Abstract Elementary Classes Generalized Quantifiers

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Topics

Two Goals

Tell

Explore the notion of Abstract Elementary Class as a way of examining extensions of first order logic

Ask

Does the work on generalized quantifiers lead to other examples that fit into this framework?

Two Streams of Model Theory

Logics

completeness, syntactic complexity and model theoretic properties, decidability, interpolation etc.

Classification Theory

Investigate the properties of classes defined in the logic.

categoricity, spectrum problem, applications to algebra, geometrical properties of models

The uhr-classification theorem

Theorem (Morley 1965)

If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.

Zilber: 2005

Quasiminimal Excellence implies Categoricity

Theorem. Suppose the quasiminimal excellent class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Zilber and $L_{\omega_1,\omega}(Q)$

The categorical class can be axiomatized in $L_{\omega_1,\omega}(Q)$. And Zilber's result could be phrased as categoricity transfer for very special sentences in $L_{\omega_1,\omega}(Q)$. Only involving $\neg Q$ and about

quasiminimal excellence.

It is likely that a version of the following theorem of Shelah holds for sentences of $L_{\omega_1,\omega}(Q)$ only have negative occurrences of Q.

Categoricity Transfer in $L_{\omega_1,\omega}$

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

WGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

A first order principle

Slogan

To study a structure A, study $\mathrm{Th}(A)$.

e.g.
The theory of algebraically closed fields to investigate $(\mathcal{C}, +, \cdot)$.

The theory of real closed fields to investigate $(\mathcal{R}, +, \cdot)$.

Not well-behaved for $L_{\omega_1,\omega}$

The $L_{\omega_1,\omega}$-theory of a structure A may have no countable model.

Theorem (Shelah)

If an $L_{\omega_1,\omega}$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete $L_{\omega_1,\omega}$-sentence ψ_0 that implies ψ and has a model of cardinality $\aleph_1$.

We can draw the same conclusion if ψ has arbitrarily large models but I think it is open if $\aleph_1$ is just replaced by $\aleph_2$.

$L_{\omega_1,\omega}(Q)$

Theorem (Shelah)

If an $L_{\omega_1,\omega}(Q)$-sentence ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then there is a complete small $L_{\omega_1,\omega}(Q)$-sentence ψ_0 that implies ψ and has a model of cardinality $\aleph_1$.

The result for $L_{\omega_1,\omega}(Q)$ is weaker that that for $L_{\omega_1,\omega}$ because the sentence ψ_0 in the last Theorem does not have a countable model (it implies $Qx(x = x)$).

Finding the right context

There is no real necessity for the 'theory' to be complete.

Maybe the syntactic notion of 'elementary submodel' is not the right relation between models.

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Strong Slogan

Classes of structures and the relations between them are more interesting than singleton structures.

ABSTRACT ELEMENTARY CLASSES defined

Definition

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an abstract elementary class: AEC if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism and satisfy the following conditions.

- **A1**. If $M \prec_{\mathbf{K}} N$ then $M \subseteq N$.
- **A2**. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
- **A3**. If $\langle A_i : i < \delta \rangle$ is $\prec_{\mathbf{K}}$-increasing chain:
  1. $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
  2. for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
  3. if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.

- **A4**. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$.
- **A5**. There is a Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and
  $|A'| \leq \mathrm{LS}(\mathbf{K}) + |\mathrm{A}|$.

Examples I

1. First order complete theories with $\prec_{\mathbf{K}}$ as elementary submodel.
2. Models of $\forall\exists$-first order theories with $\prec_{\mathbf{K}}$ as substructure.
3. L^n-sentences with L^n-elementary submodel.
4. Varieties and Universal Horn Classes with $\prec_{\mathbf{K}}$ as substructure.
5. Ext-orthogonal classes: ${}^{\perp}N$

Examples II

4. Models of sentences of $L_{\kappa,\omega}$ with $\prec_{\mathbf{K}}$ as: elementary in an appropriate fragment.
5. Models of sentences of $L_{\kappa,\omega}(Q)$ with $\prec_{\mathbf{K}}$ carefully chosen.
6. Robinson Theories with Δ-submodel
7. ‘The Hrushovski Construction’ with strong submodel
8. Other generalized quantifiers

Abelian Groups

At CRM-Barcelona in September 2006, I asked.

