The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 1

Let $G_{\mathbf{Q}} =$ Gal$(\bar{\mathbf{Q}}/\mathbf{Q})$ and p be a prime.

One knows if $F(q) = \sum_{n\geq 1} a_n q^n$ is the q-expansion of a weight k normalized cuspidal eigenform of level N and character χ, $E = \mathbf{Q}_p(\{a_n\})$ is a finite extention of $\mathbf{Q}_p$ and an odd, irreducible repesentation $\rho: G_{\mathbf{Q}} \to \mathbf{Gl}_2(E)$ unramified outide of Np such that if $\ell \not| Np$

$$\mathrm{Tr}\,(\;\mathrm{Frob}_\ell) = a_\ell \;\text{ and }\; \det(\;\mathrm{Frob}_\ell) = \chi(\ell)\ell^{k-1}.$$

One also knows the restriction of ρ to a decomposition group at p is "potentially semi-stable."

Example. On $X_0(49)$ there is a unique normalized weight 2 cusp form $F(q)$, where

$$a_2 = 1, a_{11} = 4, a_{23} = 8, a_{29} = 2, a_{37} = -6,$$

$$a_3 = a_5 = a_{13} = a_{17} = a_{19} = a_{31} = 0.$$

$$\sum_{n\geq 1} a_n n^{-s} = \prod_{\ell\neq 7}(1 - a_\ell \ell^{-s} + \ell^{-2s})^{-1}(1 + 7^{-s})^{-1}.$$

In 1993, J.M. Fontaine and B. Mazur conjectured [F-M],

Conjecture. *Suppose E is a finite extension of $\mathbf{Q}_p$ and $\rho: G_{\mathbf{Q}} \to \mathbf{Gl}_2(E)$ is a continuous odd, irreducible representation ramified at only finitely many primes whose restriction to a decomposition group at p is potentially semi-stable. Then ρ arises from a modular form.*

Mark Kisin has recently proven, using the "eigencurve" this conclusion when ρ arises from an "overconvergent form of finite slope."

Topics of course

Serre's theory of p-adic Banach spaces [S], [C2] and [B]. Overconveregent forms and the U-Operator [K], [C2]. The Canonical subgroup and the U-operator [K], [B2]. Pseudo-representations attached to overconvergent Forms, [H]. The Eigencurve, [C-M]. Fontaine's theory, [F] (see also www.math.berkeley.edu/ coleman/fontaine.html). The Fontaine-Mazur conjecture, [F-M]. Kisin's Theorem.

p-adic Banach spaces

A **Banach Algebra** is a commutative ring A with a unit element, complete and separated with respect to a non-trivial ultrametric norm $|\ |$. I.e., $|1| = 1$,

$$|a+b| \leq \max |a|, |b|, \qquad |ab| \leq |a||b|,$$

for a and $b \in A$, and moreover, $|a| = 0$ if and only if $a = 0$. A **Banach module** over A is an ultrametrically normed complete module E over A, such that $|ae| \leq |a||e|$ if $a \in A$ and $e \in E$.

An **orthonormal basis** for a Banach module E over A is a set $\{e_i : i \in I\}$ of elements of E, for some index set I, such that every element m in E can be written uniquely in the form $\sum_{i \in I} a_i e_i$ with $a_i \in A$ such that $\lim_{i\to\infty} |a_i| = 0$ and

$$|m| = Sup\{|a_i| : i \in I\}.$$

We say E is **orthonormizable** if it has an orthonormal basis.

Examples. Suppose $A = \mathbf{Q}_p$ and M is the ring of analytic functions on the unit disk.

An A-homomorphism $h: M \to N$ between two Banach A-modules is said to be **completely continuous** or **compact** if there exists a sequence of A-homomorphisms $h_j: M \to N$ of "finite rank" such that

$$\lim_{j\to\infty} (\sup_{|m|\leq 1} |(h-h_j)(m)|) = 0.$$

It turns out that if $M = N$, has an orthonormal basis and A is "nice," then h has a characteristic series (Fredholm determinant).

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Lecture 2

www.math.berkeley.edu/ coleman/Courses/Sp02/ecfm.html

Quick introduction to rigid analysis

Let K be a complete local field with absolute value $|\ |$. By $K\langle X_1, \ldots, X_n\rangle$ I mean the ring

$$\mathbf{A}_n =: \sum_{I \geq 0} a_I X^I \quad \text{where } a_I \in K,\ |a_I| \to 0 \ \text{ as } \sigma(I) \to \infty.$$

This ring is Noetherian and is called the **Tate algebra** of dimension n over K. One can think of it as functions on a polydisk of radius 1. A quotient ring of this ring is called an affinoid algebra.

Example. Consider $\{(x,y)\colon y^2 = x^3 - 1, |x| \leq 1, |y| \leq 1\}$.

If $F(X_1, \ldots, X_n) = \sum_{I \geq 0} a_I X^I$, put $||F|| = \sup_I |a_I|$. If $\alpha\colon \mathbf{A}_n \to A$ is surjective we define

$$||f||_\alpha = \inf\{||g||\colon g \in \mathbf{A}_n, \alpha(g) = f\}.$$

This is a norm on A.One can also set

$$||f||_{sup} = \inf_{n \in \mathbf{N}} ||f^n||_\alpha^{1/n}.$$

This is independent of α. The **power bounded** elements A^0 of A are the elements f such that $\{||f^n||_\alpha\}$ is bounded or equivalently $||f||_{sup} \leq 1$ and the **topological nilpotents** of A A^+ are the elements f such that $||f^n||_\alpha \to 0$ or equivalently $||f||_{sup} < 1$. If A is reduced and A^0/A^+ is an integral domain $||\ ||_{\mathrm{sup}}$ is a norm eq3uivalent to $||\ ||_\alpha$.

Example. Same as above and also $5xy = p$.

Compact operator over affinoid algebras

An A-homomorphism $L: M \to N$ between two Banach A-modules is said to be **compact** if there exists a sequence of A-homomorphisms of finite rank $h_j: M \to N$ such that $h_j \to L$. In good situations $\det(1 - Th_j)$ is defined and $\lim_{j\to\infty} \det(1 - Th_j)$ exists.

Suppose $\{e_i\}_{i\geq 0}$ is an orthonormal basis for M and $\{d_j\}_{j\geq 0}$ is an orthonormal basis for N. Suppose

$$L(e_i) = \sum_j n_{i,j} d_j.$$

Proposition. *Suppose K is a finite extension of $\mathbf{Q}_p$ and A is a reduced affinoid algebra over K. The linear map L is compact if and only if*

$$\lim_{j\to\infty} Sup_{i\geq 0} |n_{i,j}| = 0.$$

Proof. Let π_n be the projection onto the submodule E_n generated by d_j, $j \leq n$ and $L_n = \pi_n \circ L$.

Now suppose L is compact. Then for each $\epsilon > 0$ there exists an A-linear map $L': M \to N$ whose image is contained in a finitely generated submodule P and is such that $|L - L'| < \epsilon$.

We will show $P^0 =: P \cap N^0$ is finitely generated over A^0. Assume this for now. Claim: There exists an $m \geq 0$ such that

$$|\pi_m|_P - \text{ id}_P| < \epsilon.$$

It follows that

$$|L - \pi_m \circ L'| < \epsilon.$$

This implies $|n_{i,j}| < \epsilon$ for $j \notin T$ which concludes the proof. ∎

If $M = N$,

$$\det(1 - TL) = \lim_{j \to \infty} \det(1 - T(\pi_n \circ L|_{M_n})).$$

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Lecture 3

J. Tate: Rigid analytic spaces, Inv. Math. 12 (1971) 257-289.

Compact Operators

Let $L: M \to N$ be a continuous linear map between orthonormizable Banach modules over A. Suppose $\{e_i\}_{i\geq 0}$ is an orthonormal basis for M and $\{d_j\}_{j\geq 0}$ is an orthonormal basis for N and

$$L(e_i) = \sum_j n_{i,j} d_j.$$

Proposition. *Suppose K is a finite extension of $\mathbf{Q}_p$ and A is a reduced affinoid algebra over K. The linear map L is compact if and only if*

$$\lim_{j\to\infty} Sup_{i\geq 0}|n_{i,j}| = 0.$$

Proof. Let π_n be the projection onto the submodule E_n generated by d_j, $j \leq n$ and $L_n = \pi_n \circ L$.

Suppose L is compact. Then for each $\epsilon > 0$ there exists an A-linear map $L': M \to N$ whose image is contained in a finitely generated submodule P and is such that $|L - L'| < \epsilon$.

Claim: $P^0 =: \{n \in P: ||n|| \leq 1\}$ is finitely generated over A^0.

Indeed, let $n_i = \sum_i b_{ij} d_j$ $1 \leq i \leq k$ generate P. Let

$$U = \{(a_1, \ldots, a_k) \in A^k: \sum_{i=1}^k a_i n_i = 0\}.$$

Since A is Noetherian, there exists $r \geq 0$ such that $U = \mathrm{Ker}\, F_r$, where

$$F_t(a_1, \ldots, a_k) = \pi_t\big(\sum_{i=1}^k a_i n_i\big).$$

Thus, if $t \geq r$

$$0 \to U \to A^k \xrightarrow{F_t} \pi_t N \cong A^t$$

is exact. Let $B_t = F_t^{-1}\big((A^0)^t\big)$ so that in particular $(\bigcap_{t\geq 0} B_t)/U \cong P^0$. ▮

End of proof. There exists an $m \geq 0$ such that $|\pi_m|_P - \mathrm{id}_P| < \epsilon$.

It follows that

$$|L - \pi_m \circ L'| < \epsilon.$$

This implies $|n_{i,j}| < \epsilon$ for $j \notin T$ which concludes the proof. ▮

Characteristic series

Suppose π is a uniformizing parameter of K and $M = N = E$.

Theorem. *If L is a compact operator on E, then*

$$\lim_{m \to infty} \det(1 - T(\pi_m \circ L)|_{E_m})$$

exists.

We will denote it by $P_L(T)$.

Proof. First we can assume $|L| \leq 1$. Next we know that given $k \geq 0$ there exist $m_k \geq 0$ such that

$$L(e) \equiv \pi_{m_k} \circ L(e) \bmod \pi^k.$$

Theorem. *If L has norm at most $|a|$ where $a \in A$ then $P_L(T)$ is an element of $A^0[[aT]]$ and is entire in T. Also, $P_L(T)$ is characterized by:*

(i) If $\{L_n\}_{n\geq 0}$ is a sequence of completely continuous operators on E, and $L_n \to L$ then $P_{L_n} \to P_L$ coefficientwise.

(ii) If the image of L in E is contained in an orthonormizable direct factor F of finite rank over A of E such that the projection from E onto F has norm at most 1 then

$$P_L(T) = \det(1 - TL|F).$$

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Lecture 4

The Fredholm Deteminant

Theorem. *Suppose L is a compact operator on a ON Banach module E over A. If L has norm at most $|a|$ where $a \in A$, then $P_L(T)$ is an element of $A^0[[aT]]$ and is entire in T. Also, $P_L(T)$ is characterized by:*

(i) If $\{L_n\}_{n\geq 0}$ is a sequence of compact operators on E, and $L_n \to L$ then $P_{L_n} \to P_L$ coefficientwise.

(ii) If the image of L in E is contained in a direct factor F of finite rank over A of E such that the projection from E onto F is continuous then

$$P_L(T) = \det(1 - TL|F).$$

In particular, $P_L(T)$ depends only on the topology.

Proof. I will prove $P_L(T)$ is entire in T and (i). Let $(e_i)_{i\geq 0}$ be an ONB. We can suppose $|L| \leq 1$. Suppose $L(e_i) = \sum_j n_{i,j} e_j$. For a finite set S of non-negative integers and a permutation σ of S, set

$$n_{S,\sigma} = \prod_{i\in S} n_{i\,\sigma(i)}$$

Then

$$P_L(T) = 1 + c_1 T + c_2 T^2 + \cdots,$$

where

$$c_m = (-1)^m \sum_{\substack{S,\sigma \\ |S|=m}} \epsilon_\sigma n_{S,\sigma}.$$

Now let $R_1 \geq R_2 \geq \cdots$ be the numbers $r_j = \sup_{i\geq 0} |n_{ij}|$. It follows that

$$|c_m| \leq R_1 R_2 \cdots R_m,$$

so

$$|c_m|M^m \leq (R_1M)(R_2M)\cdots(R_mM).$$

Now suppose, $|L' - L| < \epsilon < 1$.

Some other key facts.

Remark. *If $L: M \to N$ is compact and $F: N \to M$ is continuous, then $L \circ F$ and $F \circ L$ are compact.*

(i) If u and v are compact operators on E,

$$\det(1 - Tu)\det(1 - Tv) = \det((1 - Tu)(1 - Tv)).$$

(ii) Suppose E_1 and E_2 are orthonormizable Banach modules over A. Suppose u is a compact homomorphism from E_1 to E_2 and $v: E_2 \to E_1$ is a continuous homomorphism. Then $P_{u \circ v}(T) = P_{v \circ u}(T)$.
(iii) if $\phi: A \to B$ is a homomorphism of Banach algebras then $\phi^*E =: E \otimes_A B$ is orthonormizable over B and

$$P_{\phi^* L}(T) = \phi(P_L(T)).$$

Given this one can define the characteristic series of a continuous operator V on M if one only asssumes M is "locally orthonormizable."

Riesz Theory

Suppose u is a compact operator on E. Let $A\{\{T\}\}$ denote the ring of entire series over A. For a polynomial of degree d whose leadin coefficient is a unit, $F(T)$, let $F^*(T) = T^d F(T^{-1})$.

Theorem. *Suppose $P_u(T) = Q(T)S(T)$ where $S \in A\{\{T\}\}$ and Q is a polynomial whose leading coeicient is a unit such that $Q(0) = 1$ and which is relatively prime to S. Then there is a unique direct sum decomposition*

$$E = N_u(Q) \oplus F_u(Q)$$

of E into closed submodules stable by u such that $N_u(Q)$ is projective of rank $\deg Q$*, $Q^*(u)N_u(Q) = 0$ and $Q^*(u)$ is invertible on $F_u(Q)$. Moreover, $N_u(Q)$ and $F_u(Q)$ are locally equivalent to orthonomizable modules and $P_{u|_{N_u(Q)}}(T) = Q(T)$ and $P_{u|_{F_u(Q)}}(T) = S(T)$.* ∎

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Lecture 5

Restants

See Lang's algebra Chapter IV §8. Let e_i be the i-th elementary symmetric polynomial of $T_1, \ldots, T_n$.

Lemma. *The subring of* $A[[T_1, \ldots, T_n]]$, $A\{\{e_1, \ldots, e_n\}\}$, *is equal to the subring of* $A\{\{T_1, \ldots, T_n\}\}$ *consisting of elements which are left invariant under permutation of the variables* T_i.

Suppose $Q(T) = T^n - a_1 T^{n-1} + \cdots + (-1)^n a_n \in A[T]$ and $P(T) \in A\{\{T\}\}$. Then $P(T_1) \cdots P(T_n) = H(e_1, \ldots e_n)$ for some $H \in A\{\{X_1, \ldots, X_n\}\}$. The **resultant** of Q and P is

$$Res(Q, P) = H(a_1, \ldots, a_n).$$

Then,

$$Res(Q, 1) = 1 \quad Res(Q, T) = (-1)^n Q(0)$$

$$Res(Q, aP) = a^n Res(Q, P)$$

$$Res(Q, PR) = Res(Q, P) Res(Q, R)$$

$$Res(Q, P + BQ) = Res(Q, P)$$

and if S is a monic polynomial of degree m,

$$Res(SQ, P) = Res(S, P) Res(Q, P)$$

$$Res(Q, S) = (-1)^{mn} Res(S, Q)$$

$$Res(Q, S^*) = Res(S, Q^*).$$

Recall $Q^*(T) = T^n Q(T^{-1})$.

Say that an element $a \in A$ is **multiplicative** if $|ab| = |a||b|$ for all $b \in A$.

Proposition. *The resultant of Q and P is a linear combination of Q and P. If Q and P have a non-constant polynomial common factor G whose leading term is multiplicative, then the resultant of Q and P is zero and is a unit if and only if Q and P are relatively prime in $A\{\{T\}\}$.*

Lemma. *If $G(T)$ is a polynomial whose leading coefficient is multiplicative and $H(T) \in A\{\{T\}\}$ such that $G(T)H(T) \in A$ then $G(T) \in A$ or $H(T) = 0$.*

Proof. Replacing $G(T)$ by $G(p^{-M}T)$ for some positive integer M we may assume that the absolute value of the leading coefficient c of G is greater than all its other coefficients. Suppose $\deg G = n$. Suppose $H(T) = \sum_k b_k T^k$ and $m \geq 0$ is such that $|b_m| \geq |b_k|$ for all k with strict inequality for $k > m$.

Now suppose $B(T) \in A[T]$, $B(0) = 0$ and $F = Q^*$ for a monic polynomial Q. Let $P(T) = 1 - XB(T)$. Let

$$D(B, F) = Res(Q, P) \in A[X].$$

Now if $B, F \in \{T\}$, $B(0) = 0$, $F(0) = 1$ let

$$D(B, F)(X) = \lim_{n \to \infty} D(B_n, F_n)(X).$$

Then $D(B, F)(X) \in A\{\{X\}\}$ and

Theorem. *If u is a compact operator on an orthonormizable Banach module E over A and $B \in TA\{\{T\}\}$ then $B(u)$ is compact and*

$$P_{B(u)}(T) = D(B, P_u)(T).$$

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Lecture 6

Correction: Lemma. If $G(T)$ is a polynomial whose leading coefficient is multiplicative and $H(T) \in A\{\{T\}\}$ such that $G(T)H(T) \in A$ then $G(T) \in A$ or $H(T) = 0$.

Riesz Theory

Suppose A is a reduced affinoid algebra over K, E is an orthonormizable Banach module over A and u is a compact operator on E.

