Finite and Infinite Model Theory
A Historical Perspective

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MODEL THEORY BEFORE 1960: FUNDAMENTALS

Theorem[Löwenheim-Skolem]. If a first order theory has an infinite model, it has a model in each infinite cardinality.

Theorem[Compactness]. If every finite subset of a collection Σ of sentences has a model then Σ has a model.

Theorem[Completeness]. The collection of valid first order sentences is recursively enumerable.

MODEL THEORY BEFORE 1960: APPLICATIONS

Theorem[Łoś-Tarski]. A class of models is closed under substructure iff it can be axiomatized by universal sentences.

Theorem[Lyndon]. A class of models is closed under homomorphism iff it can be axiomatized by positive sentences.

Theorem[Robinson]. Every submodel of a model of T is an elementary submodel iff every formula is equivalent in T to an existential formula.

VOCABULARY

- A signature L is a collection of relation and function symbols.
- A structure for that signature (L-structure) is a set with an interpretation for each of those symbols.
- The first order language associated with L is the least set of formulas containing the atomic L-formulas and closed under the Boolean operations and quantification over individuals.
- Formulas in which each variable is bound by a quantifier are called L-sentences.
- An L-structure M is a model of an L-sentence if the sentence is true in M.
- A theory T is a collection of L-sentences.
- If all models of T satisfy exactly the same L-sentences (are elementarily equivalent), T is said to be *complete*.

Model Theory in the 60's: Morley-Vaught

The collection of formulas p is a complete type over A if it satisfies one of the following equivalent conditions.

1. p is a maximal consistent set of formulas $\phi(x, \mathbf{a})$ with parameters from A.
2. p is a member of the Stone Space of the Lindenbaum algebra of A.
3. The solutions of p are an orbit of the group of automorphisms of the monster model which fix A.

For every set A contained in a model M and every n, $S^n(A)$ denotes the collection of types over A (the *nth Stone space* of A).

$S(A)$ denotes the set of such p.

Model Theory in the 60's: Morley-Vaught

A model M is κ-*saturated* if for every A contained in M with $|A| \leq \kappa$, if $p \in S(A)$ then p is realized in M.

Homogeneous-Universal Structures

The category of models and elementary embeddings.

The 'monster model' is a large saturated model.

UNDERSTANDING TYPES

Let $\mathcal{A} = (A, <, a_n)$ be the rational numbers in the open interval $(-1, 1)$ and let a_n denote $-1/n$.

Do 0 and .1 realize different types over $\{a_n : n < \omega\}$?

Are $\mathcal{A}$ and $\mathcal{A}^-$ (the result of omitting 0) isomorphic?

Model Theory in the 60's: Vaught

Theorem. There is no complete theory with exactly two countable models.

Theorem. $S(\emptyset)$ is countable if and only if there is a countable saturated model.

Theorem. [Ryll-Nardjewski] For each n, $S^n(\emptyset)$ is finite if and only if T is $\aleph_0$-categorical.

CATEGORICAL THEORIES

T is κ-categorical if all models of T with cardinality κ are isomorphic.

Example 1 The theory of an infinite set is totally categorical.

Example 2 The theory of the integers under successor is categorical in all uncountable powers.

Example 3 Let T be the theory of a vector space V over a field F. Then (V, acl) is a pregeometry. If F is finite then T is $\aleph_0$-categorical.

Model Theory in the 60's: Morley

Theorem[Morley]. A complete countable theory T is categorical in one uncountable cardinal if and only if it is categorical in all uncountable cardinalities.

A theory T is ω-stable if for every A,

$$|S(A)| \leq |A|.$$

Morley showed every $\aleph_1$-categorical theory is ω-stable.

GEOMETRIES

0.1 Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

Geometries are classified as: trivial, locally modular, non-locally modular.

STRONGLY MINIMAL THEORIES

Lemma. A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between acl-bases for models of T extends to an isomorphism of the models

Theorem. A strongly minimal theory is categorical in any uncountable cardinality.