1. Does the notion of AEC provide a general framework to describe some work in Abelian group theory?
2. Certain AEC of abelian groups provide interesting previously unknown examples for the general study of AEC. Can this work be extended?

The group group

AIM meeting July 2006
J. Baldwin, W. Calvert, J. Goodrick, A. Villaveces, & A. Walczak-Typke, & Jouko Väänänen
I described some very preliminary results of this group to emphasize the exploratory nature of this program.

Strong Submodel

Notation

Consider various subclasses $\mathbf{K}^{foo}$ *of the class* $\mathbf{K}^{ab}$ *of all abelian groups (e.g. foo* $= div, red(p), \ldots$*).*

1. *"*$\leq$*" denotes subgroup.*
2. $G \prec_{pure} H$ *means* G *is a pure subgroup of* H*:*
3. *"*$G \prec_{sum} H$*" means that* G *is a direct summand of* H*;*
4. *"*$G \prec_{foo} H$*" means that* G *is a pure subgroup of* H *and* $H/G \in \mathbf{K}^{foo}$*.*

Properties of $(\mathbf{K}^{cyc}, \prec_{sum})$

$\mathbf{K}^{cyc}$ is the class of direct sums of cyclic groups

Abbreviating $(\mathbf{K}^{cyc}, \prec_{sum})$ as $\mathbf{K}^{cyc}$, we had hoped to prove the following:

- $\mathbf{K}^{cyc}$ is not an elementary class.
- $\mathbf{K}^{cyc}$ is a tame AEC with amalgamation and Löwenheim-Skolem number $\aleph_0$.

But it fails **A3.3**

Reflection

Why do we want A.3.3?

THE PRESENTATION THEOREM

Every AEC is a PCΓ

More precisely,

Theorem

If K is an AEC with Löwenheim number $\mathrm{LS}(\mathbf{K})$ (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbf{K})$), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then
$M' \restriction L \prec_{\mathbf{K}} N' \restriction L$.

What's so great about PCΓ

We care because PCΓ gives

- Ehrehfeucht Mostowski models;
- omitting types (for Galois types);
- can construct non-splitting extensions;
- key to finding showing a sentence of $L_{\omega_1,\omega}(Q)$ that is categorical in $\aleph_1$ has a model in $\aleph_2$.

PCΓ not the right choice

Conjecture

Let X be a class of cardinals in which a reasonably defined class is categorical.

Exactly one of X and its complement is cofinal.

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Exactly one of X and its complement is cofinal.

(Note: So, PC-classes are not 'reasonable'. The class:

$$\{(M, X) : 2^{|X|} \geq |M|\}$$

is categorical only in strong limit cardinals.

Synergy

Baldwin, Eklof, Trlifaj :

Theorem

1. *For any module N, if the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class then N is a cotorsion module.*
2. *For any R-module N, over a ring R, if N is a pure-injective module then the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class.*
3. *For an abelian group N, (module over a Dedekind domain R), the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class if and only if N is a cotorsion module.*

What do the words mean?

Definition

1. $^{\perp}N = \{A : Ext^i(A, N) = 0 : i < \omega\}$
2. For $A \subseteq B$ both in $^{\perp}N$, $A \prec_N B$ if $B/A \in {}^{\perp}N$.

Generalizes the class of Whitehead groups: $Ext(G, \mathcal{Z}) = 0$.

Whitehead Groups

Definition

We say A is a Whitehead group if $Ext(A, \mathcal{Z}) = 0$. That is, every short exact sequence

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow A \rightarrow 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathcal{Z}$.

Under V=L, Whitehead groups are free; hence PCΓ. What about in ZFC?

Cotorsion Modules

Definition

A module N over a ring R is cotorsion if $\mathrm{Ext}^1(\mathrm{J}, \mathrm{N}) = 0$ for every flat module J.

We provide other equivalents below.