We need one more thing about resultants,

Lemma. *If $P(T) = R(T)S(T)$, $R, S \in A\{\{T\}\}$ and $R(0) = S(0) = 1$, then we have,*
$D(B, P) = D(B, R)D(B, S)$,
and if Q is a monic polynomial, $D(1 - Q^*, P)(1) = Res(Q, P)$.

The **Fredholm resolvant** Fr_u of u is

$$\frac{P_u(T)}{(1 - Tu)} = P_u(T) \sum_{i \geq 0} u^i T^i.$$

Proposition. *The Fredholm resolvant is "entire."*

Proof. Fr_u acts on $E \otimes_A A[[T]]$. If $P_u(T) = \sum_{m \geq 0} c_m T^m$, $Fr(u)(T) = \sum v_m T^m$, where

$$v_0 = 0 \text{ and } v_m = c_m + u v_{m-1}.$$

Let $R_1 \geq R_2 \geq \cdots$ be the numbers $r_j = \sup_{i \geq 0} |n_{ij}|$ where (n_{ij}) is the matrix for u wrt. an ONB $B = \{e_I\}$.Claim: $|v_m| \leq R_1 R_2 \cdots R_m$.

First suppose E is free of finite rank n. Then since $Fr(T)P_u(T) = \det(1 - Tu)$

Now suppose $u(E) \subseteq E_n$.

End of proof. $\pi_n \circ u \to u$.

Lemma. *Suppose $Q(T) \in A[T]$ is monic. Then $(Q, P_u) = 1$ in $A\{\{T\}\}$ if and only if $Q^*(u)$ is invertible.*

Proof. Let $v = 1 - Q^*(u)$. Suppose $(Q, P_u) = 1$.

$$(1 - vT)Fr_v(T) = P_v(T) = D(1 - Q^*, P_v)(T).$$

Last time we saw $D(1 - Q^*, P_v)(1) = Res(Q, P_u)$.

Now suppose $Q^*(u)(1 - w) = 1$.

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Lecture 7

Riesz Theory (continued)

For $R(T) = \sum_{n \geq 0} a_n T^n$, let

$$\Delta^k F(T) = \sum_{n \geq k} \binom{n}{k} a_n T^{n-k}$$

and $\Delta = \Delta^1$. If $F(T) \in A\{\{T\}\}$ and $a \in A$, say a is a zero F of **order** k if $\Delta^i F(a) = 0$ for $i < k$ and $\Delta^k F(a)$ is a unit.

Lemma. *Suppose $a \in A$ is a zero of $P_u(T)$ of order h. Then we have a unique decomposition*

$$E = N(a) \oplus F(a)$$

into closed submodules such that $1 - au$ is invertible on $F(a)$ and $(1 - au)^h N(a) = 0$.

Proof. Proof. We have

$$(1 - uT)\Delta^s Fr_u(T) - u\Delta^{s-1} Fr_u(T) = \Delta^s P_u(T).$$

So if $v_s = \Delta^s Fr_u(a)$. We get $(1 - au)^{s+1} v_s = 0$ for $s \leq h$.

Let $c = \Delta^h P_u(a)$,

$$e = c^{-1}(1 - au)v_h \quad \text{and} \quad f = c^{-1} u v_{h-1}.$$

Then

$$e + f = 1 \quad \text{and} \quad fe^h = 0.$$

The endomorphisms e^h and $\sum_{i \geq 1} \binom{h}{i} e^{h-1} f^i$ are projectors.

Theorem. *Suppose* $P_u(T) = Q(T)S(T)$ *where* $S \in A\{\{T\}\}$ *and* Q *is a monic polynomial such that* $Q(0) = 1$ *and which is relatively prime to* S*. Then there is a unique direct sum decomposition*

$$E = N_u(Q) \oplus F_u(Q)$$

of E *into closed submodules stable by* u *such that* $N_u(Q)$ *is projective of rank* $\deg Q$*,* $Q^*(u)N_u(Q\ ) = 0$ *and* $Q^*(u)$ *is invertible on* $F_u(Q)$*. Moreover,* $N_u(Q)$ *and* $F_u(Q)$ *are locally equivalent to orthonomizable modules and*

$$P_{u|_{N_u(Q)}}(T) = Q(T) \quad \text{and } P_{u|_{F_u(Q)}}(T) = S(T).$$

Proof. Let $n = \deg Q$, $B(T) = 1 - Q^*(T)$ and $v = B(u)$. Then

$$P_v(T) = D(B, P_u)(T) = D(B, Q)(T) \cdot D(B, S)(T),$$

but

$$D(B, Q)(X) = Res(Q^*, 1 - X(1 - Q^*)) = (1 - X)^n,$$

and $D(B, S)(1) = Res(Q, S)$.

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Lecture 8

Riesz Theory (continued)

Last we proved, Suppose $P_u(T) = Q(T)S(T)$ where $S \in A\{\{T\}\}$ and Q is a polynomial of degree h whose leading coefficient is a unit such that $Q(0) = 1$ which is relatively prime to S. Then there is a direct sum decomposition

$$E = N_u(Q) \oplus F_u(Q)$$

of E into closed submodules stable by u such that $Q^*(u)^h N_u(Q\ ) = 0$ (note the h) and $Q^*(u)$ is invertible on $F_u(Q)$. Moreover, if $Q(T) = (1 - bT)^h$ then $Q^*(u)N_u(Q) = 0$.

Let $F = N_u(Q)$ and e.$F = F_u(Q)$ Claim: $Q^*(u)N = 0$ in general.

Now lets prove $N_u(Q)$ is projective of rank h. Suppose we know this when A is a field.

Let $\{e_i\}$ be an ON basis for E. Let m be a maximal ideal. Let

$$f_i = \sum_{j \in I} a_{i,j} e_j \quad \text{for } 1 \le i \le h$$

be elements of N which form a basis of N_m modulo m. Then $\exists j_1, \ldots, j_h$ such that

$$g =: \det(a_{i\, j_k}) \neq 0$$

is not zero at m. Let U be an open affinoid in $\mathrm{Max}A$ where g is invertible. Claim: $f_1, \ldots, f_h$ is a basis for N_U.

Now we prove when the leaing coeffikcikent of Q is multiplicative, $\det(1 - Tu|N_u(Q) = Q(T)$.∎

Proposition. *Suppose N is a free. Then, locally, there exists a norm on E equivalent to $|| \ ||$ such that both N and F with their induced norms are orthonormizable.*

Corollary. *If u_F is the induced operator on F, u_F has a characteristic series and*

$$P_u(T) = \det(1 - Tu|_N) P_{u_F}(T).$$

It follows that There exist $H(T) \in A\{\{T\}\}$ such that

$$H(T)Q(T) = \det(1 - Tu|_N)$$

We also get $P_{u_F}(T) = S(T)$.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 9

Serre's Riesz theory

Suppose now A is a field. As usual E is an ON Banach space over A and u is a compact operator on E. Let $\{e_i\}$ be an ONB of E.

Suppose a is a zero of $P_u(T)$ of order h and $E = N \oplus F$ is the decomposition of E into u-stable Banach subspaces such that $(1 - au)^h N = 0$ and $1 - au$ is invertible on E.

Theorem. *(Serre) N is free of dimension h.*

Suppose W is d-dimensional subspace of N stable by u. Claim: $E = W \oplus G$ with G ONable.

Suppose $\dim W = 1$. Suppose $w \in W$, $||w|| = 1$. Suppose

$$w = \sum a_i e_i$$

and $|e_k| = 1$. Let $G =$ Span$\{e_i : i \neq k\}$.

Using this we see that

$$(1 - Ta)^{\dim W} | P_u(T)$$

and so $\dim N \leq h$.

We know

$$\det(1 - Tu|N) \cdot P_{u|F}(T) = P_u(T).$$

Since $1 - au$ is invertible on F, it follows that $\dim N \geq h$.

Pseudo-representations

Suppose you have a group G and functions $D, T \to G \to R$? What do you need to know about D and T to know there is ae representation $\rho\colon G \to \mathbf{Gl}_2(R)$ such that

$$D(\sigma) = \det(\rho(\sigma)) \text{ and } T(\sigma) = \text{ Tr}\,(\rho(\sigma))? \qquad (*)$$

Let S be a finite set of primes. Suppose G_S is the Galois group of the maximal Abel;ian extension of $\mathbf{Q}$ unramified outside of S and $\mathbf{c} \in \mathbf{G_S}$ a complex conjugation.

Theorem. *Then if R is an integfral domain whose quotient field K is not of characteristic $\neq 2$, there exists a ρ satisfying $(*)$ and $\rho(\mathbf{c}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if (for all $g, h, k, \ell \in G_S$):*

$$\delta(g \cdot h) = \delta(g)\delta(h) + \xi(h, g)$$

$$\xi(gh, k) = \alpha(g)\xi(h, k) + \delta(h)\xi(g, k)$$

$$\xi(g, hk) = \alpha(k)\xi(g, h) + \delta(h)\xi(g, k)$$

$$\xi(g, h)\xi(k, \ell) = \xi(g, \ell)\xi(k, h)$$

and

$$\alpha(1) = \delta(1) = 1;\ \alpha(\mathbf{c}) = -1;\ \delta(\mathbf{c}) = 1$$

where

$$\alpha(x) = \frac{T(x) + T(cx)}{2},\ \delta(x) = \frac{T(x) - T(cx)}{2}$$
$$\xi(x, y) = \alpha(xy) - \alpha(z)\alpha(y).$$

Moreover. if $R \in$ ob$(\mathcal{C})$, the category of complete notherian local $\mathbf{Z}_p$-algebras, ρ is continuous if and only if T is.

If there exist $r, s \in G_S$ such that $\xi(r, s) \neq -0$, the representation ρ is given by

$$g \mapsto \begin{pmatrix} \alpha(g) & \frac{\xi(g,s)}{\xi(r,s)} \\ \xi(r, g) & \delta(g) \end{pmatrix}.$$

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 10

One formula I left out from the previous theorem is $D(g) = \alpha(g)\delta(g) - \xi(g,g)$. Since for a pseudo-representation (T, D), D is determined by T, I will call T a pseudo-representation.

What pseudo-representations are good for

Suppose $(N,p) = 1$. Then $h_1(N, \mathbf{Z}_p) = \varprojlim h_k(\Gamma_1(Np^n), \mathbf{Z}_p)$ is independent of the weight and is topologically generated by Hecke operators $T(n)$ and $\langle a \rangle$, $(a, Np) = 1$.

Theorem. *(Hida) Suppose $A \in Ob(\mathcal{C})$, where $\mathcal{C}$ is the category of complete local noetherian $\mathbf{Z}_p$-algebras, be an integral domain with quotient field K and $\lambda: h_1(N, \mathbf{Z}_p) \to A$ is a continuous $\mathbf{Z}_p$-homomorphism. Then there is a unique semi-simple representation $\rho: G_{\mathbf{Q}} \to \mathbf{Gl}_2(K)$ such that*

(i) ρ is continuous.

(ii) ρ is unramified outside Np.

(iii) If $\ell \not| Np$ is a prime and ϕ_ℓ is a Frobenius above ℓ

$$\det(1 - \rho(\phi_\ell)X) = 1 - \lambda(T(\ell))X + \lambda(\langle \ell \rangle)\ell X^2.$$

Back to pseudo-representations

Proposition. *Suppose R is a product of finitely many objects in $\mathcal{C}$, $\mathfrak{a}$ and $\mathfrak{b}$ two ideals of R and $T_{\mathfrak{a}}: G_{\mathbf{Q}} \to R/\mathfrak{a}$ and $T_{\mathfrak{b}}: G_{\mathbf{Q}} \to R/\mathfrak{b}$ two continuous p-rs (pseudo-representtions). If there exist functions t and d on a dense subseet Σ of $G_{\mathbf{Q}}$ with values in $R/(\mathfrak{a} \cap \mathfrak{b})$ such that*

$$(T_{\mathfrak{a}}(\sigma)), D_{\mathfrak{a}}(\sigma)) \equiv (t(\sigma), d(\sigma)) \ mod\, \mathfrak{a}$$

$$(T_{\mathfrak{b}}(\sigma)), D_{\mathfrak{b}}(\sigma)) \equiv (t(\sigma), d(\sigma)) \ mod\, \mathfrak{b},$$

for $\sigma \in \Sigma$ then there exists a p-r $T_{\mathfrak{a}\cap\mathfrak{b}}: G_{\mathbf{Q}} \to R/(\mathfrak{a} \cap \mathfrak{b})$ such that

$$(T_{\mathfrak{a}\cap\mathfrak{b}}(\sigma), D_{\mathfrak{a}\cap\mathfrak{b}}(\sigma)) \equiv (t(\sigma), d(\sigma)) \ mod\, \mathfrak{a} \cap \mathfrak{b}.$$

Proof. Consider

$$0 \to R/(\mathfrak{a} \cap \mathfrak{b}) \longrightarrow R/\mathfrak{a} \oplus R/\mathfrak{b} \xrightarrow{\alpha} R/(\mathfrak{a}+\mathfrak{b}) \to 0.$$

Theorem (Wiles)**.** *Suppose R is a topological $\mathbf{Z}_p$-algebra and $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ are ideals such that $R/\mathfrak{p}_i \in \mathcal{C}$ and*

$$R = \varprojlim R/\bigcap_{i=1}^{n} \mathfrak{p}_i,$$

Σ is a dense subset of G, t, d are functions $\Sigma \to R$ and p-rs $T_i \colon G \to R/\mathfrak{p}_i$ such that

$$(T_i(\sigma), D_i(\sigma)) \equiv (t(\sigma), d(\sigma)) \mod \mathfrak{p}_i$$

for $\sigma \in \Sigma$. Then there exists a unique p-r $T \colon G \to R$ such that $T(\sigma) \equiv T_i(\sigma) \mod \mathfrak{p}_i$ for all $\sigma \in \Sigma$ and all i.

Proof.

Corollary. *If $\lambda \colon R \to A$ is a continuous $\mathbf{Z}_p$-algebra homomomorphism into an integral domain with fraction field K of characteristic different than 2, there exist a semisimple representation $\rho \colon G \to \mathbf{Gl}_2(K)$ such that*

$$\det(1-\rho(\sigma)X) = 1 - \lambda(T(\sigma))X + \lambda(D(\sigma))X^2$$

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 11

Let $G = G_{\mathbf{Q}}$. Now will prove

Theorem. *Suppose $(N,p) = 1$. Suppose $A \in Ob(\mathcal{C})$ is an integral domain with quotient field K and $\lambda: h_1(N, \mathbf{Z}_p) \to A$ is a continuous $\mathbf{Z}_p$-homomorphism. Then there is a unique semi-simple representation $\rho: G \to \mathbf{Gl}_2(K)$ such that*

(i) ρ is continuous.

(ii) ρ is unramified outside Np.

(iii) If $\ell \not| Np$ is a prime and ϕ_ℓ is a Frobenius above ℓ

$$\det(1 - \rho(\phi_\ell)X) = 1 - \lambda(T(\ell))X + \lambda(\langle \ell \rangle)\ell X^2.$$

Proof. Let $\Sigma = \{\phi_\ell : \phi_\ell \text{ is a Frobenius above } \ell\}$.

Fix $k \geq 2$. Let $R = h_1(N, \mathbf{Z}_p) = \varprojlim h_k(\Gamma_1(Np^n), \mathbf{Z}_p)$. Now $R_n = h_k(\Gamma_1(Np^n), \mathbf{Z}_p)$ is a product of finitely many objects of $\mathcal{C}$ and R_n contains finitely many minimal prime ideals $\mathfrak{p}_{ni}$ and $\bigcap_i \mathfrak{p}_{ni} = 0$.

Let $\mathfrak{p}_{ni}$ denote its inverse image in R. It follows that

$$R = \varprojlim_n R/\bigcap \mathfrak{p}_{ni}.$$

Now, one knows if $\lambda: R_n \hookrightarrow \overline{\mathbf{Q}}_p F$ there exists a weight k eigenform F on $\Gamma_1(Np^n)$ such that

$$F(q) = \sum \lambda_{n\geq 1}(T(n))q^n.$$

and by Deligne there exists an irreducible continuous representation $\pi: G \to \mathbf{Gl}_2(\overline{\mathbf{Q}}_p)$ such that $\det(\pi(\mathbf{c})) = -1$ and

$$\det(1 - \pi(\phi_\ell)X) = 1 - \lambda(T(\ell))X + \lambda(\langle \ell \rangle)\ell X^2.$$

Thus for each (n, i) we have a p-r with values in $R/\mathfrak{p}_{ni}$. Now let

$$t(\phi_\ell) = T(\ell) \text{ and } d(\phi_\ell) = \ell\langle\ell\rangle.$$

Back to Banach Modules

Let K be a finite extension of $\mathbf{Q}_p$. Let $K^0 = \{a \in K\colon |a| \leq 1\}$ and $\wp = \{a \in R\colon |a| < 1\}$. Suppose Y is a reduced irreducible affinoid such that $\widetilde{Y}$ is also reduced and we will regard $A(Y)$ as a Banach algebra with respect to the supremum norm.

For a rigid spacs X let $A(X)$ denote the ring of rigid analytic functions on X, and $|\ |$ denote the supremum semi-norm on $A(X)$ and $A^0(X)$ will denote the subring in $A(X)$ of power bounded functions on Y. Then $\wp A^0(Y)$ equals the set of topologically nilpotent elements in $A(Y)$ and $\bar{Y} = Spec(A^0(Y)/\wp A^0(Y))$. Let $\mathbf{B}^n_K$ will denote the n-dimensional affinoid polydisk over K. Then

$$A(\mathbf{B}^n_K) \cong K\langle T_1, \ldots, T_n\rangle \text{ and } A^0(\mathbf{B}^n_K) \cong K^0\langle T_1, \ldots, T_n\rangle.$$

If $a \in K$ and $r \in |\mathbf{C}_p|$ we let $B_K[a, r]$ and $B_K(a, r)$ denote the affinoid and wide open disks of radius r about a in $\mathbf{A}^1_K$.

Suppose $X \to Y$ is a morphism of reduced affinoids over K. Then $(A(X), |\ |)$ is a Banach module over $(A(Y), |\ |)$.