$\aleph_1$-CATEGORICITY

Definition. The formula ϕ is strongly minimal if in every model of T, ϕ has infinitely many solutions but for each formula ψ either $\phi \wedge \psi$ or $\phi \wedge \neg\psi$ has only finitely many solutions.

Definition. The model M of a theory T is *prime* over A if every elementary embedding of A into a model N of T extends to an embedding of M into N.

Theorem. Every model M of an $\aleph_1$-categorical theory T is prime, and even minimal, over $\phi(M)$ for a fixed strongly minimal formula ϕ.

Theorem [Baldwin-Lachlan]. An uncountably categorical countable theory has either one or $\aleph_0$ countable models.

Model Theory in the 70's: Three Guiding Problems

The *spectrum function* of T, $n(T, \kappa)$, is the number of models of T with cardinality κ.

Morley's conjecture: With the obvious exception of $\aleph_1$ but not $\aleph_0$ categorical theories, $n(T, \kappa)$ is an increasing function.

Morley's question: Are there finitely axiomatizable categorical theories?

Vaught's conjecture: A first order theory with uncountably many countable models has $2^{\aleph_0}$ countable models.

Model Theory in the 70's: Morley's Conjecture

Shelah proved Morley's conjecture by exhibiting all possible spectrum functions. He thus reformulated it as the classification problem:

Characterize those theories which admit good structure theories.

Classify theories.

Classify models.

Model Theory in the 70's: Stability

A theory T is κ-stable if for every A with $|A| \leq \kappa$, $|S(A)| \leq \kappa$.

T is *ω-stable* if it is stable in ω (and thus in every cardinality).

T is *superstable* if it is stable in all cardinals at least $2^{\aleph_0}$.

T is *stable* if it is stable in some cardinal.

In all other cases T is *unstable*.

INSTABILITY: FIRST ORDER CASE

Theorem T is unstable if any of the following hold

1. For every κ, there is an A with $|A| = \kappa$ and $|S(A)| > \kappa$.
2. There is a formula ϕ and an A with $|A| = \aleph_0$ and $|S_\phi(A)| > \aleph_0$.
3. Some formula $\phi(\overline{x}, \overline{y})$ has the strict order property.
4. Some formula $\phi(\overline{x}, \overline{y})$ has the independence property.

NOTE: These are local properties.

UNSTABLE FORMULAS

The formula $\phi(\overline{x}, \overline{y})$ has the order property if there exist $\overline{a}_i, \overline{b}_i : i < \omega$ such that

$$\phi(\overline{a}_i, \overline{b}_j) \text{ iff } i < j.$$

The formula $\phi(\overline{x}, \overline{y})$ has the strict order property.

there exist $\overline{b}_i : i < \omega$ such that

$$(\forall \overline{x}) \phi(\overline{x}, \overline{b}_i) \to \phi(\overline{x}, \overline{b}_j) \text{ iff } i < j.$$

The formula $\phi(\overline{x}, \overline{y})$ has the independence property.

there exist $\overline{b}_i : i < \omega$, $\overline{a}_\sigma : \sigma \in 2^\omega$ such that

$$\phi(\overline{a}_\sigma, \overline{b}_j) \text{ iff } \sigma(j) = 1.$$

MODEL THEORY IN THE 70's: THE MAIN GAP

Theorem[Shelah]. If T is not superstable

$$n(T, \kappa) = 2^{\kappa}.$$

Theorem[Shelah]. Either, there is a $\delta(T)$ such that

for sufficiently large α, $n(T, \aleph_\alpha) \leq \beth(\delta, \alpha)$
or for sufficiently large α, $n(T, \alpha) = 2^{\aleph_\alpha}$

Moreover, when there are few models, each model is decomposed into a tree of small models.

Theorem[Shelah]. An ω-stable first order theory with uncountably many countable models has $2^{\aleph_0}$ countable models.