Easy Part

A1 and **A2** are immediate for any R and N.
And in this context **A.3.1** easily implies **A.3.2**
And **A4** is equally immediate using that $\mathrm{Ext}^{\mathrm{i}}(-, \mathrm{N})$ is resolving:
That is,

1. it contains all projective modules;
2. whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in {}^{\perp}N$ then $Y \in {}^{\perp}N$);
3. $X \in {}^{\perp}N$ whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $Y, Z \in {}^{\perp}N$.

Interesting part

A3.1 is immediate from Eklof's Lemma:

Lemma

Let C be a module. Suppose that $A = \bigcup_{\alpha<\mu} A_\alpha$ *with* $A_0 \in {}^\perp N$ *and for all* $\alpha < \mu$, $A_{\alpha+1}/A_\alpha \in {}^\perp N$ *then* $A \in {}^\perp N$.

Very Interesting part

Fact

For abelian groups, the following are equivalent:

1. *N is cotorsion;*
2. *${}^{\perp}N$ is closed under direct limits;*
3. *$Ext(Q, N) = 0$.*

Now it is easy to show that ${}^{\perp}N$ is closed under direct limits implies ${}^{\perp}N$ satisfies **A.3.3**.

${}^{\perp}N$ satisfies **A.3.3** implies $Ext(Q, N) = 0$.

Very Very Interesting part

We do not know exactly the rings for which the hypothesis of N cotorsion is sufficient for **A.3**(3). It is sufficient when R is a Dedekind domain:

Lemma

Let R be a Dedekind domain and N a module. Then the following are equivalent:

1. N is cotorsion;
2. ${}^{\perp}N = {}^{\perp}PE(N)$ where $PE(N)$ denotes the pure–injective envelope of N;
3. ${}^{\perp}N$ is closed under direct limits;
4. **A3**(3) holds for $({}^{\perp}N, \prec_N)$.

Refinements

Definition

For any right R-module A and any cardinal κ, a
(κ, N)-refinement of length σ of A
is a continuous chain $\langle A_\alpha : \alpha < \sigma \rangle$ of submodules such that:

- $A_0 = 0$,
- $A_{\alpha+1}/A_\alpha \in {}^\perp N$, and
- $|A_{\alpha+1}/A_\alpha| \leq \kappa$ for all $\alpha < \sigma$.

Löwenheim-Skolem Number

Using the generalized Hill's Lemma, it is straightforward to show:

lemma

Let κ be a cardinal $\geq |R| + \aleph_0$. Let N be a module.

1. If every module $A \in {}^{\perp}N$ has a (κ, N)–refinement then $({}^{\perp}N, \prec_N)$ has Löwenheim–Skolem number κ.
2. If $({}^{\perp}N, \prec_N)$ has Löwenheim–Skolem number κ and satisfies **A3**(3) then every module $A \in {}^{\perp}N$ has a (κ, N)–refinement.

$({}^{\perp}N, \prec_N)$ as an AEC

We show that **A4** holds iff $(\mathbf{K}, \prec_{\mathbf{K}})$ has refinements. But the question of when refinements exist is rather complicated.

Lemma

Each member of ${}^{\perp}N$ admits a (κ, N)–refinement under any of the following conditions.

1. N is pure–injective and R is arbitrary; $\kappa = |R| + \aleph_0$.
2. N is cotorsion and R is a Dedekind domain; $\kappa = |R| + \aleph_0$.
3. (V=L) N is arbitrary and R is hereditary; $\kappa = \max\{|R|, |N|\} + \aleph_0$.

A beautiful proof

Theorem (Shelah)

If **K** is a $\aleph_0$-categorical $PC\Gamma(\aleph_0, \aleph_0)$ class that is also an AEC and has a unique model of power $\aleph_1$, then there is a model of power $\aleph_2$.

Corollary (Shelah)

An $\aleph_1$-categorical sentence ψ in $L_{\omega_1,\omega}$ has a model of power $\aleph_2$.

But what about $L_{\omega_1,\omega}(Q)$?

The Q-quantifier

Let $\mathbf{K}$ be the class of models of $L(Q)$-sentences and $\prec$ denote $L(Q)$-elementary submodel.