Lemma. *Suppose $X \to Y$ is a morphism of reduced affinoids over K and $A^0(X)/\wp A^0(X)$ is free over $A^0(Y)/\wp A^0(Y)$. Then the Banach module $A(X)$ over $A(Y)$ is orthonormizable.*

Proposition. *Suppose $f\colon Z \to X$ is a map of reduced affinoids over Y, $\tilde{X}$ is reduced and $A(X)$ is orthonormizable over $A(Y)$ and the image of $\overline{Z}$ in $\overline{X}$ is finite over $\overline{Y}$. Then the map f^* from $A(X)$ to $A(Z)$ is a compact homomorphism of Banach modules over $A(Y)$.*

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture

Nuclear Families

Robert F. Coleman

Suppose M is a Banach space over $\mathbf{C}$, M' is the continuous dual space. Then $M \otimes M'$ has a natural norm such that

$$||h \otimes e|| = ||e|| \max_{||d|| \leq 1} |h(d)|,$$

and we get a new Banach space $N_M =: M \hat{\otimes} M'$. This space has natural ring structure

$$(e \otimes h) \cdot (d \otimes f) = h(d)(f \otimes e).$$

Moreover, there is a natural "trace" map

$$\mathrm{Tr} : e \otimes h \to h(e).$$

We also have a continuous linear map $b: N_M \to \mathcal{B}(M) := \underset{cont}{Hom}(M, M)$,

$$b(e \otimes h): d \to h(d)e$$

which turns multiplication into composition and its image is an ideal. The operators in the image of b are called **nuclear.** (They are compact.) One has the Fredholm determinant, for $u \in N_M$

$$\det(1 - zu) = \exp\big(- \sum_{n=1}^{\infty} \mathrm{Tr}\,(u^n)\frac{z^n}{n}\big).$$

This series is entire and zeroes, counting multiplicity, are the inverses of the non-zero spectra of $b(u)$. (This was all extracted from Grothendieck's *La Theorie de Fredholm* (1956).)

Let $H = L^2([0,1], dt)$. Then if $k(x,y) \in L^2([0,1] \times [0,1], dt \times dt)$ we an operator K on H

$$Kf(x) = \int_0^1 k(x,y) f(y) dy.$$

These are called Hilbert-Schmidt operators. The product of two of these is nuclear.

What about families?

Suppose one has a "family" of nuclear operators. How does the spectrum vary?

Example. Suppose M is a Banach space and Z is a compact Hausdorff space. Supppose U is a nuclear operator on M and $V \in C(Z, \mathcal{B}(M))$. Then $U_x := U \circ V(x)$ is a family of nuclear operators on M. In fact, we get a Fredholm determinant $D_{U,V}(T)$ whose coefficients are in $A := C(Z)$. Call its zero locus the **spectral space** of the family.

Another way to phrase this is: Let $M_A = M \hat{\otimes} A = C(Z, M)$. Then we have an operator on M_A over A

$$e \otimes f \to \big(x \to U(x)e \otimes f(x)\big),$$

and this operator is "nuclear" over A. One can replace C with An everywhere.

Questions. *Suppose Z is a closed disk and U and V are analytic. Under what conditions is the zero locus of $D_{U,V}(T)$ a finite union of connected components and when do these components have finite genus?*

The U-operator and modular forms

Let p be a prime. The compactification $X_0(p)$ of the Riemann surface $\mathcal{H}/\Gamma_0(p)$ (one has to add tvhe cusps 0 and ∞) can be described with equations over $\mathbf{Z}$ and thought about over $\mathbf{Q}_p$. It has two natural p-adic analytic pieces W_∞ and W_0,

Let X_r be the neighborhood of X_∞ of "radius" r. For r small there is a natural finite morphism $\phi: X_r \to X_{r^{1/p}}$. We can think of points on $X_0(p)$ as pairs (E, C) where E is an elliptic curve and C is a subgroup of order p. For some elliptic curves E there is a canonical subroup of order p, $K(E)$ and

$$\phi: (E, K(E)) \to \big(E/K(E), K(E/K(E))\big).$$

Now we have "nuclear" operator on $M := A(X_r)$

$$U =: \ \mathrm{Res}_{X_r}^{X_{r^{1/p}}} \circ \ \mathrm{Tr}\,{}_{X_{r^{1/p}}}^{X_r}(\phi).$$

There is a weight $p-1$ Eisentein series E_{p-1} and therefore a function $\mathbf{E}$ on X_r (for small r) whose q-expansion is

$$E_{p-1}(q)/E_{p-1}(q^p).$$

Since this q-expansion is $\equiv 1 \bmod p$, $\mathbf{E}^s$ makes sense for $|s| \leq 1$ and is in $A(X_r)$ for small r, so we have

$$V: B[0,1] \to \mathcal{B}(M),$$

$$V(s)g = \mathbf{E}^s \cdot g$$

and so we get a family of nuclear operators U_s on M. If $k = (p-1)n$ one calls the elements of $M_k = E_{p-1}^n M$ weight k **overconvergent** modular forms. It contains the classical weight k forms on $\Gamma_0(p)$ and if F is classical

$$F \to E_{p-1}^n U_n(F/E_{p-1}^n)$$

is the classical weight k U-operator. We get a **spectral curve** S over $B[0,1]$.

The Eigencurve

There are other operators $T(n)$ for any integer n prime to p and using the fact that nuclear operators make up an ideal we can use $U \circ T(n)$ to make another spectral curve S_n. The **eigencurve** $\mathcal{E}$ is essentially the fiber product of all these spectral curves. A point x on the eigencurve corresponds to a normalized overconvergent eigenforms F_x with non-zero U-eigenvalue. These have q-expansion s.

For each eigenform mod p f there is a component $\mathcal{E}_f$ of $\mathcal{E}$ whose points correspond to normalized overonvergent eigenforms whose q-expansion s reduce to that of f.

One can attach a representation $\rho_f: G_{\mathbf{Q}} \to \mathbf{Gl}_2(\overline{\mathbf{F}}_p)$ unramified away from p such that

$$\mathrm{Tr}\,\rho_f(\phi_\ell) = a_\ell$$

if $\ell \neq p$ and $f(q) = \sum_n a_n q^n$. If ρ_f is irreducible one can attach a representation $\rho_x : G_{\mathbf{Q}} \to \mathbf{Gl}_2(\mathbf{C}_p)$ to each point x in $\mathcal{E}_f$ unramified away from p which "lifts" ρ_f such that

$$\mathrm{Tr}\, \rho_x(\phi_\ell) = A_\ell$$

if $\ell \neq p$ and $F_x(q) = \sum_n A_n q^n$.

Fontaine-Mazur and Kisin

Conjecture. *Suppose E is a finite extension of $\mathbf{Q}_p$ and $\rho: G_{\mathbf{Q}} \to \mathbf{Gl}_2(E)$ is a continuous odd, irreducible representation ramified at only finitely many primes whose restriction to a decomposition group at p is "semi-stable." Then ρ arises from a classical modular form.*

Mark Kisin has recently proven this conclusion when ρ arises from an overconvergent eigenform with non-zero U eigenvalue using the eigencurve (**Coleman-Mazur**) and the following

Theorem (C, 94). *If F is an overconergent eigenform of weight k and the valuation of its U-eigenvalue is $< k-1$ then F is classiscal.*

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 13

Li's Example

Suppose $H = \{v = \sum_{i\geq 1} a_i e_i : a_i \in \mathbf{C},\ ||v|| =: \sum_{i\geq 1} |a_i| < \infty\}$. Consider the operator $L: e_i \to \dfrac{e_i}{i}$.

$$L = \lim_{n\to\infty} b(\sum_{i=1}^{n} \frac{e_i' \otimes e_i}{i})$$

Why doesn't L have a trace?

A Compact Source

Proposition. *Suppose $f: Z \to X$ is a map of reduced affinoids over Y, $\tilde{X}$ is reduced and $A(X)$ is orthonormizable over $A(Y)$ and the image of $\overline{Z}$ in $\overline{X}$ is finite over $\overline{Y}$. Then $f^*: A(X) \to A(Z)$ is a compact homomorphism of $A(Y)$-Banach modules.*

Proof. Let $B = A^0(Y)$, $C = A^0(Z)$ and $D = A^0(X)$. Let $x_1, \ldots, x_n$ be elements of D such that the map from $B\langle T_1, \ldots, T_n\rangle$, $T_i \mapsto x_i$ is surjective onto D. There are monic polynomials $g_i(S) \in B[S]$, $1 \leq i \leq n$ such that $f^* g_i(x_i) \in \pi C$ for some $\pi \in K^0$ such that $|\pi| < 1$. We can write any element of D as

$$\sum_{I,N} a_{I,N} x^I g(x)^N,$$

where $x = (x_1, \ldots, x_n)$, $g = (g_1, \ldots, g_n)$, $I, N \in \mathbf{N}^n$ (ordered lexographically), $I < deg(g)$ and $a_{I,N} \in B$. Now let $\{e_i\}_{i\in I}$ be an ON basis for $A(X)$ over $A(Y)$. Then $e_i \in D$. Let $F_{i,m}$ be an element in the B-span of

$$\{f^*(x^I g(x)^N): I < \deg g \ \text{ and } S(N) < m\}$$

such that $F_{i,m} \equiv f^* e_i \bmod \pi^m C$. Define $L_m: A(X) \to A(Z)$ by $L_m(e_i) = F_{i,m}$. Then L_m is of finite rank and converges to f^* ∎

Call such a morphism f **inner over** Y. If $Y = \operatorname{Spec} K$ call f **inner**.

Examples.

Overconvergnce

Suppose Z is an affinoid. Then an overconvergent function f on Z is a rigid function such that there exists some inner embedding $Z \to X$ and a function F on X which extends f.

When Z has good reduction one can use the same X for any two functions.

Examples.

When f is a section of a sheaf $\mathcal{F}$ one does something similar.

Suppose $(N,p) = 1$. Then $X_1(Np)$ has a model whose reduction has two components, $X_0 =: X_0(N)$ and $X_\infty =: X_\infty(N)$, Let $W_\infty = \text{Red}^{-1} X_\infty$ and $Z_1(N) = \text{Red}^{-1} X_\infty - X_0$. Define W_0 similarly. Then $W_\infty \cap W_0$ is a untion of annuli A_s where s is a ss point of $X_1(N)$. There exist $w_s \in \mathbf{N}$ and $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to Z_1(N)$. If $x \neq 0, 1728$ or $N > 4$, $w_s = 1$

Let $W_\infty(r) =: W_\infty(N)(r)$ be the set of $x \in W_\infty$, $x \in Z_1(N)$ or s and $v(T_s(x)) \le r$. (In particular, $W_\infty((Nn)(0) = Z_1(Nn)$.)

One has a canonical sheaf ω on $X_1(Np)$ (if $Np \ge 5$).

An overconvergent form of weight k is an overconvergent section of $\omega^{\otimes k}$ on $Z_1(N)$. It extends to $W_\infty(r)$ for some r.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 14

Overconvergnce

Suppose Z is an affinoid. Then an overconvergent function f on Z is a rigid function such that there exists some inner embedding $Z \to X$ and a function F on X which extends f.

Lemma. *Overconvergent functions form a ring.*

Lemma. *If Z is reduced and has good reduction, $Z \to Y$ is inner and f is an ovewrconvergent function on Z. There exists an affinoid Y, morphisms $Z \to X \to Y$ such that $Z \to X$ is inner and a function F on X which extends f.*

Examples.

Suppose $(N,p) = 1$. Then $\overline{\mathcal{X}_1(Np)} = X_0 \cup X_\infty$, Let $W_\infty = \text{Red}^{-1} X_\infty$ and $Z_1(N) =: Z_1(Np) = \text{Red}^{-1}(X_\infty - X_0)$. There exist $w_s \in \mathbf{N}$ and $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to Z_1(N)$. Let $W_\infty(r) =: W_\infty(N)(r)$ be the set of $x \in W_\infty$, $x \in Z_1(N)$ or s and $v(T_s(x)) \le r$. There is a canonical sheaf ω on $X_1(Np)$ (if $Np \ge 5$).

An overconvergent form of weight k is an overconvergent section of $\omega^{\otimes k}$ on $Z_1(N)$.

Eisenstein Series

Suppose p is odd. Let $\pi^{p-1} = -p$. For a character $\chi: \mathbf{Z}_p^* \to \mathbf{C}_p^*$, let f_χ denote its "conductor". Let $\mathcal{W} = \underset{cont}{Hom}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ (weight space). $\mathbf{Z}$ injects naturally into $\mathcal{W}(\mathbf{Q}_p)$;

$$k \in \mathbf{Z} \to \left(a \to a^k\right).$$

Let τ denote the Teichmuller character and $\mathbf{1}$ denote the trivial character.

Suppose $\kappa \in \mathcal{W}(\mathbf{C}_p)$, $\kappa \neq \mathbf{1}$, and $n \geq 1 \in \mathbf{Z}$, let

$$\sigma^*_\kappa(n) = \sum_{\substack{d|n \\ (d,p)=1}} \kappa(d)d^{-1}, \quad \zeta^*(\kappa) = \frac{1}{\kappa(c)-1}\int_{\mathbf{Z}_p^*} \kappa(a)a^{-1}dE_{1,c}(a)$$

for any $c \in \mathbf{Z}_p^*$ such that $\kappa(c)$ is not 1. So that, when $\kappa(a) = \langle\langle a\rangle\rangle^s\chi(a)$ (is **arithmetic**) where $s \in \mathbf{C}_p$, $|s| < |\pi/p|$, and χ is a character of finite order $\zeta^*(\kappa) = L_p(1-s,\chi)$. Let

$$G^*_\kappa(q) = \frac{\zeta^*(\kappa)}{2} + \sum_{n\geq 1}\sigma^*_\kappa(n)q^n.$$

When $\kappa(a) = \langle\langle a\rangle\rangle^k\chi(a)$, where k is an integer and χ is a character of finite order on $\mathbf{Z}_p^*$ such that $\chi(-1) = 1$, $G^*_\kappa(q)$ is the q-expansion of a weight k overconvergent modular form G^*_κ on $\Gamma_1($ LCM$(p, f_\chi))$ and character $\chi\tau^{-k}$. It is classical if k is at least 1.

If $\zeta^*(\kappa) \neq 0$ and $\kappa \neq \mathbf{1}$, let $E^*_\kappa(q) = 2G^*_\kappa(q)/\zeta^*(\kappa)$ and also $E^*_{\mathbf{1}}(q) = 1$. Suppose $\kappa \in \mathcal{W}(\mathbf{C}_p)$ and κ is trivial on $\mu(\mathbf{Q}_p)$, then $|\zeta^*(\kappa)| > 1$ and $|E^*_\kappa(q) - 1| < 1$.

Let $\mathcal{B}^* = B(0, |\pi/p|)$ and $\mathcal{W}^* = \mathcal{B}^* \times \mathbf{Z}/(p-1)\mathbf{Z}$. For $s = (t,i) \in \mathcal{W}^*(\mathbf{C}_p)$ let $\kappa_s(a) = a^s =: \langle\langle a\rangle\rangle^t\tau^i(a)$. Let $E = E_{\kappa_{(1,0)}}$. Note that $E(q) \equiv 1 \bmod p$.

For $m \geq 0, N > 0$ $(N,p) = 1$ let $Z_1(Np^m)$ denote the connected component of the ordinary locus in $X_1(Np^m)$ containing ∞.

Lemma. *Suppose $\kappa(a) = \langle\langle a\rangle\rangle^k\chi(a)$ is arithmetic and χ is trivial on $\mu(\mathbf{Q}_p)$. Then E^*_κ (which converges on) does not vanish on $Z_1(p^m)$ where $p^m = LCM(p, f_\chi)$.*

Proof. First E^*_κ converges on $Z_1(p^m)$ because it is overconvergent. Next, the lemma is true for E. Now observe that $F = E^*_\kappa/E^k$ is a function on $Z_1(p^m)$ whose q-expansion is congruent to 1.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 15

Some remarks on overconvergnce

First we can define overconvergent differentials of degree d, $\Omega^{d\dagger}(X)$, on an affinoid X in the same way we defined overconvergent functions $A^\dagger(X) = \Omega^{0\dagger}(X)$ and this module is a finite rank $A^\dagger(X)$-module. Next we can sheafify these things.

Some speculation:

If $\mathcal{F}$ is a coherent sheaf on a rigid space X, an overconvergent a structure $\mathcal{F}^\dagger$ is a sheaf on X coherent over $\mathcal{O}_X^\dagger$ such that $\mathcal{F} = \mathcal{O}_X \otimes_{\mathcal{O}_X^\dagger} \mathcal{F}^\dagger$. Then we get overconvergent structures on Ω_X^d and if $(\mathcal{F}, \mathcal{F}^\dagger)$ and $(\mathcal{G}, \mathcal{G}^\dagger)$ are two coherent sheaves with OS so is $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{F}^\dagger \otimes_{\mathcal{O}_X^\dagger} \mathcal{G}^\dagger)$. Moreover, if $f\colon X \to Y$ is a proper morphism of rigid spaces $(R^n_{f_*}\mathcal{F}, R^n_{f_*}\mathcal{F}^\dagger))$ is an overconvergent structure on the coherent sheaf $R^n_{f_*}\mathcal{F}$.

Since $\omega_M = R_{f_*}\Omega^1_{E_1(M)/X_1(M)}$, if $M \geq 5$, we get a canonical overconvergent structures on $\omega^{\otimes n}$ where $\omega = \omega_{Np}|_{Z_1(Np)}$.

Back to Eisenstein Series

Suppose p is odd, $\pi^{p-1} = -p$.

