_1}$ and

$$\langle M_\alpha, G_\alpha, j_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

and

$$\langle M'_\alpha, G'_\alpha, j'_{\alpha,\gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

are two distinct iterations of M, then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\alpha,$$

where α is least such that $G_\alpha \neq G'_\alpha$.

G_α not defined for $\alpha = \omega_1$.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an abstract elementary class: AEC if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;

Analytically Presented

$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is Analytically Presented if the class of countable models and the $\prec_{\boldsymbol{K}}$ on countable models are analytic sets.

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;
3. Downward Löwenheim-Skolem.

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A much baptized concept

Definition

An AEC $\boldsymbol{K}$ is $PC\Gamma(\aleph_0, \aleph_0)$-presented:

if the models are reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types
and the submodel relation is given in the same way.

AKA:

1. Keisler: PC_δ over $L_{\omega_1,\omega}$
2. Shelah: $PC(\aleph_0, \aleph_0)$, $\aleph_0$-presented

More Precisely

Theorem

If $\boldsymbol{K}$ is AEC then $\boldsymbol{K}$ can be analytically presented iff and only if its restriction to $\aleph_0$ is the restriction to $\aleph_0$ of a $PC\Gamma(\aleph_0, \aleph_0)$-AEC.

Proof remarks

The countable case is basically folklore.
The proof that this gives an aec in all cardinals combines the countable result with ideas from the proof of the presentation theorem.

Galois Types

Let $M \prec_{\mathbf{K}} N_0$, $M \prec_{\mathbf{K}} N_1$, $a_0 \in N_0$ and $a_1 \in N_1$ realize the same Galois Type over M iff
there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0 : N_0 \rightarrow N$ and $f_1 : N_1 \rightarrow N$ such that $f_0|M = f_1|M$ and $f_0(a_0) = f_1(a_1)$.

Realizing the same Galois type (over countable models) is an equivalence relation

$$E_M$$

if $\boldsymbol{K}_{\aleph_0}$ satisfies the amalgamation property.

The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property for models of cardinality at most $\aleph_0$

and has at most $\aleph_1$-Galois types over models of cardinality $\leq \aleph_0$

then there is an $\aleph_1$-monster model $\mathbb{M}$ for $\boldsymbol{K}$ and the Galois type of a over a countable M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

So E_M is an equivalence relation on $\mathbb{M}$.

Some stability notions

Definition

1. The abstract elementary class $(\mathbf{K}, \prec)$ is said to be Galois ω-stable if for each countable $M \in \mathbf{K}$, E_M has countably many equivalence classes.
2. The abstract elementary class $(\mathbf{K}, \prec)$ is almost Galois ω-stable if for each countable $M \in \mathbf{K}$, no E_M has a perfect set of equivalence classes.

Almost Galois Stable

Well-orders of type at most $\aleph_1$ under end-extension are an AEC where countable models have only $\aleph_1$ Galois types.

Galois equivalence is Σ^1_1

On an analytically presented AEC, having the same Galois type over M is an analytic equivalence relation, E_M. So by Burgess's theorem we have the following trichotomy.

Theorem

An analytically presented abstract elementary class $(\mathbf{K}, \prec)$ is

1. Galois ω-stable or
2. almost Galois ω-stable or
3. has a perfect set of Galois types over some countable model

Again basically folklore.

Keisler for Galois types

Theorem: (B/Larson)

Suppose that

1. **K** is an analytically presented abstract elementary class;
2. N is a **K**-structure of cardinality $\aleph_1$, and N_0 is a countable structure with $N_0 \prec_{\mathbf{K}} N$;
3. P is a perfect set of E_{N_0}-inequivalent members of ω^ω;
4. N realizes the Galois types of uncountably many members of P over N_0.

Then there exists a family of $2^{\aleph_1}$ many **K**-structures of cardinality $\aleph_1$, each containing N_0 and pairwise realizing just countably many P-classes in common.

Recovering Keisler's result

ZFC-Corollary: (Keisler, new proof Larson)

Let F be a countable fragment of $L_{\omega_1,\omega}(\mathrm{aa})$.
If there exists a model of cardinality $\aleph_1$ realizing uncountably many F-types,
there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many F-types in common.

$L_{\omega_1,\omega}$-case

Fact: Hyttinen-Kesala, Kueker

If a sentence in $L_{\omega_1,\omega}$, satisfying amalgamation and joint embedding, is almost Galois ω-stable
then it is Galois ω-stable.

Using the extending models of set theory:

Fact: B-Larson-Shelah

Suppose $\boldsymbol{K}$ is an analytically presented AEC, that satisfies amalgamation and joint embedding, and
has only countably many models in $\aleph_1$
If $\boldsymbol{K}$ is almost Galois ω-stable
then it is Galois ω-stable.