Is $(\mathbf{K}, \prec)$ an AEC?

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Is $(\mathbf{K}, \prec)$ an AEC?

1. In $\aleph_0$ interpretation, yes.
2. In $\aleph_1, \aleph_2$, equicardinal NO

Infinitary Logic and Omitting Types

Key Insight

(Chang, Lopez-Escobar) Any sentence of $L_{\kappa^+,\omega}$ can be coded by omitting a set of types in a first order theory.

Recall that it takes a further serious argument to insure that we can choose this theory and set of types to both be satisfied by a given uncountable structure and have a countable model.

Translating to first order

Let ψ_0 be a small $L_{\omega_1,\omega}(Q)$-complete sentence with vocabulary τ in the countable fragment L^* of $L_{\omega_1,\omega}(Q)$. Form τ' by adding predicates for formulas but also add for each formula $(Qx)\phi(x,\mathbf{y})$ a predicate $R_{(Qx)\phi(x,\mathbf{y})}$ and add the axiom

$$(\forall x)[(Qx)\phi(x,\mathbf{y}) \leftrightarrow R_{(Qx)\phi(x,\mathbf{y})}].$$

Let ψ' be the conjunction of ψ_0 with the $L_{\omega_1,\omega}(Q)$-τ'-axioms encoding this expansion. Let $\mathbf{K}_1$ be the class of atomic τ'-models of $T(\psi)$, the first order τ' theory containing all first order consequences of ψ'.

$\prec_{\mathbf{K}}$ for $L_{\kappa,\omega}(Q)$

1. Let $\leq^*$ be the relation on $\mathbf{K}_1$: $M \leq^* N$ if $M \prec_{T'} N$ and for each formula $\phi(x, \mathbf{y})$ and $\mathbf{m} \in M$, if $M \models \neg R_{(Qx)\phi(x,\mathbf{m})}$ then $R_{\phi(x,\mathbf{m})}$ has the same solutions in M and N.
2. Let $\leq^{**}$ be the relation on $\mathbf{K}_1$: $M \leq^{**} N$ if $M \prec_{L'} N$ and for each formula $\phi(x, \mathbf{y})$ and $\mathbf{m} \in M$, $M \models \neg R_{(Qx)\phi(x,\mathbf{m})}$ if and only if $R_{\phi(x,\mathbf{m})}$ has the same solutions in M and N.

AEC???

Abusing notation,
$(\mathbf{K}, \leq^*)$ is an aec in the $\aleph_1$-interpretation but with Löwenheim number $\aleph_1$.

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$(\mathbf{K}_1, \leq^*)$ is an aec with Löwenheim number $\aleph_0$.

AEC???

Abusing notation,
$(\mathbf{K}, \leq^*)$ is an aec in the $\aleph_1$-interpretation but with Löwenheim number $\aleph_1$.

$(\mathbf{K}_1, \leq^*)$ is an aec with Löwenheim number $\aleph_0$.

but
$(\mathbf{K}_1, \leq^{**})$ is NOT an AEC.

An unsatisfactory solution

Shelah proved the existence of a model in $\aleph_2$ for $L_{\omega_1,\omega}(Q)$. But the extension is an <u>ad hoc</u> argument.

What is the difficulty?

The two roles of the union axiom conflict.

$(\mathbf{K}_1, \leq^*)$ is an AEC with Löwenheim number $\aleph_0$.
To get Löwenheim number $\aleph_0$, we allow models of $\mathbf{K}_1$ that are not models of ψ.

Unfortunately,
We may also have gained uncountable models of $\mathbf{K}_1$ that are not models of ψ. Working with $(\mathbf{K}_1, \leq^*)$, one cannot show that many models for $\mathbf{K}_1$ implies many models of ψ.

$(\mathbf{K}_1, \leq^{**})$ solves this problem.
But, $(\mathbf{K}_1, \leq^{**})$, does not satisfy A.3.3.

Smoothing things out

Coppola introduced the notion of a Q-aec which has two notions of submodel. This provides a completely axiomatic proof of this result.