For $\kappa \in \mathcal{W}(\mathbf{C}_p)$, $\kappa \neq \mathbf{1}$, and $n \geq 1 \in \mathbf{Z}$,

$$\sigma^*_\kappa(n) = \sum_{\substack{d|n \\ (d,p)=1}} \kappa(d)d^{-1}, \quad \zeta^*(\kappa) = \frac{1}{\kappa(c)-1}\int_{\mathbf{Z}_p^*} \kappa(a)a^{-1}\mathrm{d}E_{1,c}(a)$$

for any $c \in \mathbf{Z}_p^*$ such that $\kappa(c)$ is not 1 and

$$G^*_\kappa(q) = \frac{\zeta^*(\kappa)}{2} + \sum_{n\geq 1}\sigma^*_\kappa(n)q^n.$$

If $\zeta^*(\kappa) \neq 0$ and $\kappa \neq \mathbf{1}$, let $E^*_\kappa(q) = 2G^*_\kappa(q)/\zeta^*(\kappa)$ and also $E^*_{\mathbf{1}}(q) = 1$. Suppose $\kappa \in \mathcal{W}(\mathbf{C}_p)$ and κ is trivial on $\mu(\mathbf{Q}_p)$, then $|\zeta^*(\kappa)| > 1$ and $|E^*_\kappa(q) - 1| < 1$.

Let $\mathcal{B}^* = B(0, |\pi/p|)$ and $\mathcal{W}^* = \mathcal{B}^* \times \mathbf{Z}/(p-1)\mathbf{Z}$. For $s = (t, i) \in \mathcal{W}^*(\mathbf{C}_p)$ let $\kappa_s(a) = a^s =: \langle\langle a \rangle\rangle^t \tau^i(a)$. If $E = E_{\kappa_{(1,0)}}$. $E(q) \equiv 1 \bmod p$.

For $m \geq 0, N > 0$, $(N, p) = 1$, let $Z_1(Np^m)$ denote the connected component of the ordinary locus in $X_1(Np^m)$ containing ∞.

q is a parameter at ∞ and any section of $\omega^{\otimes k}$ has a q-expansion .

Lemma. *Suppose* $\kappa(a) = \langle\langle a \rangle\rangle^k \chi(a)$ *and* χ *is trivial on* $\mu(\mathbf{Q}_p)$. *Then* E^*_κ *(which converges on) does not vanish on* $Z_1(p^m)$ *where* $p^m = LCM(p, f_\chi)$.

Proof. First E^*_κ converges on $Z_1(p^m)$. Next, the lemma is true for E. Now observe that $F = E^*_\kappa/E^k$ is a function on $Z_1(p^m)$ whose q-expansion is congruent to 1. -

$X_1(Np) = W_0(N) \cup W_\infty(N)$. $W_\infty \cap W_0 = \bigcup_s A_s$. Suppose $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to Z_1(Np)$. Let $W_\infty[r] =: W_\infty(N)[r]$ be the set of $x \in W_\infty$, $x \in Z_1(Np)$ or $x \in A_s$ for some s and $v(T_s(x)) \leq rw_s$.

If $d \in \mathbf{Z}_p^*$ we have an operator $\langle d \rangle$ in $E_1(Np)/X_1(Np)$ and hence on ω_{Np} and ω.

If k is an integer, and $s = (k, i)$ an overconvergent form F of **weight-character** κ_s are sections of ω^k on $Z_1(Np)$ which extend to $W_\infty[r]$ for some $r > 1$ and satisfy

$$\langle d \rangle F = \tau^i(d) F.$$

Frobenius

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Lecture 16

Weight-Characters

$X_1(Np) = W_0(N) \cup W_\infty(N)$. $W_\infty \cap W_0 = \bigcup_s A_s$. Suppose $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to Z_1(Np)$. Let $W_\infty[r] =: W_\infty(N)[r]$ be the set of $x \in W_\infty$, $x \in Z_1(Np)$ or $x \in A_s$ for some s and $v(T_s(x)) \le rw_s$.

If $d \in \mathbf{Z}_p^*$ we have an operator $\langle d \rangle$ in $E_1(Np)/X_1(Np)$ and hence on ω_{Np} and ω.

If k is an integer, and $s = (k, i)$ an overconvergent form F of **weight-character** κ_s are sections of ω^k on $Z_1(Np)$ which extend to $W_\infty[r]$ for some $r > 1$ and satisfy

$$\langle d \rangle F = \tau^i(d) F.$$

In particular, E_{κ_s} has weight-character κ_s.

Frobenius

Suppose $N > 4$ and $n \ge 1$ are integers such that $(N, p) = 1$. Let $A = E^{p-1}$.

Let $E_1(N)(v)$ denote the pullback of $E_1(Np)$ to $X_1(N)(v)$. Then, for $v < 1/(p+1)$. If E is an elliptic curve with a canonical sdubgroup, denote this subgroup $K(E)$.

Theorem. *There is a commutative diagram of rigid morphisms;*

$$\begin{array}{ccc} E_1(N)(v) & \xrightarrow{\Phi} & E_1(N)(pv) \\ \downarrow & & \downarrow \\ X_1(N)(v) & \xrightarrow{\phi} & X_1(N)(pv) \end{array}$$

$$\phi(E, \iota_N, \alpha) = (\beta_E(E), \beta_E \circ \iota_N, \alpha')$$

where $\beta_E\colon E \to \beta_E(E) =: E/K(E)$ *and* $\alpha'(\zeta) = \beta_E(a)$ *where* $pa = \alpha(\zeta)$ *and* $\alpha'(\mu_p) \subset K(\beta_E(E))$.

Call the above diagram Φ/ϕ, a morphism from

$$E_1(Nn)(v)/X_1(N)(v) \text{ to } E_1(N)(pv)/X_1(N)(pv).$$

Proof. Let U be the family of kernels of reduction and if $r \in p^{\mathbf{Q}} < 1$, $U[r]$ the subfamily of affinoid disks of radius r. If $s < p/(p+1)$, Frank has shown that there exists an $r < 1$ such that

$$F_s = \big(E_1[N][p] \cap U[r]\big)_{X_1(N)(s)}$$

is the family K_s of canonical subgroups over $X_1(N)(s)$.

Lemma. *F_s is finite over $X_1(N)(s)$.*

Proof. Frank showed that $K(E)$ equals the zero locus of $z^p - t_{can}(E)z$. Using Weiersträss preparartion (Theorem 5.2..2/1) one sees that t_{can} is a locally analytic function on $X_1(N)(s)$.

Now use Stein factorization (Theorem 9.6.2/5 of [BGR]).

From this we get a morphism

$$\Gamma/\gamma\colon E_1(N)(v)/X_1(N)(v) \to E_0(N)(pv)/X_0(N)(pv).$$

We have a section of order p, $\sigma\colon X_1 \to E_1$. Define $\tau\colon X_1(pv) \to E_1(pv)$ by

$$\tau(X_1(pv)) = \Gamma\big(p^{-1}\sigma(X_1) \cap \Gamma^{-1}(K_0(pv))\big)$$

[BGR] Bosh, S., U. Güntzer and R. Remmert, *Non-Archimedian Analysis*, Springer-Verlag, (1984).

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Lecture 17

Frobenius

If $n \geq 0$ and $v < p/p^n(p+1)$ and E corresponds to a point in $X_1(N)(v)$, there exists a unique cyclic subgroup of E, $K_n(E)$, of order p^{n+1} such that

$$K_0(E) = K(E), pK_n(E) = K_{n-1}(E) \text{ and } K_n(E)/K(E) = K_{n-1}\big(E/K(E)\big).$$

Theorem. *Suppose $N > 4$ and $v < 1/(p+1)$. There is a commutative diagram of rigid morphisms;*

$$\begin{array}{ccc} E_1(N)(v) & \xrightarrow{\Phi} & E_1(N)(pv) \\ \downarrow & & \downarrow \\ X_1(N)(v) & \xrightarrow{\phi} & X_1(N)(pv) \end{array}$$

$$\phi(E, \iota_N, \alpha) = (\beta_E(E), \beta_E \circ \iota_N, \alpha')$$

where $\beta_E \colon E \to \beta_E(E) =: E/K(E)$ and $\alpha'(\zeta) = \beta_E(a)$ where $a \in K_1(E)$ and $pa = \alpha(\zeta)$.

Proof. Let U be the family of kernels of reduction and if $r \in p^{\mathbf{Q}} < 1$, $U[r]$ the subfamily of affinoid disks of radius r. If $s < p/(p+1)$, there exists an $r < 1$ such that

$$K_s = \big(E_1[N][p] \cap U[r]\big)_{X_1(N)(s)}$$

is the family of canonical subgroups over $X_1(N)(s)$.

Is K_s finite over $X_1(N)(s)$?

Proof. Frank showed that, after choosing a good parameter, z, on E, $K(E)$ equals the zero locus of $z^p - t_{can}(E)z$. For x a supersingular point, T_s our parameter on A_x and $r \in \mathbf{Q}$, $0 < r < 1$, let $C_x(N)(r)$ be the circle in A_x of points y such that $v(T_x(y)) = rw_x$. Using

Weierstrass Preparation ([BGR] Theorem 5.2.2/1). *Suppose* $F(X,Y) = \sum_{n\geq 0} a_n(X)Y^n \in K\langle X,Y\rangle$, $a_d(X)$ *is a unit and* $|a_d| \geq |a_n|$ *for all* n *with strict inequality for* $n > d$. *Then there exists a unique monic polynomial of degree* d, $P(X,Y)$, *in* $R\langle X\rangle[Y]$ *and* $U(X,Y) \in K\langle X,Y\rangle^*$ *such that* $F(X,Y) = P(X,Y)U(X,Y)$.

one sees that t_{can} is analytic on every residue disk in $X_1(N)(0)$ or $C_x(N)(r)$ if $0 < r < p/(p+1)$.

Theorem (Proposition 6.3.2/1 of [BGR]). *If* $f\colon X \to Y$ *is a morphism of reduced affinoids and* $\tilde{f}$ *is finite, then* f *is finite.*

We get a (homo)morphism

$$\Gamma/\gamma\colon E_1(N)(v)/X_1(N)(v) \to E(N,p)(v)/X(N,p)(v).$$

Pick a p-th root of unity. Then we have a section of order p, $\sigma\colon X_1 \to E_1$. Define $\tau\colon X_1(pv) \to E_1(pv)$ by

$$\tau(X_1(pv)) = \Gamma\big(p^{-1}\sigma(X_1) \cap \Gamma^{-1}(K_0(pv))\big).$$

[BGR] Bosh, S., U. Güntzer and R. Remmert, *Non-Archimedian Analysis*, Springer-Verlag, (1984).

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Lecture 18

Review and improvements

Let $X(n,p)$ be the modular curve whose points correspond to triples $(E.\iota, C)$ where $\iota: \mu_N \to E$ is an embedding and C is a subgroup of order p. Then $X(N,p) = W_0(N) \cup W_\infty(N)$. $W_\infty \cap W_0 = \bigcup_s A_s$. Suppose $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to Z(N,p) =: Z(n,p)(0) =: W_\infty(N) - W_0(N)$. For $1 > v > 0$ let $Z(N,p)(v)$ be the set of $x \in W_\infty$, $x \in Z(N,p)$ or $x \in A_s$ for some s and $v(T_s(x)) \le rw_s$. We can also well define $Z_1(N)(v)$ for $0 \le v < 1$,

If $n \ge 0$ and $v < p/p^n(p+1)$ and E corresponds to a point in $X(N,p)(v)$, there exists a unique cyclic subgroup of E, $K_n(E)$, of order p^{n+1} such that $K_0(E) = K(E), pK_n(E) = K_{n-1}(E)$ and$K_n(E)/K(E) = K_{n-1}\big(E/K(E)\big)$.

Moduli problems See Katz–Mazur

Let $\mathcal{E}$ be the category of elliptic curves over rigid spaces. A moduli problem $\mathcal{P}$ on $\mathcal{E}$ is a functor from $\mathcal{E}$ to sets. $\mathcal{P}$ is said to be representible if there is an object $E(\mathcal{P})/M(\mathcal{P})$ in $\mathcal{P}$ such that for every $E/S \in \mathcal{E}$

$$\mathcal{P}(E/S) = Hom_{\mathcal{E}}(E/S, E(\mathcal{P})/M(\mathcal{P})).$$

If $N > 4$ anid $(N,p) = 1$, the moduli problem E/S goes to pairs (ι, C) where $\iota: S \times \mu_N \to E/S$ is an embedding C is a subgroup of E/S flat over S of rank p is representble by a pair $E(N,p)/X(N,p)$.

Frobenius

Theorem. *Suppose $N > 4$ and $v < 1/(p+1)$. There is a commutative diagram of rigid morphisms;*

$$\begin{array}{ccc} E(N,p)(v) & \xrightarrow{\Phi} & E(N,p)(pv) \\ \downarrow & & \downarrow \\ X(N,p)(v) & \xrightarrow{\phi} & X(N,p)(pv) \end{array}$$

$$\phi(E,\iota,C) = (E/C, \beta_E \circ \iota, C')$$

where $\beta_E\colon E \to E/K(E)$ and $C' = K(E/C)$ (which exists).

Lemma. *If $v < p/(p+1)$ there exists a unique section of $X(N,p) \to X_1(N)$. Moreover, $s(X_1(N))(v) = X(N,p)(v)$.*

Proof.

Let V be the family of subgroups o order p of $E(N,p)/X(N,p)$.

Let U be the family of kernels of reduction in $E_1(N)$ and if $r \in p^{\mathbf{Q}} < 1$, $U[r]$ the subfamily of affinoid disks of radius r. If $v < p/(p+1)$, there exists an $r < r' < 1$ such that

$$K_v = \big(E_1[N][p] \cap U[t]\big)_{X_1(N)(v)}$$

with $t = r$ or r' is the family of canonical subgroups over $X_1(N)(v)$. In particular $(K_v)_\infty = \mu_p$.

Proposition. $s^*V = K_v$.

Proof. Claim: $K_v|_{Z_1(0)}/Z_1(0)$ is finite.

Let $\pi_2\colon X(N,p) \to X_1(N)$ be $(E,\iota,C) \mapsto (E/C, \iota \bmod C)$. Now we can define ϕ as $s \circ \pi_2 \circ s$.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 19

Notation

In $X(n,p)$ for $0 \leq v < 1$, we have subspaces $Z(N,p)(v)$ defined as follows: $Z(N,p) =: Z(n,p)(0) =: W_\infty(N) - W_0(N)$. Suppose $T_s : A_s \cong A(p^{-w_s}, 1)$ such that $|T_s(x)| \to 1$ as $x \to$. Then if $1 > v > 0$, $Z(N,p)(v)$ be the set of $x \in W_\infty$, $x \in Z(N,p)$ or $x \in A_s$ for some s and $v(T_s(x)) \leq rw_s$. We can also well define $Z_1(N)(v)$ for $0 \leq v < 1$,

Frobenius

Theorem. *Suppose $N > 4$ and $v < 1/(p+1)$. There is a commutative diagram of rigid morphisms;*

$$\begin{array}{ccc} E(N,p)(v) & \xrightarrow{\Phi} & E(N,p)(pv) \\ \downarrow & & \downarrow \\ X(N,p)(v) & \xrightarrow{\phi} & X(N,p)(pv) \end{array}$$

$$\phi(E, \iota, C) = (E/C, \beta_E \circ \iota, C')$$

where $\beta_E\colon E \to E/K(E)$ and $C' = K(E/C)$ (which exists).

Proof.

Proposition. *There exists a section t of $X(N,p) \to X_1(N)$ over $Z_1(N)(v)$ if $v < p/(p+1)$. Moreover, in this case, $t(Z_1(N)(v)) = Z(N,p)(v)$.*

We will use

Lemma. *If $f\colon X \to Y$ is a morphism of reduced curves over K and $U \subset X$ and $V \subset Y$ are affinoid subdomains such that $f(U) \subseteq V$ and $\bar{f}\colon \bar{U} \to \bar{V}$ is an isomorphism. Then there exists a strict neighborhood Z of V in Y and a section $Z \to X$ of F.*

and

Lemma. *If* $f\colon A(p^{-1},1) \to B(0,1)$ *is a finite morphism of degree* $p+1$ *and* $\deg_{A[r]} f = 1$ *for* r *near 1, then there exist a section of* f *on* $A(p^{-\frac{p}{p+1}},1)$

Proof of proposition

Our ϕ will be $t \circ \pi_2$ where All we have to show is that $t(A,\alpha) = (A,\alpha,K(A))$

Let V be the family of subgroups of order p of $E(N,p)/X(N,p)$.

Let U be the family of kernels of reduction in $E_1(N)$ and if $r \in p^{\mathbf{Q}} < 1$, $U[r]$ the subfamily of affinoid disks of radius r. If $v < p/(p+1)$, there exists $r < r' < 1$ such that

$$K_v = \big(E_1[N][p] \cap U[t]\big)_{X_1(N)(v)}$$

with $t = r$ or r' is the family of canonical subgroups over $X_1(N)(v)$. In particular $(K_v)_\infty = \mu_p$.

Proposition. $s^*V = K_v$.

Proof. Claim: $K_v|_{Z_1(0)}/Z_1(0)$ is finite.

A (little) higher level

Let $X_1(Np)(v)$ be the inverse image of $X(N,p)(v)$ under the forgetful map f.

Theorem. *Suppose* $N > 4$ *and* $v < 1/(p+1)$. *There is a commutative diagram of rigid morphisms;*

$$\begin{array}{ccc} E_1(Np)(v) & \xrightarrow{\Phi} & E_1(Np)(pv) \\ \downarrow & & \downarrow \\ X_1(Np)(v) & \xrightarrow{\phi} & X_1(Np)(pv) \end{array}$$

$$\phi(E,\iota,\alpha) = (\beta_E(E), \beta_E \circ \iota, \alpha')$$

where $\beta_E\colon E \to \beta_E(E) =: E/K(E)$ *and* $\alpha'(\zeta) = \beta_E(a)$ *where* $a \in K_1(E)$ *and* $pa = \alpha(\zeta)$.

Proof. We have, $\begin{array}{ccc} E_1(Np)(v) & \stackrel{\Phi\circ F}{\longrightarrow} & E(N,p)(pv) \\ \downarrow & & \downarrow \\ X_1(Np)(v) & \stackrel{\phi\circ f}{\longrightarrow} & X(N,p)(pv) \end{array}$ so all we need is a rigid map $\beta: X_1(Np)(pv) \to K_{pv}$ compatible with tihe other maps, of order p.