Example

Groupable partial orders (Jarden varying Shelah)

Let $(\boldsymbol{K}, \prec)$ be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order (equivalently admits a group structure) with $M \prec N$ if $M \subseteq N$ and no component is extended.

This AEC is analytically presented.
Add a binary function and say it is a group on each component.
It has $2^{\aleph_1}$ models in $\aleph_1$ and $2^{\aleph_0}$ models in $\aleph_0$.
But is almost Galois ω-stable.

Is there an analytically presented AEC with few models in $\aleph_1$ that is almost Galois ω-stable but not Galois ω-stable?

Many models in $\aleph_1$
Pseudo-algebraicity and Pseudo-minimality

Fixing the context

Fact

There is a 1-1 correspondence between the models of Scott sentence in a vocabulary τ and the class of atomic models of a first order theory T in an expanded vocabulary τ^*.

$\boldsymbol{K}_T$ is the class of atomic models of the countable first order theory T.

Definition

The atomic class $\boldsymbol{K}_T$ is extendible if there is a pair $M \preceq N$ of countable, atomic models, with $N \neq M$.

We assume throughout that $\boldsymbol{K}_T$ is extendible. We work in the monster model of T, which is usually not atomic.

A new notion of closure

Definition

An atomic tuple **c** is in the pseudo-algebraic closure of the finite, atomic set B ($\mathbf{c} \in \mathrm{pcl}(B)$)
if for every atomic model M such that $B \subseteq M$, and $M\mathbf{c}$ is atomic, $\mathbf{c} \subseteq M$.

When this occurs, and **b** is any enumeration of B and $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, we say that <u>$p(\mathbf{x}, \mathbf{b})$ is pseudo-algebraic.</u>

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Example I

Our notion, pcl of algebraic differs from the classical first-order notion of algebraic as the following examples show:

Example

Suppose that an atomic model M consists of two sorts. The U-part is countable, but non-extendible (e.g., U infinite, and has a successor function S on it, in which every element has a unique predecessor). On the other sort, V is an infinite set with no structure (hence arbitrarily large atomic models). Then, if an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

Definability of pseudo-algebraic closure

Strong ω-homogeneity of the monster model of T yields:

Fact

If $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, then

$$\mathbf{c} \in \mathrm{pcl}(\mathbf{b}) \quad \text{if and only if} \quad \mathbf{c}' \in \mathrm{pcl}(\mathbf{b}')$$

for any $\mathbf{c}'\mathbf{b}'$ realizing $p(\mathbf{x}, \mathbf{y})$. In particular, the truth of $c \in \mathrm{pcl}(\mathbf{b})$ does not depend on an ambient atomic model.

Extending non-algebraic types

Lemma

Let N be an atomic model containing $\mathbf{b}\boldsymbol{a}$. If $\mathbf{b}$ is not pseudoalgebraic over $\boldsymbol{a}$ then $\mathrm{tp}(\mathbf{b}/\boldsymbol{a})$ is realized in $N - \mathrm{pcl}(\boldsymbol{a}\mathbf{b})$.

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Pseudo-minimal sets

Definition

1. A type q over **b** is pseudominimal if pcl satisfies exchange on the realizations of q (even over external parameters).
2. M is pseudominimal if $x = x$ is pseudominimal in M.

'Density'

Definition

$\boldsymbol{K}_T$ satisfies 'density' of pseudominimal types if for every atomic $\mathbf{e}$ and atomic type $p(\mathbf{e}, \mathbf{x})$ there is a $\mathbf{b}$ with $\mathbf{eb}$ atomic and $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending p such that q is pseudominimal.

Failing 'density'

If $\boldsymbol{K}_T$ fails 'density' of pseudominimal types there is a

1. nested elementary chain of countable models M_i
2. a, $\langle c_i : i < \omega \rangle$ and $\langle \mathbf{d}_i : i < \omega \rangle$ such that:
3. for every i, $c_{i+1} \in M_{i+1} - M_i$, $\mathbf{d}_i \in M_i$ and $c_{i+1} \in \mathrm{pcl}(\mathbf{d}_i a)$.

This gives us an asymmetric relation which we extend to a linear order.
Shelah calls this notion 'failure of algebraic symmetry'.

Goals

Known

If (strongly) pseudo-minimal types are dense in $\boldsymbol{K}_T$ then $\boldsymbol{K}_T$ has a model in the continuum.

Known

If the universe of the countable model is pseudo-minimal then $\boldsymbol{K}_T$ has a model in the continuum.

Known

If $\boldsymbol{K}_T$ has few models in $\aleph_1$ then pseudo-minimal types are dense in $\boldsymbol{K}_T$.

