STABILITY AND EMBEDDED FINITE MODELS

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TOPICS

I. Embedded Finite Model Theory

II. Stable L^n theories

III. L^n axiomatizability

QUERIES

Let $\mathbf{U}$ be an infinite set, and let S be a finite relational language.

A *query* on $\mathbf{U}$, Q, is a collection of S-structures with domain a finite subset of $\mathbf{U}$.

A *generic query* on $\mathbf{U}$ is a query that is closed under S-isomorphism.

An *abstract query* Q is a collection of isomorphism types of finite S-structures.

Given an abstract query Q and set $\mathbf{U}$, Q defines a generic query $Q_{\mathbf{U}}$ by considering all finite S-structures embedded in $\mathbf{U}$ whose isomorphism type is in Q.

RELATIONAL DATA BASES

View a data base of employment records as a ternary relation containing triples: name, position, salary.

Each data base is a finite relational structure. But in the back ground we make queries such as,

‘List all employees who make over \$50,000 per year.’

Does this facility increase our expressive power?

EMBEDDED FINITE MODEL THEORY

Let L be a language disjoint from S, and let M be an L-structure with domain $\mathbf{U}$.

For any (finite) S-structure A with domain contained in $\mathbf{U}$, let $M(A)$ denote the unique $L \cup S$-structure that expands M and agrees with A on the interpretation of the predicates in S.

Such an A is what we mean by an *embedded finite model.*

Given a first-order sentence ϕ in $L \cup S$, the *query defined by* ϕ is the set of finite S-structures A with $A \subset \mathbf{U}$ such that $M(A) \models \phi$.

ϕ is generic if the query defined by it is generic.

A $\mathbf{U}$-query Q is *first-order over* M if there is $\phi \in L \cup S$ that defines Q.

An abstract query Q, is first-order over M if there is a first-order $\phi \in L \cup S$ that defines $Q_{\mathbf{U}}$.

GENERIC COLLAPSE TO EQUALITY: VERSION I

Theorem [Hull-Su, Stoulbouskin et al]. Let **U** be infinite. If Q is a first-order definable generic query over M, then Q is a pure first-order query.

Proof idea: Same proof as Lindstrom's Theorem.

i) There is an $L \cup S$ sentence ψ and for each n structures A_n and B_n which are equivalent for the first n L (equality) sentences but $A_n \models \psi$ and $B_n \models \neg\psi$.

ii) Construct A and B such that $A \models \psi$ and $B \models \neg\psi$ but A and B are the same size (and the equality isomorphism) extends to an $L \cup S$ isomorphism.

Step 1: compactness. Step 2: A first order description of winning Ehrenfreuht-Fraisse games leads via compactness and Lowenheim-Skolem to a back and forth.

GENERIC COLLAPSE TO EQUALITY
STABLE CASE

Theorem 1 *Let* $\mathbf{U}$ *be infinite and* M *be a stable* L*-structure with domain* $\mathbf{U}$*. If* Q *is a first-order definable generic query over* M*, then* Q *is a pure first-order query.*

Proof idea:

i) find a set I of indiscernibles so that every permutation of I extends to a back-and-forth on the model M.

ii) Move the argument for the equality case to the set of indiscernibles.

Step 1 is a complicated argument in stability theory.

All the new results here are joint with Michael Benedikt.

UNSTABLE THEORIES

A model M is *unstable* if it admits a parametrically definable linear-order (on k-tuples). That is, there is a $\phi(\vec{x}, \vec{y})$ such that it is consistent with theory of M that there are $\langle \vec{a}_n : n \in \omega \rangle$ such that

$$\phi(\vec{a}_m, \vec{a}_n) \leftrightarrow m < n.$$

A model M has the independence property if it admits a parametrically definable family of independent sets. That is, there is a $\phi(\vec{x}, \vec{y})$ such that it is consistent with theory of M that there are $\langle \vec{a}_n : n \in \omega \rangle$ and every Boolean combination of the sets $\phi(\vec{x}; \vec{a}_n)$ is consistent.

If M has the independence property then M is unstable.

ORDER QUERIES

An abstract order query is a collection of linearly ordered $S \cup \{<\}$-structures which is closed under $\{S, <\}$-isomorhism.

A *local map* is a partial function on a set $\mathbf{U}$.

A query Q on $\langle \mathbf{U}, < \rangle$ is *locally order generic* if it is preserved by local order-preserving maps on $\langle \mathbf{U}, < \rangle$.

For any infinite linearly ordered structure O and abstract query Q, the query Q_O induced on O is locally order generic. Conversely, any locally order generic query naturally determines an abstract order query.

Let L be a language disjoint from S and let M be an $L \cup \{<\}$-structure with domain $\mathbf{U}$.