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Lecture 20

Frobenius "finished"

Last time we proved,

Proposition. *There exists a unique section t of $X(N,p) \to X_1(N)$ over $Z_1(N)(v)$ if $v < p/(p+1)$. Moreover, in this case, $t(Z_1(N)(v)) = Z(N,p)(v)$.*

Also,

Lemma. *If $\pi_2: X(N,p) \to X_1(N)$ is the map $(E,\iota,C) \to (E/C, \iota \bmod C)$ then $\pi_2(Z(N,p)(v)) = Z_1(N)(pv)$.*

Proof.

Our ϕ will be $t \circ \pi_2$. All we have to show is that $t(A,\alpha) = (A,\alpha,K(A))$.

Let V be the family of subgroups of order p of $E(N,p)/X(N,p)$ and U the family of kernels of reduction in $E_1(N)$ and if $r \in p^{\mathbf{Q}} < 1$, $U[r]$ the subfamily of affinoid disks of radius r. If $v < p/(p+1)$, there exists $r < r' < 1$ such that

$$K_v = \big(E_1[N][p] \cap U[t]\big)_{X_1(N)(v)}$$

with $t = r$ or r' is the family of canonical subgroups over $X_1(N)(v)$. In particular $(K_v)_\infty = \mu_p$.

Proposition. $t^*V = K_v$.

Proof. Claim: $K_v|_{Z_1(N)}/Z_1(N)$ is finite. Fix a residue class U. Using what Frank showed K_U equals the zero locus of $z^p - t_{can} z$ for a some good family of parameters z at the origin on $E_1(N)_U$ and some invertible function t_{can} on U.

I am leaving the details of Φ/ϕ on $E_1(Np)/X_1(Np)$ as an exercise. One can also deal with $N \leq 4$.

The U operator

For $v \geq 0$, let $M_k(N,v) = \omega^k(X_1(N)[v])$. Now, $M_k(N,v)$ has a natural structure as a Banach space over K and when $0 \leq v < \frac{p}{p+1}$ there is an operator on this space, $U_{(k)}$. Let $F \in M_k(N,v)$, $v < \frac{p}{p+1}$. Suppose $x \in X_1(N)[v]$ corresponds to (E, ι_n, α). Then, pointwise,

$$U_{(k)}(F)(x) = \frac{1}{p} \sum_{\phi(y)=x} \check{\beta}_y^* F(y).$$

$$\sum a_n q^n \to \frac{1}{p} \sum a_{np} q^n.$$

Why is this analytic?

First, $U_{(0)} = \frac{1}{p} \operatorname{Tr}_{\phi}$. Now recall, we have a weight one Eisenstein series E on $X_1(p)$ which we can consider as an element of $M_1(N,v)$. Considered as a form ν_E on $E_1(N,p)(v)$, on $(\mathbf{G}_m/q^{\mathbf{Z}}, \iota_{Np})$ it is

$$E(q)\frac{\mathrm{d}T}{T}.$$

Now $\Phi^*\nu_E$ has q-expansion $pE(q^p)\frac{\mathrm{d}T}{T}$. Let E^ϕ be the section of $M_1(N,v)$, $v < 1/(p+1)$, with q-expansion $E(q^p)$. For v close enough to 1, $1/E^\phi \in M_{-1}(N,v)$. Then,

$$U_{(k)}F = E^k U_0(F/(E^\phi)^k).$$

$U_{(k)}$ is compact.

Proof.

N $\leq$ 4

Suppose $A, B \in \mathbf{Z}$, $A, B > 4$, $(AB, p) = 1$ and $(A, B) = N$, we define $M_{N,k}(v)$ with the intersection of the images via the forgetful maps of $M_{A,k}(v)$ and $M_{B,k}(v)$ in $M_{LCM(A,B),k}(v)$. One has to show that these are all canonically isomorphic.

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Lecture 21

"Continuity" explained

Suppose $v < p/(p+1)$. First $W = t^*V$ is finite over $X_1(N)(v)$ as is each connected component. Finally, if $r < r' < 1$ are such that

$$K_v = \big(E_1[N][p] \cap U[t]\big)_{X_1(N)(v)}$$

with $t = r$ or r', $U[r]$ and $E_1(N) - U[r']$ are disconnected and

$$W = \big(U[r] \cap W\big) \cup \big((E_1(N) - U[r']) \cap W\big).$$

The U operator

For $v \geq 0$, let $Z_1(Np)(v) = \pi^{-1}(Z(N,p)(v))$ and let $M_k(N,v) = \omega^k(Z_1(Np))$. Now, $M_k(N,v)$ has a natural structure as a Banach space over K and when $0 \leq v < \frac{p}{p+1}$ there is an operator on this space, $U_{(k)}$. Let $F \in M_k(N,v)$, $v < \frac{p}{p+1}$. Suppose $x \in Z_1(Np)(v)$ corresponds to (E,ι,α). Then, pointwise,

$$U_{(k)}(F)(x) = \frac{1}{p}\sum_{\phi(y)=x} \check{\beta}_y^* F(y),$$

where $\beta_y \colon E_y \to E_y/K(E_y) = E$. Also, if $E = \mathbf{G}_m/q^{\mathbf{Z}}$ and α is the natural embeddings and $F(x) = (\sum a_n(\iota)q^n)(\frac{\mathrm{d}T}{T})^k$ then

$$U_{(k)}(F)(x) = \big(\sum a_{np}(\iota^{p^{-1}})q^n\big)\big(\frac{\mathrm{d}T}{T}\big)^k.$$

Why is this analytic?

First, $M_0((N,v)) = A(Z_1(Np)(v))$ and $U_{(0)}$ is

$$\frac{1}{p}\ \mathrm{Tr}\,_\phi|^{Z_1(Np)(\frac{v}{p})}_{Z_1(Np)(v)} \circ \ \mathrm{Res}^{Z_1(Np)(v)}_{Z_1(Np)(\frac{v}{p})}.$$

Now recall, we have a weight one Eisenstein series E on $X_1(p)$,

$$E(q) = 1 + \frac{2}{L_p(0,\mathbf{1})}\sum_{n\geq 1}\Bigg(\sum_{\substack{d|n\\(d,p)=1}} \tau^{-1}(d)\Bigg)q^n,$$

which we can consider as an element ν_E of $M_1(N,v)$. Now $\Phi^*\nu_E$ has q-expansion $pE(q^p)\frac{\mathrm{d}T}{T}$. Let E^ϕ be the section of $M_1(N,v)$, $v < 1/(p+1)$, with q-expansion $E(q^p)$. For v close enough to 0, we showed $1/E^\phi \in M_{-1}(N,v)$ (in fact, $v < 1/(p+1)$ is enough). Then,

$$U_{(k)}F = E^k U_0(F/(E^\phi)^k).$$

$U_{(k)}$ is compact.

Proof.

w $M_k(N,v)$ is pretty big and one can show $\det(1-TU_{(k)})$ has infinitely many zeroes. However,

Theorem. *If $F \in M_k(N,v)$ is an eigenvector of $U_{(k)}$ with eigenvaluew α and $v(\alpha) < k-1$ then F is classical.*

(The proof is now on the web.)

$\mathbf{N \leq 4}$

Suppose $A, B \in \mathbf{Z}$, $A, B > 4$, $(AB,p) = 1$ and $(A,B) = N$, we define $M_k(N,v)$ to be the intersection of the images of $M_{A,k}(v)$ and $M_{B,k}(v)$ in $M_{AB,k}(v)$. One has to show that these are all canonically isomorphic.

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Lecture 22

"Another" definition of U

First, $M_0((N,v)) = A(Z_1(Np)(v))$ and $U_{(0)}$ is

$$\frac{1}{p}\,\mathrm{Tr}\,\phi|^{Z_1(Np)(\frac{v}{p})}_{Z_1(Np)(v)} \circ \mathrm{Res}^{Z_1(Np)(v)}_{Z_1(Np)(\frac{v}{p})}.$$

Recall, we have a weight one Eisenstein series E on $X_1(p)$,

$$E(q) = 1 + \frac{2}{L_p(0,\mathbf{1})}\sum_{n\geq 1}\Bigg(\sum_{\substack{d|n\\(d,p)=1}} \tau^{-1}(d)\Bigg)q^n,$$

which we can consider as an element ν_E of $M_1(N,v)$. Now $\Phi^*\nu_E$ has q-expansion $pE(q^p)\frac{\mathrm{d}T}{T}$. Let E^ϕ be the section of $M_1(N,v)$, $v < 1/(p+1)$, with q-expansion $E(q^p)$. For v close enough to 0, we showed $1/E^\phi \in M_{-1}(N,v)$ (in fact, $v < 1/(p+1)$ is enough). Then, define

$$U_{(k)}F = E^k U_0(F/(E^\phi)^k).$$

$U_{(k)}$ is compact.

Proof.

Now $M_k(N,v)$ is pretty big and one can show $\det(1 - TU_{(k)})$ has infinitely many zeroes. However,

Theorem. *If $F \in M_k(N,v)$ is an eigenvector of $U_{(k)}$ with eigenvaluew α and $v(\alpha) < k-1$ then F is classical.*

(The proof is "Classical and Overconvergent Forms" which is now on the web.)

The U operator in families

We defined $U_{(k)}(F) = E^k U_0(F/(E^\phi)^k)$. Let $\mathcal{E} = E/E^\phi$. This is a function close to 1 on $Z_1(N)(v)$ for v small. In fact, for $v < 1/(p+1)$,

$$|\mathcal{E} - 1| \leq p^{(p+1)v-1}.$$

So if u_k is the operator on $M_{(0)}(N, v)$, $G \mapsto U_{(0)}(G \cdot \mathcal{E}^k)$,

$$E^{-k} U_{(k)}(F) = u_k(F/E^k),$$

but since $\mathcal{E}$ is close to 1, u_k makes sense for any $k \in \mathbf{C}_p$ which is not too big. Suppose $|s| < |\pi/p|$ then $\exists v$ such that

$$|\mathcal{E} - 1| < |\pi/s|$$

on $Z_1(Np)(v)$ this means

$$\mathcal{E}^s = 1 + (\mathcal{E} - 1) + \cdots + \binom{s}{n}(\mathcal{E} - 1)^n + \cdots$$

converges on $Z_1(Np)(v)$.Thus, if $r \in p^{\mathbf{Q}} < |\pi/p|$ and $|\mathcal{E} - 1| < |\pi|/r$ on $Z_1(Np)(v)$, we get an operator $\mathcal{U}_{r,v}$ over $A(B[0,r])$ on $M(r,v) =: A(B[0,r] \times Z_1(Np)(v))$ which is

$$(U_{(0)} \otimes 1) \circ m_{\mathcal{E}^s}.$$

We know this operator is compact. Thus we get characteristic series $P_{r,v}(T)$ for every (r, v), as above. But they are all "the same."

Theorem. *There is a unique rigid analytic function $P(s,T) = P_N(s,T)$ on $\mathcal{B}^* \times \mathbf{C}_p$ defined over $\mathbf{Q}_p$, i.e. $P(s,T)$ is a power series over $\mathbf{Q}_p$ in s and T, which converges for $|s| < |\pi/p|$, such that for $k \in \mathbf{Z}$ and $v \in \mathbf{Q}$ such that $0 < v < p/(p+1)$,*

$$P(k,T) = \det(1 - TU_{(k)}|M_k(v)).$$

$\mathbf{N \leq 4}$

Suppose $A, B \in \mathbf{Z}$, $A, B > 4$, $(AB, p) = 1$ and $(A, B) = N$, we define $M_k(N, v)$ to be the intersection of the images of $M_{A,k}(v)$ and $M_{B,k}(v)$ in $M_{AB,k}(v)$. One has to show that these are all canonically isomorphic.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 23

Classical forms

Suppose $F(q) = \sum_{n\geq 0} a_n q^n$ is the q-expansion of a normalized weight k eigenform on $X_1(N)$ of character χ. Associated to F there are (at most) two eigenforms (oldforms) on $X(N,p)$ whose U_p eigenvalues are the roots of

$$X^2 - a_p X + \chi(p)p^{k-1}$$

The Spectral Curve

Theorem. *There is a unique rigid analytic function $P(s,T) = P_N(s,T)$ on $\mathcal{B}^* \times \mathbf{C}_p$ defined over $\mathbf{Q}_p$, i.e. $P(s,T)$ is a power series over $\mathbf{Q}_p$ in s and T, which converges for $|s| < |\pi/p|$, such that for $k \in \mathbf{Z}$ and $v \in \mathbf{Q}$ such that $0 < v < p/(p+1)$,*

$$P(k,T) = \det(1 - TU_{(k)}|M_k(v)).$$

Proof. Because "If $\phi\colon A \to B$ is a homomorphism of Banach algebras then $\phi^* E =: E \otimes_A B$ is orthonormizable over B and

$$P_{\phi^* L}(T) = \phi(P_L(T))."$$

our $P_{r,v}(T)$ is "independent of r." Now because ϕ is finite, if $p/(p+1) > v \geq v' \geq v/p$,

$$T_{v'}^{v'/p} \circ R_{v'/p}^{v'} = R_{v'}^{v} \circ T_v^{v/p} \circ R_{v/p}^{v'},$$

where $T =$ Tr and $R =$ Res. As

$$(T_v^{v/p} \circ R_{v/p}^{v'} \circ m_{\mathcal{E}^s}) \circ R_{v'}^{v} = T_v^{v/p} \circ R_{v/p}^{v} \circ m_{\mathcal{E}^s} = U_{r,v},$$

the "independence" of v follows from: "Suppose E_1 and E_2 are orthonormizable Banach modules over A. Suppose u is a compact homomorphism from E_1 to E_2 and $v\colon E_2 \to E_1$ is a continuous homomorphism. Then $P_{u\circ v}(T) = P_{v\circ u}(T)$."

Now $D =: (\mathbf{Z}/p\mathbf{Z})^*$ acts on $Z(N,p)(v)$ and

$$M(t,v) = \bigoplus_{\epsilon\in\hat{D}} M(t,v,\epsilon)$$

and

$$P(s,T) = \prod_{\epsilon\in\hat{D}} P_\epsilon(s,T)$$

where

$$P_\epsilon(s,T)|_{B[0,t]\times\mathbf{C}_p} = \det(1 - T\mathcal{U}_{t,v}|M(t,v,\epsilon)).$$

Thus we get an entire function on $\mathcal{W}^*\times\mathbf{C}_p$. Its zero locus is the fiber of the **spectral curve** of U over $\mathcal{W}^*$.

A Formula

Theorem. *Suppose $N\geq 4$. Then*

$$T\frac{d}{dT}P_N(T)/P_N(T) = \sum_{m\geq 1} A_m T^m$$

where A_m is the element of $\mathbf{Z}_p[[\mathbf{Z}_p]]\subset A(\mathcal{W}^*)$, expressed by the finite sum,

$$A_m = \sum_{\gamma\in W_{p,m}}\ \sum_{\mathcal{O}\in O_\gamma} h(\mathcal{O})B_N(\mathcal{O},\gamma)\cdot\frac{[\gamma]}{\gamma^2 - p^m}$$

where $B_N(\mathcal{O},\gamma)$ is the number of elements of $\mathcal{O}/N\mathcal{O}$ of order N fixed under multiplication by $\overline{\gamma}$.

For an order $\mathcal{O}$ in a number field, let $h(\mathcal{O})$ denote the class number of $\mathcal{O}$. If γ is an algebraic integer, let O_γ be the set of orders in $\mathbf{Q}(\gamma)$ containing γ. Finally, for m an integer, let $W_{p,m}$ denote the finite set of $\gamma\in\mathbf{Q}_p$ such that $\mathbf{Q}(\gamma)$ is an imaginary quadratic field, γ is an algebraic integer, $Norm_{\mathbf{Q}}^{\mathbf{Q}(\gamma)}(\gamma) = p^m$ and $v(\gamma) = 0$.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 24

"Review"

As always $(N,p) = 1$. $\mathcal{W}(N)$ is the rigid analytic space whose $\mathbf{C}_p$ points are continuous characters from $(\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$ into $\mathbf{C}_p^*$. $\mathcal{W}^*(N)$ is the open subspace of characters of the form $\chi \cdot \langle\langle\ \rangle\rangle^s$ where χ is a character on $(\mathbf{Z}/Np\mathbf{Z})^*$ and $|s| < |\pi/p|$. We call the corresponding spaces of character on $1 + p\mathbf{Z}_p$, $\mathcal{B}$ and $\mathcal{B}^*$. If $D(M) = Hom((\mathbf{Z}/M\mathbf{Z})^*, \mathbf{C}_p^*)$,

$$\mathcal{W}(N) = D(N) \times \mathcal{B} \ \text{ and } \mathcal{W}^*(N) = D(N) \times \mathcal{B}^*.$$

Also $\mathcal{B} \cong B(1,1)$ and $\mathcal{B}^* \cong B(0, p^{\frac{p-2}{p-1}})$. Now $(\mathbf{Z}/Np\mathbf{Z})^*$ acts on $Z_1(Np)(v)$ (by "diamond operators"). For each $v > 0$, $t > 0$ and $\chi \in D(Np)$ let $M(v,\chi)$ and $M(v,t,\chi)$ be the spaces of rigid analytic functions on $Z_1(Np)(v)$ and $B(0,t) \times Z_1(Np)(v)$ with character χ. These spaces are affinoids if $v \in \mathbf{Q}$ and $t \in p^{\mathbf{Q}}$. We have a compact U-operator on all these spaces $(U_{(0)} \otimes 1) \circ m_{\mathcal{E}^t}$ if v is sufficiently small ($< p/(p+1)$) and $t < p^{\frac{p-2}{p-1}}$.