MODEL THEORY IN THE 70's: AXIOMS OF INFINITY

Sentences with only infinite models:

1. infinite linear order
2. $f(x)$ is an injective function; exactly one element does not have a predecessor.
3. $t(x, y)$ is a pairing function

A *complete* axiom of infinity is a first order sentence ϕ such that

- the consequences of ϕ are a complete first order theory
- which has an infinite model.

It is easy to extend 1) linear order to a complete sentence; the others are more difficult.

Model Theory in the 70's: Categorical Axioms of Infinity

Theorem [Zilber, Cherlin-Harrington-Lachlan]. No first order sentence is categorical in all infinite cardinalities.

Theorem [Peretyatkin]. There is an $\aleph_1$ categorical first order sentence.

Peretyatkin was motivated by trying to capture a tiling problem but his example really seems to capture 'pairing'.

Open Question. Is there a finitely axiomatizable strongly minimal set?

MODEL THEORY IN THE 80's: O-MINIMALITY

A theory is O-minimal if the universe is linearly ordered and in every model every definable subset is a union of intervals.

Examples:
$(R, +, \cdot)$ (Tarski)

$(R, +, \cdot, e^x)$ (Wilkie)

Many other expansions of the reals: (Van Den Dries, Miller, Marker, MacIntyre ...)
There is an extremely highly developed theory of O-minimal structures.

MODEL THEORY IN THE 80's/90's: ALGEBRA

Work of Pillay, Van Den Dries, Marker, Chatzidakis, etc. and most of all Hrushovski produces increasingly strong connections and applications between model theory and such areas of traditional geometry as Real or Complex Algebraic Geometry and Diophantine Geometry.

Most of this is not yet relevant to our theme.

THREE THEMES OF INFINITE MODEL THEORY

Study:

- definable subsets of a single structure
- models of a single theory
- theories with a common property

KEY IDEAS OF STABILITY THEORY

Study:

- Classify theories by counting types.
- Count models of stable theories by understanding dimension relations.
- Instability is caused by combinatorial properties of formulas.

FINITE MODEL THEORY

Theorem [Trahtenbrot]. The collection of sentences valid on finite models is not recursively enumerable.

Theorem. Compactness fails on the class of finite models.

Corollary. The preservation theorems fail.

Examples are due to Tait, Gurevich, Shelah, ...

First order logic is too strong - categorical on finite models.

First order logic is too weak - doesn't support recursion or counting.

DESCRIPTIVE COMPLEXITY

Fact: If $\mathbf{K}$ is a class of finite structures that is defined by a first order sentence then $\mathbf{K}$ is decided in polynomial time.

Theorem. (Fagin)
On finite structures $\mathbf{\Sigma^1_1} = \mathbf{NP}$.

Definition: **FP** denotes the logic obtained by adding to first order logic formulas expressing the fixed point of inductive definition over a formula $\phi(\overline{x}, X)$ where X occurs positively.

Theorem. (Immerman/Vardi)
On *ordered structures* $\mathbf{FP} = \mathbf{PTIME}$.

MAJOR PROBLEM: Is there a logic which captures polynomial time on arbitrary finite structures?

THEMES FROM THE 70'S: GENERALIZED QUANTIFIERS

Generalized quantifiers:

There exists uncountably many x such that $\phi(x)$

There exists an uncountable set which is homogeneous for $\phi(x, y)$

A general theory of such quantifiers is due to Lindstrom. There were many results on the model theory of such logics and a general theory of interpolation.

THEMES FROM THE 70'S: LINDSTROM'S THEOREM

Theorem. [Lindstrom] No proper extension $\mathcal{L}_\infty$ of first order logic (FO) satisfies both the compactness theorem and the Lowenheim-Skolem theorem.

Proof idea:

1. There is an $\mathcal{L}$-sentence ψ and for each n structures A_n and B_n which are equivalent for the first n FO sentences but $A_n \models \psi$ and $B_n \models \neg\psi$.
2. Construct A and B such that $A \models \psi$ and $B \models \neg\psi$ but A and B are the same size (and the equality isomorphism) extends to an $L \cup S$ isomorphism.