An abstract order query Q is *first-order over* M if there is a $\phi \in L \cup \{S, <\}$ that defines Q.

An abstract order query Q is *order definable* if it is defined over the structure $\langle \mathbf{U}, < \rangle$ by an $\{S, <\}$-formula ψ.

GENERIC COLLAPSE TO ORDER

Theorem 2 *Let* $(\mathbf{U}, <)$ *be an infinite linear order and* M *an* $L \cup \{<\}$*-structure that expands* $(\mathbf{U}, <)$*, which does not have the independence property. If* Q *is a first-order definable locally order generic query over* M*, then* Q *is an order-definable query.*

Theorem 3 *Let* $\mathbf{U}$ *be infinite and* M *be an* L*-structure with domain* $\mathbf{U}$ *that does not have the independence property. If* Q *is a first-order definable generic query over* M*, then* Q *is definable over any dense linear order without endpoints.*

A model satisfying the second sentence of Theorem 3 is said to have *generic collapse to order*.

APPLICATIONS

These kinds of collapse results were earlier done for the O-minimal theories by Benedikt et al.

Corollary 4 *Let* $\mathbf{U}$ *be infinite and* M *be an* L*-structure with domain* $\mathbf{U}$ *that does not have the independence property. If* Q *is a first-order definable generic query over* M*, then the abstract ordered query defined by* Q *(i.e the class of linearly ordered expansions of isomorphism types in* Q*) is definable by a formula in* $\{S, <\}$*.*

Corollary 5 *Let* S *contain a single binary relation symbol. If* M *does not have the independence property, then the following queries are not first-order definable over* M*: parity, transitive or deterministic transitive closure of a graph; maximal matching in a bipartite graph; Eulerian cycle.*

FROM INFINITE TO FINITE STABILITY IN L^n

See the extended discussion in the talk: Infinite and Finite Model Theory in April 99 at the Workshop in Logic and Computer Science at Philadelphia. The slides from that talk are on the same web page as this. An expanded version with references is with my mathematical preprints.

FROM FINITE TO INFINITE L^n-AXIOMATIZABILITY

Fact. An $\aleph_0$-categorical first order theory is L^n axiomatizable (for some n) if and only if it is finitely axiomatizable.

Proof: There is a first order Scott sentence.

Theorem [Hedman].

i) Eliminating functions for relations does not affect finite variable axiomatizability.

ii) If T is a strongly minimal theory with trivial algebraic dependence then T is axiomatized by sentences in some L^n in a language with (infinitely many) constants.

TRIVIAL STRONGLY MINIMAL SETS

Let (G, R) be a strongly minimal graph. G is *trivial* if $a \in \mathrm{acl(b, c)}$ implies $a \in \mathrm{acl(b)}$ or $a \in \mathrm{acl(c)}$.

In any structure M define $d_M(a, b) = 1$ if for some relations symbol R and some sequence $\mathbf{e}$ including a, b, $R(\mathbf{e})$ holds. By induction $d_M(a, b)$ is $n + 1$ if b is directly connected to a point c with $d_M(a, c) = n$

a and b are said to be *generic* if they are algebraically independent.

CLASSIFYING STRONGLY MINIMAL SETS

Theorem [Hedman]. In any strongly minimal set, two generic points a, b satisfy one of the following.

Type I. $d_M(a, b) = 1$

Type II. $d_M(a, b) = 2$

Type III. $d_M(a, b) = \infty$

Theorem [Hedman]. Every trivial strongly minimal set is bidefinable with one of type III.

INVOLVED GROUPS

Hrushovski defines a notion of a finite group G being *involved* in T. Roughly, G occurs as a Galois group of B/A for $A \subset B$ contained in a model of T. If A contains the algebraic closure of the empty set G is *strongly involved*.

Theorem [Hrushovski]. If an $\aleph_1$-categorical T is finitely axiomatizable only finitely many simple groups are involved in T.

Theorem [Hedman]. If only finitely many simple groups are involved in a trivial strongly minimal theory then T is axiomatized with finitely many variables.

k-VARIABLE AXIOMATIZABILITY

Theorem [Hedman]. If T is a type III trivial strongly minimal theory only finitely many simple groups are strongly involved in T.

Theorem [Hedman]. If T is a trivial strongly minimal theory then T is axiomatized by finitely many variables over constants.

THEOREMS AND CONJECTURES

Theorem [Hrushovski]. If a strongly minimal set is finitely axiomatized then the geometry is locally modular.

Conjecture [A. Robinson]. There is no k such that the complex field is L^k axiomatizable.

Conjecture [Hedman]. If T is an L^k-axiomatized strongly minimal theory then the geometry is locally modular.