Theorem. *There are unique rigid analytic functions $P_\chi(s,T)$ on $\mathcal{B}^* \times \mathbf{C}_p$ defined over $\mathbf{Q}_p$, such that for $k \in \mathbf{Z}$ and $v \in \mathbf{Q}$ such that $0 < v < p/(p+1)$,*

$$P_\chi(k,T) = \det(1 - TU_{(k)} | M_k(v,\chi)).$$

Let Q be the rigid function on $\mathcal{W}^*(N) \times \mathbf{C}_p$ defined by $Q(\chi, s, z) = P_\chi(s,z)$, for $\chi \in D(Np)$, $s \in \mathcal{B}^*$ and $z \in \mathbf{C}_p$.

Theorem. *Q extends analytically to a function on $\mathcal{W}(N) \times \mathbf{C}_p$.*

See "On the coefficients of the characteristic series of the U-operator," which is now on the course webpage.

The key object(s) to consider is the q-expansion $\mathbf{E}(q)$ which at $\kappa \in \mathcal{B}$ is

$$E_\kappa(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n\geq 1} \sigma_\kappa^*(n) q^n.$$

Proposition. *There is an an analytic function $\mathbf{E}_p$ on a "strict" neighborhood of $Z_{\mathcal{B}} =: \mathcal{B} \times Z_1(p)$ in $\mathcal{B} \times X_1(p)$ with q-expansion at κ $E_\kappa(q)/E_\kappa(q^p)$ bounded by 1 on $Z_{\mathcal{B}}$.*

We may now use the operator $U =: (U_{(0)} \otimes 1) \circ m_{\mathbf{E}_p}$ on $M^\dagger(N)$, the space of q-expansions $F(q)$ with coeficients in $A(\mathcal{B})$ such that $F(q)/\mathbf{E}(q)$ is the q-expansion of an analytic function which converges on a "strict" neighborhood of $Z_{\mathcal{B}} =: \mathcal{B} \times Z_1(pN)$ in $\mathcal{B} \times X_1(pN)$.

We also get to define: A series $\sum_{n\geq 1} a_n q^n$, $a_n \in K$ is "the q-expansion of an OC form of **type** $\alpha = \chi \cdot \kappa$" if $F(q)/E_\kappa(q)$ is the q-expansion of an OC function on $Z_1(Np)$ with charater χ. When $\kappa(a) = a^k$, $k \in \mathbf{Z}$, $F(q)$ will be the q-expansion of an OC form of weight k and character $\chi \cdot \omega^{-k}$.

Hecke Operators

First, if $l \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, $\kappa \in \mathcal{B}$,

$$\big(F|\langle l\rangle^*(q)\big)|_\kappa = \kappa(\langle\langle l\rangle\rangle) E_\kappa(q) \big(\frac{F|_\kappa}{E_\kappa}|\langle l\rangle\big)(q).$$

When $\kappa(a) = a^j$, $k \in \mathbf{Z}$,

$$(F|\langle l\rangle^*)_k = l^k F|_k \langle l\rangle.$$

For prime ℓ, let ψ_ℓ be the operator on $A(\mathcal{B})[[q]]$

$$\psi_\ell(\sum_n a_n q^n) = \sum_n a_{n\ell} q^n.$$

Proposition. *For each prime number l there is a unique continuous operator $T(\ell)$ on $M^\dagger(N)$ such that, for $F \in M^\dagger(N)$, when $\ell = p$,*

$$(F|T(p))|_\kappa = E_\kappa \cdot U\big(\frac{F_\kappa}{E_\kappa}\big), \quad \text{when } l|N \quad F|T(\ell)(q) = \psi_\ell(F(q))$$

and when $l \not| Np$

$$(F|T(l))(q) = \psi_\ell(F(q)) + \ell^{-1}(F|\langle\ell\rangle^*)(q^\ell).$$

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 25

Hecke opertors

We now have the operator $\mathbf{T}(p)$ on $M^\dagger(N)$ which is

$$F(q) \mapsto \mathbf{E}(q)(U_0 \otimes 1)(\frac{F(q)}{\mathbf{E}(q^p)}).$$

If κ is arithmetic, $\kappa = \psi \cdot \langle\langle\ \rangle\rangle^k$ where ψ is a character on $1 + p\mathbf{Z}_p$ of finite order and $k \in \mathbf{Z}$, $F_\kappa(q)$ is the q-expansion of weight k modular form G, of tame level N, and

$$(F(q)|\mathbf{T}(p))_\kappa = (G|T(p))(q).$$

If $\ell \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, $\kappa \in \mathcal{B}$,

$$\big(F|\langle l\rangle^*(q)\big)|_\kappa = \kappa(\ell)E_\kappa(q)(\frac{F_\kappa}{E_\kappa}|\langle\ell\rangle)(q).$$

When κ is arithmetic, as above,

$$(F(q)|\langle\ell\rangle^*)_\kappa = \ell^k F|_\kappa\langle\ell\rangle(q).$$

For prime ℓ, let ψ_ℓ be the operator on $A(\mathcal{B})[[q]]$

$$\psi_\ell(\sum_n a_n q^n) = \sum_n a_{n\ell} q^n.$$

Proposition. *For each prime number ℓ there is a unique continuous operator $\mathbf{T}(\ell)$ on $M^\dagger(N)$ such that, for $F \in M^\dagger(N)$, when $\ell|Np$*

$$F|T(\ell)(q) = \psi_\ell(F(q))$$

and when $\ell \not| Np$

$$(F|T(l))(q) = \psi_\ell(F(q)) + \ell^{-1}(F|\langle\ell\rangle^*)(q^\ell).$$

Suppose $\ell \neq p$. For any prime ℓ we have a function $\mathbf{E}_\ell$ on a strict neighborhood of $\mathcal{W} \times Z(\ell)$ with q-expansion $\mathbf{E}(q)/\mathbf{E}(q^\ell)$.

Proof. Let $M = Np$. We first look at $X(M;\ell)$ the modular xurve which classifies triples (E, α_M, C) where $|C| = \ell$ and Image $(\alpha_M) \cap C = 0$ and define $Z(M;\ell)(v)$. We have

$$g_1, g_2 \colon Z(M;\ell)(v) \to Z_1(M)(v)$$

and

$$\ell^{-1} \operatorname{Tr}_{g_1} \circ g_2^*$$

on $A^\dagger(M)$ which on q-expansions if $\ell|N$ is ψ_ℓ. Now

$$\psi_\ell(F(q)) = \mathbf{E}(q)\psi_\ell\left(\frac{F(q)}{\mathbf{E}(q)} \cdot \mathbf{E}_\ell(q)\right).$$

We define $\mathbf{T}(n)$, for positive integers n by:

$$\sum_{n \geq 1} \frac{\mathbf{T}(n)}{n^t} = \prod_{l|Np} (1 - \mathbf{T}(\ell)\ell^{-t})^{-1} \prod_{(\ell,Np)=1} (1 - \mathbf{T}(\ell)\ell^{-t} + \langle \ell \rangle^* \ell^{-1-2t})^{-1},$$

where the products are over primes ℓ.

Let $\mathbf{T}$ be the ring generated by the operators $\mathbf{T}(\ell)$ and $\langle d \rangle^*$, $(d, Np) = 1$.

We will use Q, $\mathbf{T}$ and Riesz theory to build the eigencurve.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 26

Building the Eigencurve

We now have a space $M^\dagger(N)$ of "families of q-expansions of overconvergent forms of tame level N." It is a module over $A(\mathcal{B})$ and we have an $A(\mathcal{B})$ algebra $\mathbf{T} =: \mathbf{T}(N)$ generated by operators $\langle d\rangle^*$, $d \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, and $\mathbf{T}(n)$, $n_{>0} \in \mathbf{Z}$. If X is any affinoid in $\mathcal{B}$, $M_X =: M^\dagger(N)\hat{\otimes}A(X)$ is the direct limit of Banach submodules N_n on which $\mathbf{T}(p)$ acts compactly. In fact, for each character $\chi \in D(Np) = \big((\mathbf{Z}/Np\mathbf{Z})^*\big)\hat{}$ there is an power series $P_\chi(s,T) \in A(\mathcal{B})[[T]]$ whose restiction to X is $\det(1-\mathbf{T}(p)|N_n(\chi))$.

Let

$$S_\chi = \{(b,z) \in \mathcal{B}\times\mathbf{C}_p : P_\chi(s,z) = 0\}.$$

Fix χ and let $S = S_\chi$ and $P = P_\chi$.

Lemma. *Suppose $X \subset \mathcal{B}$ and $Y \subset S_X$ are affinoid subdomains, Y and $S - Y$ are disconnected and Y is finite over X. Then there exists $R(T) \in A(X)[T]$ and $Q(T) \in A(X)[[T]]$ such that R is monic Q is entire, $R(0)$ is a unit, $Q(0) = 1$, $(R^*(T), Q(T)) = 1$,*

$$P(T) = R^*(T)Q(T)$$

and Y is the zero locus of R. ($A(Y) = A(X)[T]/R(T)$.)

Riesz theory tells us

$$M_X = N(Y) \oplus F(Y),$$

where $N(Y)$ is projective of rank $d_Y =: \deg R$, $R^*(\mathbf{T}(p))$ annihilates $N(Y)$ and is invertible on $F(Y)$. In particular, $\mathbf{T}$ acts on $N(Y)$.

Let $\mathbf{T}_Y =: \mathbf{T}_Y(N)$ denote the image of $\mathbf{T}$ in $\mathrm{End}_{A(X)}(N(Y))$.

Proposition. $\mathbf{T}_Y$ *is finite of degree* d_Y *over* $A(X)$.

Proof. Define

$$\langle\ ,\ \rangle: \mathbf{T}_Y \times N(Y) \to A(X) \quad by$$
$$\langle h, F\rangle = a_1(F|h).$$

This pairing is perfect. The key point is that

$$\langle \mathbf{T}(n), F\rangle = a_n(F).$$

Thus we get an affinoid $E_Y(N)$ finite over X, $\kappa: E_Y(N) \to X$.

Gouvêa-Mazur

Suppose $\chi \in D(Np)$, $\rho \in \mathcal{B}$, $r_{<1} \in p^{\mathbf{Q}}$ and $\alpha_{\geq 0} \in Q$. First

$$Y =: \{(\tau, z) \in S: |\tau(1+p) - \rho(1+p)| \leq r, v(z) = \alpha\}$$

is an affinoid subdomain of S quasi-finite over $X = B[\rho, r]$. In fact, if r is small enough it is finite and Y is disconnected from $S - Y$.

Proposition. *Suppose,* $\rho = \psi \cdot \langle\langle\ \rangle\rangle^k$ *where* ψ *has finite order,* $k \in \mathbf{Z}$ *and* $\alpha < k-1$. *Then the degree of* $E_Y(N)$ *over* X *equals the number of classical eigenforms of tame level* N *and character* $\chi \cdot \psi \cdot \omega^{-k}$. *Moreover, if* $x \in E_Y(N)(L)$, $\kappa(x) = \psi \cdot \langle\langle\ \rangle\rangle^j \in X$, $j \in \mathbf{Z}$, $\alpha < j-1$.

$$\sum_{n\geq 1} \mathbf{T}(n)(x)q^n$$

is the q*-expansion of a classical eigenform (minus its constant term), of tame level* N *of character* $\chi \cdot \psi \cdot \omega^{-j}$.

Conjecture. *One can take* $r = p^{-\alpha}$.

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 27

Comments on Last Time

First, inside $M^\dagger(N)$ we have $C^\dagger(N)$ which are the elements with constant term 0. $\mathbf{T}$ acts on $C^\dagger$.

(Recall, we've fixed χ.) Suppose X is an affinoid in $\mathcal{B}$ and Y is a "clopen" affinoid in S_X finite of degree d over X. Then we got a projective module $N(Y)$ of rank d in M_X and we defined $\mathbf{T}_Y$ to be the image of $\mathbf{T}$ in $\mathrm{End}_X(N(Y))$.We started proving,

Proposition. *$\mathbf{T}_Y$ is finite of degree d over $A(X)$.*

Let $N^0(Y) = C_X \cap N(Y)$. Then $N^0(Y)$ is projective of rank $d-\delta$ where $\delta = 1$ or 0. Also let $\mathbf{T}_Y^0$ the image of $\mathbf{T}$ in $\mathrm{End}_X(N^0(Y))$.

We proved, $\mathbf{T}_Y^0$ is finite of degree $d-\delta$ over $A(X)$.

$$0 \to I \to \mathbf{T}_Y \to \mathbf{T}_Y^0 \to 0 \text{ and } 0 \to N^0(Y) \to N(Y) \to J \to 0$$

and we have perfect pairing $(i,j) \mapsto a_0(j|i)$.

Also the conjecture stated in the last lecture is true by Hida when $\alpha = 0$.

Glueing

For every $Y \subset S = S_\chi$ such that Y is finite over $X \subset \mathcal{B}$ and "clopen" in S_X we found an affiniod $E_Y(N)$ which is finite over X and such that $A(E_Y(N)) = \mathbf{T}_X(N)$. Let $\mathcal{C}$ be the collection of these Y.

Proposition. *S is admissibly covered by $\mathcal{C}$.*

This means if $A(S) =: A(\mathcal{B})[[T]]^{entire}/P(T)$ and $h\colon A(S) \to A$ is a continuous homomorphism into an affinoid algebra. There exists a finite collection $Y_i \in \mathcal{C}$ such that if $f \in A(S)$ vanishes on all the Y_i, $h(f) = 0$.

Proposition. *Suppose* $Y_1, Y_2 \in \mathcal{C}$. *(i)* $Y_3 =: Y_1 \cap Y_2 \in \mathcal{C}$. *(ii)* $E_{Y_3}(N)$ *is naturally a subdomain of* $E_{Y_1}(N)$ *and* $E_{Y_2}(N)$.

Proof.

Now we make E_χ is $\coprod_{Y \in \mathcal{C}} E_Y(N)$ with the identifications $\beta_1(x) = \beta_2(x)$ if $x \in E_{Y_1 \cap Y_2}(N)$ and

$$\beta_i \colon E_{Y_1 \cap Y_2}(N) \to E_{Y_i}(N)$$

is the natural morphism. We can also mske $E_\chi^0 \subset E_\chi$.

Properties of the Eigencurve

I. We have a natural surjective morphism $\kappa \colon E_\chi \to \mathcal{B}$. v

II. There are analytic functions $\langle d \rangle^*$, $d \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, and $\mathbf{T}(n)$, $n_{>0} \in \mathbf{Z}$ and $\mathbf{T}(p)$ is invertible.

III. If $\sum_{n \geq 0} a_n q^n$ is the q-expansion of an overconvergent eigenform on $X_1(Np^n)$ of weight k and character $\chi \cdot \psi \cdot \omega^{-k}$ such that $a_p \neq 0$ then there exists a point $x \in E_\chi$ such that

$$a_n = \mathbf{T}(n)(x) \quad \text{for } n > 0.$$

IV. If $x \in E_\chi$ and $\kappa(x) = \psi \cdot \langle\langle\ \rangle\rangle^k$,

$$\sum_{n \geq 1} \mathbf{T}(n)(x) q^n$$

is the q-expansion of an OC eigenform (minus its constant term), of tame level N and character $\chi \cdot \psi \cdot \omega^{-k}$.

V. The morphism $x \to (\kappa(x), \mathbf{T}(p)(x))$ a locally finite from E_χ onto S_χ.

VI. There exists a pseudo-repreentation $\rho = (T, D) \colon G_{\mathbf{Q}} \to \mathbf{T}$ such that, if $(\ell, Np) = 1$,

$$T(\mathrm{Frob}_\ell) = \mathbf{T}(\ell) \quad \text{and} \quad D(\mathrm{Frob}_\ell) = \langle \ell \rangle^* / \ell.$$

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 28

Glueing

For every $Y \subset S = S_\chi$ such that Y is finite over $X \subset \mathcal{B}$ and "clopen" in S_X we found an affiniod $E_Y(N)$ which is finite over X and such that $A(E_Y(N)) = \mathbf{T}_X(N)$. Let $\mathcal{C}$ be the collection of these Y.

Proposition. *S is admissibly covered by $\mathcal{C}$.*

This means that thw image of every morphism of an affinoid into S isa coved9 by finitely many elements of $\mathcal{C}$.

Proposition. *Suppose $Y_1, Y_2 \in \mathcal{C}$. (i) $Y_3 =: Y_1 \cap Y_2 \in \mathcal{C}$. (ii) $E_{Y_3}(N)$ is naturally a subdomain of $E_{Y_1}(N)$ and $E_{Y_2}(N)$.*

Proof.

Now we make E_χ is $\coprod_{Y \in \mathcal{C}} E_Y(N)$ with the identifications $\beta_1(x) = \beta_2(x)$ if $x \in E_{Y_1 \cap Y_2}(N)$ and

$$\beta_i : E_{Y_1 \cap Y_2}(N) \to E_{Y_i}(N)$$

is the natural morphism. We can also mske $E_\chi^0 \subset E_\chi$.

Properties of the Eigencurve

I. We have a natural surjective morphism $\kappa\colon E_\chi \to \mathcal{B}$. v

II. There are analytic functions $\langle d\rangle^*$, $d \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, and $\mathbf{T}(n)$, $n_{>0} \in \mathbf{Z}$ and $\mathbf{T}(p)$ is invertible.

III. If $\sum_{n\geq 0} a_n q^n$ is the q-expansion of an overconvergent eigenform on $X_1(Np^n)$ of weight k and character $\chi\cdot\psi\cdot\omega^{-k}$ such that $a_p \neq 0$ then there exists a point $x \in E_\chi$ such that

$$a_n = \mathbf{T}(n)(x) \quad \text{for } n > 0.$$

IV. If $x \in E_\chi$ and $\kappa(x) = \psi \cdot \langle\langle\ \rangle\rangle^k$,

$$\sum_{n\geq 1} \mathbf{T}(n)(x) q^n$$

is the q-expansion of an OC eigenform (minus its constant term), of tame level N and character $\chi\cdot\psi\cdot\omega^{-k}$.