Step 1 is a diagonal argument. For the second step note that a first order description of winning Ehrenfreuht-Fraisse games leads via compactness and Lowenheim-Skolem to a back and forth.

THEMES FROM THE 80'S/90's: CLASSIFICATION IN INFINITARY LOGIC

Shelah, Makkai, Grossberg, Hart, Hyttinen, Baldwin, Lessmann develop a model theory for classes defined infinitary classes.

Characteristics:

1. dependence on set theory
2. compactness fails
3. importance of combinatorics
4. abstract formulation

INFINITARY LOGIC ON FINITE MODELS ???

FO^k denotes first order logic with only k-variables *free or bound.*

$L^k_{\omega_1,\omega}$ denotes $L_{\omega_1,\omega}$ with only k-variables *free or bound.*

$L^\omega_{\omega_1,\omega} = \bigcup_k L^k_{\omega_1,\omega}$.

Theorem: (Barwise) **FP** is contained in $L^\omega_{\omega_1,\omega}$.

For finite A, B,
if $A \equiv_{FO^k} B$ then $A \equiv_{L^k_{\omega_1,\omega}} B$

It is more convenient to analyze $L^\omega_{\omega_1,\omega}$ by means of 'pebble games' than to work directly with fixed-point logics.

THE SEARCH FOR A LOGIC FOR P

Considerable work has attempted to find a logic which captures polynomial time, e.g. by adding Lindstrom quantifiers.

Theorem: (Cai, Furer, Immerman) No extension of $L^{\omega}_{\omega_1,\omega}$ by a finite number of Lindstrom Quantifiers captures polynomial time.

FINITE VARIABLE LOGIC

Definition.

1. For $m \leq k$ and A finite, $\overline{a} \in A^m$, an m-type in FO^k, p, is the collection of FO^k formulas satisfied by $\overline{a} \in A$.
2. $S_k^k(A)$ is the collection of k-ary FO^k-types realized in A.
3. The k-size of A is $|S_k^k(A)|$.

We have Scott sentences for finite models.

Lemma. For any finite structure A and any finite k there is a sentence of FO^k which characterizes A up to FO^k-equivalence. Moreover, the quantifier rank of the Scott sentence is less than $|S_k^k(A)|$.

LOWENHEIM SKOLEM FOR FINITE VARIABLE LOGIC

Lemma. (Dawar) For finite A there are functions

1. (Lowenheim) $L_k(n)$ such that $|S_k^k(A)| \leq n$ implies there is a model B with $B \equiv_k A$ and $|B| < L_k(n)$.
2. (Hanf) and $H_k(n)$ such that $|S_k^k(A)| \leq n$ and there is a model B with $B \equiv_k A$ and $|B| > H_k(n)$ implies there are arbitrarily large finite models k-equivalent to A.

Proof sketch: E.g.

$$L_k(n) = \max\{|A| : k - size(A) \leq n$$
$$\text{and A is smallest in its } \equiv_k \text{ class}\}.$$

Problem: Grohe has shown L_k can not be recursive; what about H_k?

STABILITY AND FINITE MODEL THEORY: INTERACTIONS

I. Embedded finite model theory

II. Stability Theory for L^k.

III. L^k axiomatizability of first order theories.

IV. 0-1 laws

STABLE FORMULAS

Lemma. (Baldwin-Dawar) If a complete theory T in FO^k has arbitrarily large finite models then no FO^k-formula has the strict order property.

Lemma. (Shelah) If the k-variable formula ϕ does not have the have the independence property then there is a k'-formula ψ with the strict order property.

Questions: Must k' be greater than k? Suppose a formula does not have the strict order property but is unstable; can you derive a formula with the independence property; how complicated? Need to consider (a la Grossberg/Ensley) the n-order property etc.

Is there a reasonable definition of a k-variable type *with parameters*?