Q-aec

A Q-Abstract Elementary Class is a collection of τ-structures K equipped with a notion of submodel $\prec_{\mathbf{K}}$, a refined notion of submodel to build chains $\prec^{\cup}_{\mathbf{K}}$, $\mathcal{K} = (K, \prec_{\mathbf{K}}, \prec^{\cup}_{\mathbf{K}})$ such that

- **A0** $K, \prec_{\mathbf{K}}, \prec^{\cup}_{\mathbf{K}}$ are closed under isomorphism, i.e.
  a. If $M \in K$ and $M \approx M'$ then $M' \in K$;
  b. If $M \prec_{\mathbf{K}} N$ and $f : N \tilde{\hookrightarrow} N'$, then $f(M) = M' \prec_{\mathbf{K}} N'$;
  c. If $M \prec^{\cup}_{\mathbf{K}} N$ and $f : N \tilde{\hookrightarrow} N'$, then $f(M) = M' \prec^{\cup}_{\mathbf{K}} N'$;
- **A1** $\prec_{\mathbf{K}}$ is a partial order, and $\prec^{\cup}_{\mathbf{K}}$ is transitive on K;
- **A2** $\prec_{\mathbf{K}}$ refines the notion of substructure, $\prec^{\cup}_{\mathbf{K}}$ refines $\prec_{\mathbf{K}}$;

- A3 If $M_0 \subset M_1$ and $M_0, M_1 \prec_{\mathbf{K}} N$ then $M_0 \prec_{\mathbf{K}} M_1$ (coherence for $\prec_{\mathbf{K}}$);
- A4 There is a Löwenheim -Skolem number, $LS(\mathbf{K})$ such that for all $N \in \mathcal{K}$ and $A \subseteq N$ there is $M \prec_{\mathbf{K}}^{\cup} N$ containing A of size at most $|A| + \mathrm{LS}(\mathbf{K})$;
- A5 If $(M_i : i < \lambda)$ is $\prec_{\mathbf{K}}^{\cup}$-increasing, continuous, then
  a. $M = \bigcup_{i<\lambda} M_i \in \mathcal{K}$;
  b. $M_i \prec_{\mathbf{K}}^{\cup} \bigcup_{i<\lambda} M_i$, for each $i < \lambda$;
  c. If $M_i \prec_{\mathbf{K}} N$ for each $i < \lambda$, then $\bigcup_{i<\lambda} M_i \prec_{\mathbf{K}} N$.
- $\mathcal{K}$ satisfies Assumptions I,II,III (below).

Model Homogeneity

Definition

M is μ-model homogenous if for every $N \prec_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \prec_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.

To emphasize, this differs from the homogenous context because the N must be in $\mathbf{K}$. It is easy to show:

Monster Model

Lemma

(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then
$M_1 \approx M_2$.

Theorem

If $\mathbf{K}$ has the amalgamation property and $\mu*^{<\mu*} = \mu*$ and $\mu* \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu*$ which is $\mu*$-model homogeneous.

GALOIS TYPES: General Form

Define:

$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

GALOIS TYPES: General Form

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$$(M, a, N) \cong (M, a', N')$$

if there exists N'' and strong embeddings f, f' taking N, N' into N'' which agree on M and with

$$f(a) = f'(a').$$

'The Galois type of a over M in N' is the same as 'the Galois type of a' over M in N''
if (M, a, N) and (M, a', N') are in the same class of the equivalence relation generated by $\cong$.

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathcal{M}$ and $a \in \mathcal{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathcal{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathcal{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- **locality** Any increasing chain of types has at most one upper bound;
- **tameness** two distinct types differ on a finite set;
- **compactness** an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Galois and Syntactic Types

Work in $(\mathbf{K}^{ab}, \leq)$.

Lemma

Suppose that G_1 is a subgroup of both G_2 and G_3, $a \in G_2 - G_1$, and $b \in G_3 - G_1$. the following are equivalent:

1. $\text{ga-tp}(a, G_1, G_2) = \text{ga-tp}(b, G_1, G_3)$;
2. *There is a group isomorphism from $\langle G_1, a \rangle_{G_3}$ onto $\langle G_1, b \rangle_{G_3}$ that fixes G_1 pointwise;*
3. $\mathrm{tp}_{qf}(a/G_1) = \mathrm{tp}_{qf}(b/G_1)$.