V. The morphism $x \to (\kappa(x), \mathbf{T}(p)(x))$ a locally finite from E_χ onto S_χ.

VI. There exists a pseudo-representation $\rho = (T, D)\colon G_{\mathbf{Q}} \to \mathbf{T}$ such that, if $(\ell, Np) = 1$,

$$T(\mathrm{Frob}_\ell) = \mathbf{T}(\ell) \quad \text{and} \quad D(\mathrm{Frob}_\ell) = \langle \ell\rangle^*/\ell.$$

This requires,

Theorem (corrected) (Wiles). *Suppose R is a topological $\mathbf{Z}_p$-algebra and $\{\mathfrak{p}_i\}_{i=1}^\infty$ are ideals such that $R/\mathfrak{p}_i \in \mathcal{C}$ and*

$$R = \varprojlim R/\bigcap_{i=1}^n \mathfrak{p}_i,$$

Σ is a dense subset of G, t, d are functions $\Sigma \to R$ and p-rs $T_i\colon G \to R/\mathfrak{p}_i$ such that

$$(T_i(\sigma), D_i(\sigma)) \equiv (t(\sigma), d(\sigma)) \bmod \mathfrak{p}_i$$

for $\sigma \in \Sigma$. Then there exists a unique p-r $T\colon G \to R$ such that $T(\sigma) \equiv T_i(\sigma) \bmod \mathfrak{p}_i$ for all $\sigma \in \Sigma$ and all i.

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 29

Properties of the Eigencurve

I. We have a natural morphism $\kappa\colon \mathbf{E}_\chi \to \mathcal{B}$.

II. There are analytic functions $\langle d\rangle^*$, $d \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, and $\mathbf{T}(n)$, $n_{>0} \in \mathbf{Z}$ and $\mathbf{T}(p)$ is invertible. Let $\mathbf{T}_\chi$ be $\varprojlim_{Y\in\mathcal{C}} \mathbf{T}_Y(N)$.

III. If $\sum_{n\geq 0} a_n q^n$ is the q-expansion of an overconvergent eigenform on $X_1(Np^n)$ of weight k and character $\chi\cdot\psi\cdot\omega^{-k}$ over K such that $a_p \neq 0$ then there exists a point $x \in \mathbf{E}_\chi(K)$ such that

$$\kappa(x) = \chi\cdot\psi\cdot\langle\langle\ \rangle\rangle \text{ and } a_n = \mathbf{T}(n)(x) \quad \text{for } n > 0.$$

IV. If $x \in \mathbf{E}_\chi$ and $\kappa(x) = \psi\cdot\langle\langle\ \rangle\rangle^k$,

$$F_x(q) =: \sum_{n\geq 1} \mathbf{T}(n)(x) q^n$$

is the q-expansion of an OC eigenform (minus its constant term), of tame level N and character $\chi\cdot\psi\cdot\omega^{-k}$.

V. The morphism $x \to (\kappa(x), \mathbf{T}(p)(x))$ a locally finite from E_χ onto S_χ.

VI. There exists a pseudo-representation $(T, D)\colon G_{\mathbf{Q}} \to \mathbf{T}_\chi$ such that, if $(\ell, Np) = 1$,

$$T(\mathrm{Frob}_\ell) = \mathbf{T}(\ell) \quad \text{and} \quad D(\mathrm{Frob}_\ell) = \langle\ell\rangle^*/\ell.$$

Theorem (corrected) (Wiles)**.** *Suppose R is a topological $\mathbf{Z}_p$-algebra and $\{\mathfrak{p}_i\}_{i=1}^\infty$ are ideals such that $R/\mathfrak{p}_i$ is a local complete $\mathbf{Z}_p$-algebra and $R = \varprojlim R/\bigcap_{i=1}^n \mathfrak{p}_i$, Σ is a dense subset of G, t, d are functions $\Sigma \to R$ and p-rs $(T_i, D_i)\colon G \to R/\mathfrak{p}_i$ such that*

$$(T_i(\sigma), D_i(\sigma)) \equiv (t(\sigma), d(\sigma)) \ mod\, \mathfrak{p}_i$$

for $\sigma \in \Sigma$. Then there exists a unique p-r $(T, D): G \to R$ such that $(T(\sigma), D(\sigma)) \equiv (T_i(\sigma), D_i(\sigma))$ mod
for all $\sigma \in \Sigma$ and all i.

Lemma. *If $F(T) \in 1 + TA(\mathcal{B})[[T]]^{entire}$ and U is a connected component of the zero locus of F in $\mathcal{B} \times \mathbf{C}_p$, the complement of the image of U in $\mathcal{B}$ is finite.*

Proof of VI. Let $\mathcal{D}$ be the subset of $Y \in \mathcal{C}$ such that $\exists$ a p-r $(T_Y, D_Y): \mathbf{G_Q} \to \mathbf{E}_Y$, $\mathbf{E}_Y = \mathbf{E}_Y(N)$, such that

$$T_Y(\mathrm{Frob}_\ell) = \mathbf{T}(\ell)|_{\mathbf{E_Y}} \quad \text{and} \quad D_Y(\mathrm{Frob}_\ell) = \langle \ell \rangle^* / \ell |_{\mathbf{E}_Y}.$$

Now, $\bigcup_{Y \in \mathcal{D}} Y$ is a union of connected components of S.
Suppose $x \in \mathbf{E}_\chi$, $\kappa(x) = \psi \cdot \langle\langle\ \rangle\rangle^k$, $k_{\geq 2} \in \mathbf{Z}$ and $v(\mathbf{T}(p)(x)) < k-1$. Then, by Deligne, there exists a rep $\rho_x: \mathbf{G_Q} \to \mathbf{Gl}_2(\mathbf{Q}_p(x))$ such that if $(\ell, Np) = 1$,

$$\begin{aligned} \mathrm{Tr}\, \rho_x(\mathrm{Frob}_\ell) &= \mathbf{T}(\ell)(x) \quad \text{and} \\ \det \rho_x(\mathrm{Frob}_\ell) &= \chi(\ell)\psi(\ell)\ell^{k-1} \\ &= \langle \ell \rangle^*(x)/\ell. \end{aligned}$$

What about $\mathbf{a_p}$?

Suppose E is finite extention of $\mathbf{Q}_p$ and $x \in \mathbf{E}_\chi(E)$. We define the **weight** $k(x)$ of x to be $1 + \frac{\log(\kappa(x)(1+p))}{\log(1+p)}$. There is a subring B^+_{cris} of B^+_{DR} which contains $W(R)$ and $t = \log[\epsilon]$ on which $G_{\mathbf{Q}_p}$ acts whith a Frobenius endomorphism ϕ which commutes with $\mathbf{G}_{\mathbf{Q}_p}$ such that

$$\phi(\alpha b) = \alpha^\sigma \phi(b) \text{ and } \phi t = pt, \quad \alpha \in W(R).$$

and

Theorem (Kisin)**.** *Suppose $a_p =: \mathbf{T}(p)(x)$, and $\rho: G_\mathbf{Q} \to Aut_E(V)$ is a representation attached to x, then there exists a non-zero $\mathbf{G}_{\mathbf{Q}_p}$-equivariant E-linear map*

$$V \to (B^+_{cris} \otimes_{\mathbf{Q}-p} E)^{\phi = a_p}.$$

The Eigencurve and the Fontaine-Mazur Conjecture

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Lecture 30

The Faemily of Peudo-reps

Theorem. *There exists a pseudo-representation*

$$(T, D)\colon G_{\mathbf{Q}} \to \mathbf{T}_\chi$$

such that, if $(\ell, Np) = 1$,

$$T(Frob_\ell) = \mathbf{T}(\ell) \quad \text{and} \quad D(Frob_\ell) = \langle \ell \rangle^*/\ell.$$

Ingrediants of the proof.

Theorem (Wiles). *Suppose R is a topological $\mathbf{Z}_p$-algebra and $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ are ideals such that $R/\mathfrak{p}_i$ is a local complete $\mathbf{Z}_p$-algebra and $R = \varprojlim R/\bigcap_{i=1}^n \mathfrak{p}_i$, Σ is a dense subset of G, t, d are functions $\Sigma \to R$ and p-rs $(T_i, D_i)\colon G \to R/\mathfrak{p}_i$ such that*

$$(T_i(\sigma), D_i(\sigma)) \equiv (t(\sigma), d(\sigma)) \bmod \mathfrak{p}_i$$

for $\sigma \in \Sigma$. Then there exists a unique p-r $(T, D)\colon G \to R$ such that $(T(\sigma), D(\sigma)) \equiv (T_i(\sigma), D_i(\sigma))$ mod$\mathfrak{p}$ for all $\sigma \in \Sigma$ and all i.

Let $\mathbf{T}_\chi^0$ be the subring of $\mathbf{T}_\chi$ which is the completion of the ring generated over $A^0_{\mathbf{Q}_p(\chi)}(\mathcal{B}) \cong R_\chi[[T]]$ by $\langle d \rangle^*$, $d \in (\mathbf{Z}/N\mathbf{Z})^* \times \mathbf{Z}_p^*$, and $\mathbf{T}(n)$, $n_{>0} \in \mathbf{Z}$.

Proposition. $\mathbf{T}_\chi^0$ *is compact.*

This comes down to,

Theorem (after Hida). *If $k_{\in \mathbf{Z}} \geq 2$ and $h_k(Np^\nu)$ is the Hecke algebra acting on weight k modular forms of level Np^ν over $\mathbf{Z}_p$.*

$$\bigoplus_\chi \mathbf{T}_\chi^0 \cong R_\chi \otimes_{\mathbf{Z}_p} \varprojlim_\nu h_k(Np^\nu)$$

Suppose $x \in \mathbf{E}_\chi$, $\kappa(x) = \psi \cdot \langle\langle\ \rangle\rangle^k$, $k_{\in \mathbf{Z}} \geq 2$, and $v(\mathbf{T}(p)(x)) < k-1$. Then, by Deligne, there exists a rep $\rho_x \colon \mathbf{G}_\mathbf{Q} \to \mathbf{Gl}_2(\mathbf{Q}_p(x))$ such that if $(\ell, Np) = 1$,

$$\mathrm{Tr}\,\rho_x(\mathrm{Frob}_\ell) = \mathbf{T}(\ell)(x) \quad \text{and} \quad \det \rho_x(\mathrm{Frob}_\ell) = \chi(\ell)\psi(\ell)\ell^{k-1} = \langle \ell \rangle^*(x)/\ell.$$

For each x ae above we get a prime ideal $\mathfrak{p}_x$ of $\mathbf{T}^0_\chi$

What about $\mathbf{a_p}$?

Suppose E is finite extention of $\mathbf{Q}_p$ and $x \in \mathbf{E}_\chi(E)$. We define the **weight** $k(x)$ of x to be $1 + \frac{\log(\kappa(x)(1+p))}{\log(1+p)}$. There is a subring B^+_{cris} of B^+_{DR} which contains $W(R)$ and $t = \log[\epsilon]$ on which $G_{\mathbf{Q}_p}$ acts with a Frobenius endomorphism ϕ which commutes with $\mathbf{G}_{\mathbf{Q}_p}$ such that

$$\phi(\alpha b) = \alpha^\sigma \phi(b) \text{ and } \phi t = pt, \quad \alpha \in W(R).$$

and

Theorem (Kisin). *Suppose $a_p =: \mathbf{T}(p)(x)$, and $\rho \colon G_\mathbf{Q} \to Aut_E(V)$ is a representation attached to x, then there exists a non-zero $\mathbf{G}_{\mathbf{Q}_p}$-equivariant E-linear map*

$$V \to (B^+_{cris} \otimes_{\mathbf{Q}_p} E)^{\phi = a_p}.$$

Preview of B_{cris} and B_{st}.

Let $B_{cris} = B^{+}_{cris}[1/t]$. This embeds naturally in B_{DR}. Set $\mathrm{Fil}^i B_{cris} = B_{cris} \cap \mathrm{Fil}^i B_{DR}$. Also $((B_{cris})^{G_K} = K_0$.

We need to consider another ring B_{st} which is $B^{+}_{st}[1/t]$ where

$$B^{+}_{st} = B^{+}_{cris}[\{\ell(u) \colon u \in \mathrm{Frac}(R)^*\}],$$

where

$$\ell(wv) = \ell(w) + \ell(v) \quad \text{and } \ell(u) = \log \frac{[u]}{u^{(0)}} + \log u^{(0)},$$

if $v(u^{(0)} - 1) > 0$. We extend ϕ to B_{st} by setting $\phi(\ell(u)) = p\ell(u)$ and let N be the unique derivation over B_{cris} on B_{st} such that

$$N\,1 = 0 \quad \text{and } N\,\ell(u) = v(u^{(0)}).$$

$N \circ \phi = p\phi \circ N$ and

$$0 \to B_{cris} \to B_{st} \xrightarrow{N} B_{st} \to 0$$

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 31

Preview of B_{st}

We know $B_{cris} = B^+_{cris}[1/t]$ and this embeds naturally in B_{DR}. Set

$$\mathrm{Fil}^i B_{cris} = B_{cris} \cap \ \mathrm{Fil}^i B_{DR}.$$

$$\mathrm{Gr} B_{cris} = B_{HT} \ \text{ and } \ \mathrm{Gr} B^+_{cris} = B^+_{HT}.$$

We need to consider another ring B_{st} which is $B^+_{st}[1/t]$ where

$$B^+_{st} = B^+_{cris}[\{\ell(u) \colon u \in \ \mathrm{Frac}(R)^*\}],$$

and

$$\ell(wv) = \ell(w) + \ell(v) \ \text{ and } \ell(u) = \log \frac{[u]}{u^{(0)}} + \log u^{(0)},$$

if $v(u^{(0)} - 1) > 0$. We extend F to B_{st} by setting $F(\ell(u)) = p\ell(u)$ and let N be the unique derivation over B_{cris} on B_{st} such that

$$N\,1 = 0 \ \text{ and } N\,\ell(u) = v(u^{(0)}).$$

Then $NF = pFN$ and

$$0 \to B_{cris} \to B_{st} \xrightarrow{N} B_{st} \to 0$$

Periods of Classical Eigenforms

Suppose $x \in E_\chi$, $M =: \mathbf{Q}_p(x) \subset L$ and $F_x(q)$ is classical eigenform of weight k. Let $\rho \colon G_{\mathbf{Q}} \to \mathbf{Gl}(V)$ be a reepreesentation "attached" to x where V is a two dimensional vector space over M. Then

Theorem (Faltings)**.** $V \otimes \mathbf{C}_p \cong \mathbf{C}_p \oplus \mathbf{C}_p(k-1)$.

Suppose L is a finite extension of $\mathbf{Q}_p$. The **Weil group**, W_L, is the subgroup of G_L consisting of elements w whose restriction to L^{nr} is an integral power, $\alpha(w)$, of absolute Frobenius. The **Weil-Deligne group** of L is a group scheme WD_L over $\mathbf{Q}$ which is the semi-direct product of the constant group sheme W_L and $\mathbf{G}_a$ on which W_L acts by

$$wxw^{-1} = p^{\alpha(w)}x.$$

If M is a field, a representation of WD_L over M is an M vector space V with homomorphism of group schemes $\psi: WD_L(M) \to \mathbf{Gl}(V)$. These are equivalent to representations ρ_0 of W_L on an M-vector space Δ together with an M-linear operator N on Δ satisfying

$$N \circ \rho_0(w) = p^{-\alpha(w)}\rho_0(w) \circ N.$$

Indeed, $\psi(x) = \exp(xN_L)$ for $x \in \mathbf{G}_a$.

Let $V^* = Hom(V, L)$ and set

$$D_{pst}(V) = \bigcup_{L'/L} (B_{st} \otimes V^*)^{G'_M}$$

. Now WD_L operates on $D_{pst}(V)$ whih finite dimensional over K^{nr}. First W_L acts and second

$$N_L m = Nm.$$

N_L acts nilpotently on $D_{pst}(V)$. Let $J(V)$ denote the the invriants by inertia in the kernel of N_L. Let σ be the inverse of relative Frobenius.

Therorem (Saito). $(1 - a_p p^{-s})^{-1}$ *divides* $\det(1 - \sigma p^{-s}|J)^{-1}$.

Frank will prove,

Theorem . *There exists a non-zero* $\mathbf{G}_{\mathbf{Q}_p}$*-equivariant* E*-linear map*

$$V \to (B^+_{cris} \otimes_{\mathbf{Q}_p} E)^{\phi = a_p}.$$

After Faltings it is enough to prove, There exists a non-zero $\mathbf{G}_{\mathbf{Q}_p}$-equivariant E-linear map

$$V \to (B_{cris} \otimes_{\mathbf{Q}_p} E)^{\phi = a_p}.$$

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 32

Clarification of Weil-Deligne

Suppose ρ_0 of W_L on a finite dimensional M-vector space Δ and N is aan M-linear operator on Δ satisfying

$$N \circ \rho_0(w) = p^{-\alpha(w)} \rho_0(w) \circ N.$$

Then N is nilpotent and $\rho(x,w) = \exp(xN)\rho_0(w)$, $(x,w) \in WD_L$ is a representation.

Periods of Classical Eigenforms

Suppose $x \in E_\chi(\overline{\mathbf{Q}}_p)$, $M =: \mathbf{Q}_p(x)$. Let $\rho_x : G_{\mathbf{Q}} \to \mathbf{Gl}(V_x)$ be a repreesentation "attached" to x where V_x is a two dimensional vector space over M. We want to prove,

Theorem. *Let $a_p = \mathbf{T}(p)(x)$. There exists a non-zero $\mathbf{G}_{\mathbf{Q}_p}$-equivariant M-linear map*

$$V_x \to (B^+_{cris} \otimes_{\mathbf{Q}_p} M)^{\phi = a_p}.$$

Frank explained why this is true when $F_x(q)$ is classical. I will now explain its conection to Fontaine-Mazur.

Fontaine-Mazur

Let $\theta \sum a_q^n = \sum n a_n q^n$ and $\chi\colon G_{\mathbf{Q}_p} \to \mathbf{Z}_p^*$ be the cyclotomic character.