Conjecture

If $\boldsymbol{K}_T$ has few models in $\aleph_1$ then (strongly) pseudo-minimal types are dense in $\boldsymbol{K}_T$

The relevant forcing

The Theorem

Main Theorem

If $\boldsymbol{K}_T$ fails 'density of pseudo-minimal types' (fails algebraic symmetry) then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

Proof Outline

1. Start with a model $\mathcal{N}_1$ of enough set theory and an infinitary τ-sentence ψ that fails 'density' and satisfies Martin' axiom, MA.
2. In $\mathcal{N}_1$, force the existence of models $M^{S,T}$ that code the pair (S,T) of disjoint stationary sets by a formula $\theta(S, T)$.
3. Form a tree of $2^{\aleph_1}$ such models $\mathcal{M}_\eta$ containing pairwise non-equivalent stationary sets S_η and construct in $\mathcal{M}_\eta$ a model M^{S_η,T_η}.
4. Conclude the models M^{S_η,T_η} are pairwise non-isomorphic in V.

Analogy

Constructing many non-isomorphic $\aleph_1$-like dense linear orders.

M^S is the direct sum $\boldsymbol{I}$ of orderings X_α where
$X_\alpha \approx Q$ if $\alpha \notin S$
and
$X_\alpha \approx 1 + Q$ if $\alpha \notin S$.
So a model can be thought of as $\{a_t : t \in \boldsymbol{I}\}$.

We recognize S as the set of α such $M_{<\alpha}$ has a least upper bound.

The version here:

0th try

replace the $a_t \in I$ by $\boldsymbol{a}_t = \langle a_{t,0} \ldots a_{t,n} \rangle \in M$ such that

1. $X = \langle \boldsymbol{a}_t : t \in \boldsymbol{I} \rangle$ is an $\aleph_1$-like linear order
2. each $a_{t,i}$ is inter pseudo-algebraic with $a_{t,0}$ (over lower)
3. Nothing is pseudo algebraic in lower levels.

The forcing completes the diagram of M.
Distinguish S by adding the requirement that for $t \in S$ there a sequence of $a_{s,0}$ with s tending to t such that $a_{s,0} \in \mathrm{pcl}(\mathrm{a}_{\mathrm{t},0} \cup \mathrm{X}_{<\mathrm{t}})$.

More Complicated

Definition

Considers linear orders I equipped with a subset P and a binary relation E such that

1. I is $\aleph_1$-like with first element.
2. E is an equivalence relation on I such that
   a. If t is min(I) or in P, t/E is $\{t\}$
   b. Otherwise t/E is convex dense subset of L with neither first nor last element.
3. I/E is a dense linear order such that both $\{t/E : t \in P\}$ and $\{t/E : t \notin P\}$ are dense in it,

Coding by Catching and Strong Catching

Definition

Let $M \prec N \in \boldsymbol{K}_T$ and $a \in N - M$.

1. We say that *a* catches *M* in *N* if $b \in \mathrm{pcl}(\mathrm{Ma}, \mathrm{N}) - \mathrm{M}$ implies $a \in \mathrm{pcl}(\mathrm{Mb}, \mathrm{N})$.
2. If *M* has an ***I*** filtration and *J* is an initial segment of *I*, we say that *a* strongly catches M_J in *M* if $a \in M$ catches M_J in *M* and for every large enough $s \in J$,

$$\mathrm{pcl}(\mathrm{M}_{<\mathrm{s}}\mathrm{a}) \cap \mathrm{M}_{\mathrm{J}} = \mathrm{M}_{<\mathrm{s}}.$$

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Coding by Catching and Strong Catching: limit points catch but don't strongly catch

Suppose $M = M_G$.

Lemma: Catch not strongly catch

If J is an initial segment of $\boldsymbol{I}$ which has a least upper bound in $M - M_J$, there is an $a \in M - M_J$ such that a catches M_J but a does not strongly catch M_J.

Lemma: Catch implies strongly catch

If J is an initial segment of $\boldsymbol{I}$ with no least upper bound and with no least E- class above J and $b \in M - M_J$ catches M_J then b strongly catches M_J.

Properties of $\theta(S, T)$

Given $\psi \in L_{\omega_1,\omega}(\tau)$, the formula $\theta(S, T) \in L_{\omega_1,\omega}(Q)(\tau^*)$ implies a first order τ^*-formula $\theta_1(P_1, P_2)$ which expresses:

a If $\alpha \in P_1$ then there is an $a \in M - M_{J_\alpha}$ which catches M_{J_α} but does not strongly catch M_{J_α}.

b If $\alpha \in C - (P_1 \cup P_2)$ every $a \in M - M_{J_\alpha}$ which catches M_{J_α} strongly catches M_{J_α}.

Overview

Form a tree of $2^{\aleph_1}$ models of ZFC +MA of $\mathcal{M}_\eta$ containing pairwise non-equivalent stationary sets S_η and construct in $\mathcal{M}_\eta$ a model $M^{S_\eta, T\eta}$

θ tells us we can recognize the S_η, $T\eta$. So the $M^{S_\eta, T\eta}$ are non-isomorphic in V.