But this equivalence is far from true of all AEC's.
Even when there is a well defined notion of syntactic type it can be different from the Galois type (even over models).

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\mathbf{K}$ is (χ, μ)-tame if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \upharpoonright N_0 = q \upharpoonright N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ^+-categorical and $(\lambda, < \infty)$-tame then $\mathbf{K}$ is categorical in all $\theta \geq \lambda^+$.

Theorem (Lessmann)

If $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\mathbf{K}$ is categorical in all uncountable cardinals

Two Examples that are not tame

1. 'Hiding the zero'
   For each $k < \omega$ a class which is $(\aleph_0, \aleph_{k-3})$-tame but not $(\aleph_{k-3}, \aleph_{k-2})$-tame. Baldwin-Kolesnikov (building on Hart-Shelah)
2. Coding EXT
   A class that is not $(\aleph_0, \aleph_1)$-tame.
   A class that is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame. (Baldwin-Shelah)

Very Complicated Example

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.

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2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
4. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
5. ϕ_k is not $\aleph_{k-2}$-Galois stable;
6. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;
7. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Locality and Tameness

Definition

K has (κ, λ)-local galois types if for every continuous increasing chain $M = \bigcup_{i<\kappa} M_i$ of members of **K** with $|M| = \lambda$ and for any $p, q \in \mathcal{S}(M)$: if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.

Lemma

If $\lambda \geq \kappa$ and $\mathrm{cf}(\kappa) > \chi$, then (χ, λ)-tame implies (κ, λ)-local. If particular, $(\aleph_0, \aleph_1)$-tame implies $(\aleph_1, \aleph_1)$-local.

Key Example

Shelah constructed (page 228 of Eklof-Mekler, first edition) of a group with the following properties.

Fact

There is an $\aleph_1$-free group G of cardinality $\aleph_1$ which is not Whitehead.
Moreover, there is a countable subgroup R of G such that G/R is p-divisible for each prime p.

THE AEC EXAMPLE

Let $\mathbf{K}$ be the class of structures $M = \langle G, Z, I, H \rangle$, where each of the listed sets is the solution set of one of the unary predicates $(\mathbf{G}, \mathbf{Z}, \mathbf{I}, \mathbf{H})$.
G is a torsion-free Abelian Group. Z is a copy of $(Z, +)$. I is an index set and H is a family of infinite groups.

Each model in **K** consists of

1. a torsion free group G,
2. a copy of $\mathcal{Z}$
3. and a family of extensions of Z by G.

Each of those extensions is coded by single element of the model so the Galois type of a point of this kind represents a specific extension. The projection and embedding maps from the short exact sequence are also there.

$M_0 \prec_{\mathbf{K}} M_1$ if

M_0 is a substructure of M,

but $\mathbf{Z}^{M_0} = \mathbf{Z}^{M}$

and $\mathbf{G}^{M_0}$ is a pure subgroup of $\mathbf{G}^{M_1}$.

NOT LOCAL

Lemma

$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_1, \aleph_1)$-local. That is, there is an $M^0 \in \mathbf{K}$ of cardinality $\aleph_1$ and a continuous increasing chain of models M_i^0 for $i < \aleph_1$ and two distinct types $p, q \in \mathcal{S}(M^0)$ with $p \restriction M_i^0 = q \restriction M_i$ for each i.

Let G be an Abelian group of cardinality $\aleph_1$ which is $\aleph_1$-free but not a Whitehead group. There is an H such that,

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow G \rightarrow 0$$

is exact but does not split.

WHY?

Let $M_0 = \langle G, \mathcal{Z}, a, G \oplus Z \rangle$

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Let $p = \mathrm{tp}(t_1/M^0, M^1)$ and $q = \mathrm{tp}(t_2/M^0, M^2)$. Since the exact sequence for $\mathbf{H}^{M^2}$ splits and that for $\mathbf{H}^{M^1}$ does not, $p \neq q$.