Proposition. *If $F(q)$ is the q-expansion of a weight $2-k$ OC form where $k \in \mathbf{Z}$, $k \geq 2$, $\theta^{k-1}F(q)$ is the q-expansion of a weight $k-1$ OC form.*

A rep $\rho\colon G_K \to \mathbf{Gl}(V)$ is called potentially semi-stable (pst) if

$$\dim_{K^{nr}} D_{pst}(V) = \dim_M V.$$

Theorem (Kisin). *Suppose V_x when viewed as a $G_{\mathbf{Q}_p}$-rep is pst. Then,*
(i) $k =: k(x) \in \mathbf{Z}$ and $\alpha =: v(\mathbf{T}(p)(x)) \leq \max\{0, k-1\}$. (ii) If $k \geq 2$, either $F_x(q)$ is classical or $\alpha = k-1$ and $\exists$ OC G of weight $2-k$ such that $F_x = \theta^{k-1}G$ and $V_x \cong \epsilon_1 \oplus \epsilon_2 \chi^{k-1}$.

Corollary. *If ρ_x is semi-stable and irreducible, then x is classical.*

Proof of Theorem. First PST implies HT

$$\dim_K (V \otimes B_{HT})^{G_K} = \dim_M V.$$

and Hodge-Tate reps have integral weight.

Suppose $V =: V_x$ is ST over a finite Galois extension K of $\mathbf{Q}_p$ and let

$$D =: D_{st}(V^*) = Hom_{G_K}(V_x, B_{st}) = (V_x^* \otimes B_{st})^{G_K}.$$

Claim:

$$D_{dr}(V^*) = (D_{st}(V^*) \otimes_{K_0} K)^{\,\mathrm{Gal}(K/\mathbf{Q}_p)}$$

This follows from the fact that

$$B_{st} \otimes_{K_0} K \hookrightarrow B_{dr}.$$

Thus $D_{dr}(V^*)$ is a 2 dimensional M-space and it has an M-linear $\phi^{[K_0:\mathbf{Q}_p]}$-action. Thus $D = D_{dr}(V^*) \otimes_{\mathbf{Q}_p} K_0$ is a free $M \otimes K_0$ module of rank 2 and its Newton polygon has at most two slopes of the same run $[M, \mathbf{Q}_p]$.

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 33

Fontaeine-Mazur

Assuming,

Theorem. *Let* $a_p = \mathbf{T}(p)(x)$. *There exists a non-zero* $\mathbf{G}_{\mathbf{Q}_p}$*-equivariant* M*-linear map*

$$V_x \to (B^+_{cris} \otimes_{\mathbf{Q}_p} M)^{\phi = a_p}.$$

we'll prove,

Theorem. *Suppose* V_x *when viewed as a* $G_{\mathbf{Q}_p}$*-rep is pst. Then,* $k =: k(x) \in \mathbf{Z}$ *and* $\alpha =: v(\mathbf{T}(p)(x)) \leq \max\{0, k-1\}$.

To simplify notation, I will assume $\mathbf{Q}_p(x) = M = \mathbf{Q}_p$ and that V_x is semistable.

We already checked that $k(x) \in \mathbf{Z}$. One of the facts we used was that B_{st} embeds in B_{dr}. Recall, $B_{st} = B^+_{st}[1/t]$ where

$$B^+_{st} = B^+_{cris}[\{\ell(u) \colon u \in \ \mathrm{Frac}(R)^*\}],$$

$$\ell(wv) = \ell(w) + \ell(v) \quad \text{and } \ell(u) = \log \frac{[u]}{u^{(0)}} + \log u^{(0)},$$

if $v(u^{(0)} - 1) > 0$. We already know how to embed B_{cris} into B_{dr}. Choose a branch of the logarithm log. Then send $\ell(u)$ to

$$\log \frac{[u]}{u^0} + \log u^{(0)}.$$

This makes sense since $\theta([u]) = u^{(0)}$.

Suppose D is a filtered (F, N)-module over K, i.e. a K_0-module D with a σ-linear isomorhism F and an endomorphism N such that $NF = pFN$ as well as a decreasing, exhaustive, separated filtration on D_K, D^i, like $D(V^*) =: (V^* \otimes_M B_{st})^{G_{\mathbf{Q}_p}}$. The Hodge numbers of D are

$$h_H(D, i) = \dim D^i / D^{i+1}$$

If $D = D_K(V^*)$, $h_H(D, i) = 1$ if $i = 0$ or $k-1$ and is zero otherwise .For a rational number $\alpha = r/s$ let $D_{[\alpha]}$ be the K_0-subspace of $\bar{K}_0 \otimes_{K_0} D$ spanned by the elements x such that $(\sigma \otimes F)^s x = p^r x$. The Newton numbers are

$$h_N(D, \alpha) = \dim_{K_0} D_{[\alpha]}.$$

Suppose $\dim_{K_0} D < \infty$. If $D = D(V^*)$, $h_N(D, [v(a_p)]) \geq 1$. We also know $h_N(D, k-1-[v(a_p)]) = h_N(D, [v(a_p)])$.

Put,

$$t_H(D) = \sum_{i \in \mathbf{Z}} i h_H(D, i) \text{ and } t_N(D, \alpha) = \sum_{\alpha \in \mathbf{Q}} \alpha h_N(D, \alpha).$$

Then D is **weakly admissible** if $t_H(D') \leq t_N(D')$ for all K^0-subspaces D' of D stable by F and N with equality when $D' = D$.

Theorem. *If W is PST then $D_{pst}(W)$ is WA.*

Suppose $D = D(V^*)$. This is WA. Also the submodule $\sum_{\beta \leq 0} D_{[\beta]}$ is (F, N)-stable and thus

$$0 \leq \sum_{\substack{\alpha \in \mathbf{Q} \\ \alpha \leq 0}} \alpha h_N(D, \alpha)$$

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 34

Fontaine-Mazur

Theorem. *Let* $a_p = \mathbf{T}(p)(x)$. *There exists a non-zero* $\mathbf{G}_{\mathbf{Q}_p}$*-equivariant* M*-linear map*

$$V_x \to (B^+_{cris} \otimes_{\mathbf{Q}_p} M)^{\phi=a_p}.$$

Assuming this, and "continuity o Hodge-Tate-Sen" weights, we'll prove,

Theorem. *Suppose* V_x *when viewed as a* $G_{\mathbf{Q}_p}$*-rep is pst. Then,* $k =: k(x) \in \mathbf{Z}$ *and* $\alpha =: v(\mathbf{T}(p)(x)) \leq \max\{0, k-1\}$.

Suppose D is a finite dimensional filtered (F, N)-module over K, i.e., a K_0-module D with a σ-linear isomorhism F and an endomorphism N such that $NF = pFN$ as well as a decreasing, exhaustive, separated filtration on D_K, D^i. The Hodge polygon of D is the lower convex hull of the vertices

$$(\sum_{i \leq j} \dim D^i/D^{i+1}, \sum_{i \leq j} i \dim D^i/D^{i+1})$$

For a rational number $\alpha = r/s$ let $D_{[\alpha]}$ be the K_0-subspace of $\bar{K}_0 \otimes_{K_0} D$ spanned by the elements x such that $(\sigma \otimes F)^s x = p^r x$.

The Newton polygon of D is the lower convex hull of

$$(\sum_{\beta \leq \alpha} \dim D_{[\beta]}, \sum_{\beta \leq \alpha} \beta \dim D_{[\beta]})$$

Then D is **weakly admissible** if the Newton polygon of D' lies above the Hodge polygon of D' for all (F, N)-submodules with induced filtration and these polygons have the same endpoints when $D = D'$.

Theorem. *If W is PST then $D_{pst}(W)$ is WA.*

Proof of Kisin's Theorem

To simplify notation, I will assume $\mathbf{Q}_p(x) = M = \mathbf{Q}_p$ and that V_x is semistable.

Suppose $D = D(V_x^*)$. This is WA. Using Sen theory (which I'll discus next week), we know when $k \neq 1$, $D_{HT}(V) \cong \mathbf{C}_p(0) \oplus \mathbf{C}_p(k-1)$. Suppose $a \leq b$. Then the Hodge polygon of D is

F and N for Tate Elliptic curves

Suppose $E = \mathbf{C}_p^*/q^{2\mathbf{Z}}$, where $q_{\neq 0} \in p\mathbf{Z}_p$. Then $E = U \cup V$ and $U \cap V = A \cup B$. We have

$$H^0_{DR}(A) \oplus H^0_{DR}(B) \to H^1_{DR}(E) \to H^1_{DR}(U) \oplus H^1_{DR}(V) \to H^1_{DR}(A) \oplus H^1_{DR}(B)$$

Then N is

$$H^1_{DR}(E) \to H^1_{DR}(A) \oplus H^1_{DR}(B) \xrightarrow{Res} H^0_{DR}(A) \oplus H^0_{DR}(B) \to H^1_{DR}(E)$$

To get F all we have to do is "split"

$$H^0_{DR}(A) \oplus H^0_{DR}(B) \to H^1_{DR}(E).$$

Suppose $(\{\omega_U, \omega_V\}, \{f_A, f_B\})$ is a 1-cocycle ($\omega_U - \omega_V = \mathrm{d}f$). If we choose a branch of log we can solve

$$\mathrm{d}F_U = \omega_U \quad \text{and } \mathrm{d}F_V = \omega_V$$

Let

$$c_A = (F_U - F_V)|_A - f_A \quad \text{and } c_B = (F_U - F_V)|_B - f_B.$$

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 35

Sen-Polynomials

Let $\chi\colon \mathbf{G}_{\mathbf{Q}_p} \to \mathbf{Z}_p^*$ be the cyclotomic character and $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$. Suppose E a finite extension of $\mathbf{Q}_p$ contained in K. Finally, let $\Gamma(K) = \mathrm{Gal}(K_\infty/K)$ where $K_\infty = K(\mu_{p^\infty})$. Sen's proves in Contiuous Cohomology and p-adic Galois representaions, (Invent. Math. **62** (1980)),

Theorem. *Suppose V is a finite dimensional vector space over E and $\rho\colon G_K \to GL_E(V)$ is a continuous representation. There exists a finite extension L of K in K_∞ and an $M \in GL_{\mathbf{C}_p}(\mathbf{C}_p \otimes_E V)$ such that $\sigma \mapsto \tau(\sigma) =: M^{-1}\rho(\sigma)\sigma(M)$ is a representation of G_L into $GL_L(L \otimes_E V)$ which factors through $\Gamma(L)$. Moreover, if $\sigma \in G_L$ and its image in Γ is non-trivial*

$$S_\rho(T) =: \det\left(T - \frac{\log \tau(\sigma)}{\log \chi(\sigma)}\right)$$

is independent of the choices of L and σ and lies in $K[T]$. In fact, this polynomial is independent of K or E.

Eg. (i) (CFT) Suppose $n = 1$ and $K = E = \mathbf{Q}_p$. Then if $\gamma \in \Gamma$ sufficiently close to 1,

$$\tau(\gamma) = \rho(\sigma), \qquad \text{if } \sigma \mapsto \gamma \text{ and } T - e(\rho) =: S_\rho(T).$$

(ii) (Hodge-Tate) Suppose A is an Abelian variety of dimension g over K and $\rho\colon G_K \to GL_{2g}(\mathbf{Q}_p)$ coming from the p-Tate module of A. Then

$$S_\rho(T) = T^g(T-1)^g.$$

(iii) (Faltings) Suppose ρ is the restriction to a decomposition group above p of a represention coming from a weight k modular form. Then,

$$S_\rho(T) = T(T-(k-1)).$$

(iv) Suppase $V \otimes_E \mathbf{C}_p \cong \mathbf{C}_p(a_1) \oplus \cdots \mathbf{C}_p(a_n)$. Then,

$$S_\rho(T) = (T - a_1) \cdots (T - a_n).$$

Variation.

Let C be a topologically finitely generated complete local ring over $R =: \mathcal{O}_E$ whose residue field is a finite extension of $k = R/\pi_E R$, $C = R[[T_1, \ldots, T_n]]/I$. Let $\langle C \rangle$ be the rigid space associated to C.

Suppose $\rho: G_K \to GL_n(C)$ is a continuous representation.
Eg. Suppose k is a finite field of characteristic p and α: $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(k)$ is a representation. Then, Mazur has shown there exists a topologically finitely generated complete local ring M_α over $\mathbf{Z}_p$ and a versal deformation of α

$$\tilde{\alpha}\colon \ \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \ \mathrm{Gl}_2(M_\alpha)$$

(which is universal when α is absolutely irreducible). When α is odd, $\langle M_\alpha \rangle$ is conjectured to have dimension 3.

A slight improvemewnt of Sen's result in The Analytic Variation of p-adic Hodge Structure (Ann. Math. **127** (1988)) is,

Theorem. *There is a unique monic polynomial, $f_\rho(T)$, whose coefficients are analytic functions on the nilreduction of $\langle C \rangle_K$ and whose specialization to $x \in \langle C \rangle(\overline{E})$ is $S_{\rho_x}(T)$.*

Corollary. *If α is modular and $x \in \langle M_\alpha \rangle$,*

$$S_{\tilde{\alpha}_x}(T) = T(T - e(\det \tilde{\alpha}_x)).$$

Let E_α be the component of the eigencurve such that for $x \in E_\alpha(\overline{\mathbf{Q}}_p)$, ρ_x is a deformation of α.

Corollary. *If V_x is pst $k(x) \in \mathbf{Z}$ and $V_x \otimes \mathbf{C}_p \cong \mathbf{C}_p \oplus \mathbf{C}_p(k(x) - 1)$.*

The Eigencurve and the Fontaine-Mazur Conjecture

Robert F. Coleman

Lecture 36

Application of Sen's Theory

For a representation $\rho: G_K \to \mathbf{Gl}(V)$ where V is vector space over a finite extention of $\mathbf{Q}_p$ let $S_\rho(T)$ be the Sen polynomial. We know if ρ is attached to ae weight k modular form $S_\rho(T) = T(T-(k-1))$. Also if $V \otimes_E \mathbf{C}_p \cong \mathbf{C}_p(a_1) \oplus \cdots \mathbf{C}_p(a_n)$. Then, $S_\rho(T) = (T-a_1)\cdots(T-a_n)$.

Proposition. *If x is a point on an eigencurve and V_x is pst $k(x) \in \mathbf{Z}$ and $V_x \otimes \mathbf{C}_p \cong \mathbf{C}_p \oplus \mathbf{C}_p(k(x)-1)$.*

Variation.

Suppose $C \cong \mathcal{O}_E[[T_1,\ldots,T_n]]/I$. Let $\langle C\rangle$ be the rigid space associated to C. Suppose $\rho: G_K \to GL_n(C)$ is a continuous representation.
Eg. Suppose k is a finite field of characteristic p and α: $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(k)$ is a representation. Then, Mazur has shown there exists a topologically finitely generated complete local ring M_α over $\mathbf{Z}_p$ and a versal deformation of α

$$\tilde{\alpha}\colon\ \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(M_\alpha).$$

Theorem. *There is a unique monic polynomial, $f_\rho(T)$, whose coefficients are analytic functions on the nilreduction of $\langle C\rangle_K$ and whose specialization to $x \in \langle C\rangle(\overline{E})$ is $S_{\rho_x}(T)$.*

Suppose the above α is modular of level N and let E_α be the component of the eigencurve E_N such that for $x \in E_\alpha(\overline{\mathbf{Q}}_p)$, ρ_x is a deformation of α.

The Galois interpretation of a_p

Theorem. *Suppose* $\psi \in Hom((\mathbf{Z}/Np\mathbf{Z})^*, \mathbf{C}_p^*)$, $x \in E_\psi$. *Let* $a_p = \mathbf{T}(p)(x)$ *and* $M = \mathbf{Q}_p(x)$. *There exists a non-zero* $\mathbf{G}_{\mathbf{Q}_p}$*-equivariant* M*-linear map*

$$V_x \to (B^+_{cris} \otimes_{\mathbf{Q}_p} M)^{\phi=a_p}.$$

(Also, see forthcoming paper of Stevens and Iovita.)

Suppose $y \in E_\psi(K)$, $k(y) \in \mathbf{Z}$. We'll the following simplifying assumption: There exists an affinoid X in E_ψ defined over K containing y which is isomorphic via κ to a closed disk in $\mathcal{W}$, and a free rank 2 module $\mathbf{V}$ over $R =: A(X)$ with an action of Galois whose restriction $\mathbf{V}_x$, $x \in X$ to $k(x) \in \mathbf{Z}$, $k(x) >> 0$, is classical and crystaline.

Lemma. *After removing the weight one points (if they exist) from* X,

$$\mathbf{V}^* \hat{\otimes}_K \mathbf{C}_p \cong (R \hat{\otimes}_K \mathbf{C}_p) \oplus ((R \hat{\otimes}_K \mathbf{C}_p)(\chi/\kappa)$$

as G_K*-modules where* χ *is the cyclotomic character.*

Indeed, $W_\infty = (\mathbf{V}^* \hat{\otimes}_K \mathbf{C}_p)^{H_K}$ has a basis over a finite extention L of K such that the R_L-module W_L spanned by this basis is Galois stable and if $\gamma_{\neq 1} \in \Gamma(L)$ the linear operator on W_L

$$\Phi = \frac{\log \gamma}{\log \chi(\gamma)}$$

has characterististic polynomial $T(T + (k(x) - 1))$.

Lemma. *Suppose* $j > 0$. *After removing a finite set of points* S_j *from* X, $(\mathbf{V}^* \otimes \hat{B}^+_{dr}/B^j_{dr})^{G_K}$ *is a free* $R_J =: A(X - S_j)$*-module of rank 1.*

Corollary. *Suppose* $j > 0$ *and* $x \in X - S_j$. *There exists a non-zero* G_K*-equivariant map* $\alpha_x: V_x \to B^+_{dr}/B^j_{dr}$.

Proof of Lemma. We need Tate's Theorem $\mathbf{C}_p(k)^{G_K} = 0$ (p-Divisible Groups, in Proceedings of a Conference on Local Fields, Driebergen 1966, pp 158-183, Springer (1967).)) unless $k = 0$. Suppose $j = 1$.