NOT $\aleph_1$-LOCAL

But for any countable $M_0' \prec_{\mathbf{K}} M_0$, $p \restriction M_0' = q \restriction M_0'$, as

$$0 \to Z \to H_i' \to G' \to 0. \tag{1}$$

splits.
$G' = \mathbf{G}(M_0')$, $H' = \pi^{-1}(t_i, G')$.

NOT $\aleph_0$-TAME

It is easy to see that if $(\mathbf{K}, \prec_{\mathbf{K}})$ is $(\aleph_0, \aleph_0)$-tame then it is $(\aleph_1, \aleph_1)$-local, so
$(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\aleph_0, \aleph_0)$-tame.
So in fact, $(\mathbf{K}, \prec_{\mathbf{K}})$ is not $(\chi, \aleph_0)$-tame for any χ.

Question

Could this example be formulated more naturally as
$\{Ext(G, Z) : G \text{ is torsion-free } \}$
(with projection and injection maps?

Incompactness

Theorem

Assume $2^{\aleph_0} = \aleph_1$*, and* $\diamondsuit_{\aleph_1}, \diamondsuit_{S_1^2}$ *where*

$$S_1^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}.$$

Then, the last example fails either $(\aleph_1, \aleph_1)$ *or* $(\aleph_2, \aleph_2)$*-compactness.*

BECOMING TAME ??

Grossberg and Van Dieren asked for $(\mathbf{K}, \prec_{\mathbf{K}})$, and $\mu_1 < \mu_2$ so that $(\mathbf{K}, \prec_{\mathbf{K}})$ is not (μ_1, ∞)-tame but is (μ_2, ∞)-tame.

Admits intersection

We say the AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ admits intersections if for every $X \subseteq M \in \mathbf{K}$, there is a minimal closure of X in M. That is, $M \restriction \bigcap\{N : X \subseteq N \prec_{\mathbf{K}} M\} = \mathrm{cl}_M(X) \prec_{\mathbf{K}} M$.

Tameness gained

Theorem

There is an AEC in a countable language that admits intersections with Löwenheim-Skolem number $\aleph_0$ which is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame.

Proof Sketch: Repeat the previous example but instead of letting the quotient be any torsion free group

1. insist that the quotient is an $\aleph_1$-free group;
2. add a predicate R for the group R G/R is divisible by every prime p where G is Shelah's example of a non-Whitehead group.

This forces $|G| \leq 2^{\aleph_0}$ and then we get $(2^{\aleph_0}, \infty)$-tame. But $\aleph_1$-free groups fail amalgamation ??

Lemma

For any AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ *which admits intersections there is an associated AEC* $(\mathbf{K}', \prec_{\mathbf{K}})$ *with the same (non) locality properties that has the amalgamation property.*

Theorem

There is an AEC with the amalgamation property in a countable language with Löwenheim-Skolem number $\aleph_0$ *which is not* $(\aleph_0, \aleph_1)$*-tame but is* $(2^{\aleph_0}, \infty)$*-tame.*

Summary

The true significance of first order stability theory became clear when one found a wide variety of mathematically interesting theories at various places in the stability hierarchy.

Zilber's work and $({}^{\perp}N, \prec_N)$ suggest we may find a similar future for AEC.

A new Lindstrom Theorem

Theorem (Hart)

First order continuous logic is the maximal logic for continuous structures such that satisfies:

1. closure under ultraproducts
2. the DLS property
3. closure under unions of elementary chains of substructures

Lindstrom Theorem questions

Question 1

Is there a Lindstrom theorem for $L_{\omega_1,\omega}$?

Question 2

Suppose an AEC is categorical in all uncountable models? Must it be $L_{\omega_1,\omega}$ definable?

Main questions

Do any of the generalized quantifiers you study, give examples of AEC?
What are some examples of naturally occurring classes of mathematical structures which are only axiomatizable in some of these extended logics?

References

Much is on the web at www.math.uic.edu/jbaldwin including:

1. Categoricity: a 200 page monograph introducing AEC,
2. Some examples of Non-locality (with Shelah)
3. Categoricity, amalgamation and Tameness (with Kolesnikov)
4. And see Grossberg, VanDieren, Shelah

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