RESEARCH SUMMARY

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My primary research focus is on the geometry and topology of 3-manifolds. One appealing aspect of this subject is the wide variety of techniques which can be brought to bear on it, and the many related subjects which both contribute to and benefit from its study. A spectacular recent example of this phenomenon is the use of geometric flows by G. Perelman in proving W.P. Thurston's geometrization conjecture ([53],[55],[54]). One consequence is that many famous questions about the topology of 3-manifolds (e.g. the "virtually Haken" and "virtually fibered" conjectures) reduce to the case of *hyperbolic* 3-manifolds: those which admit Riemannian metrics with constant negative sectional curvatures. This is the richest and most mysterious class of geometric 3-manifolds, and much of my research draws connections between geometry and topology among its members.

Section 1 below addresses *commensurability* of hyperbolic 3-manifolds: the property of sharing finite-degree covers (or, when referring to fundamental groups, finite-index subgroups). Many desirable properties (like the "virtual" properties mentioned above) are shared by all members of a commensurability class. We first describe a joint paper of E. Chesebro, H. Wilton and myself [15] that identifies a large class of hyperbolic 3-manifolds whose fundamental groups are commensurable with subgroups of right-angled Artin groups. To show this we use criteria of Haglund–Wise [33]; it follows from work of Agol [3] that manifolds in this class are virtually fibered. I describe an approach to use the results of [15] and other methods to extend these conclusions to larger classes of hyperbolic 3-manifolds.

We then consider the question of when hyperbolic knots in S^3 have commensurable complements. For topological reasons this appears to be a special property, and known classes of commensurable knot complements are both rare and small. Indeed, [11] shows that if a commensurability class contains more than three hyperbolic knot complements, then these knot complements have "hidden symmetries." There are only three such among the over 59,000 hyperbolic knots with up to 14 crossings. I outline an idea for finding commensurable knot complements, related by mutation, that have hidden symmetries. This is based on a joint paper with Chesebro [14], which constructs a family of link complements with similar properties.

Section 2 addresses volumes of hyperbolic 3-manifolds, the theme being that volume increases with topological complexity. The P.I has proved results reflecting this theme, joint with M. Culler and P. Shalen [19] and with Shalen [25], giving lower bounds on the volume of a hyperbolic 3-manifold that depend on its mod-2 homology. The second paper also establishes a new bound on the volumes of certain hyperbolic 3-manifolds with totally geodesic boundary, borrowing techniques from Kojima–Miyamoto [40] and Gabai–Meyerhoff–Milley [30]. This might be extended to classify low-volume hyperbolic 3-manifolds with totally geodesic boundary, along the lines of [30].

The third section is concerned with ranks of fundamental groups of hyperbolic 3-manifolds, with a particular focus on the *rank gradient*, which measures the rate at which rank grows in families of finite covers. This is related to the classical "rank vs. Heegaard genus" question and also, by work of Abert–Nikolov [1], to the "fixed price" question in topological dynamics. I have proved that cyclic covers not associated to a fiber have positive rank gradient [24], using 3-manifold topology and an acylindrical accessibility theorem for groups acting on trees due to R. Weidmann [69]. We outline a program to understand rank among many families of 3-manifold groups that split by improving pre-existing tools, such as Grushko's theorem, using geometric methods developed by M. White, J. Souto and others.

Besides establishing geometrization, geometric flows may yield new information even about 3-manifolds known *a priori* to be hyperbolic. For instance, Agol–Storm–Thurston used the Ricci flow with surgery to give new lower bounds on volumes of hyperbolic Haken 3-manifolds [5]; an appendix due to Agol–Dunfield gives estimates crucial in [30]. In joint work, D. Knopf, A. Young and I used the "cross curvature flow" to study a class of negatively curved metrics on the solid torus arising from Dehn surgery on hyperbolic 3-manifolds with finite volume [22]. We are motivated in part by a question about connectedness of the "Teichmüller space" of negatively curved metrics on manifolds of dimension at least 3, posed and also answered in sufficiently high dimensions, by Farrell–Ontaneda [27].

Section 4 lists "tangential questions," and sketches some possible answers, that arose in connection with the investigations described above. These do not directly fit into the programs described in the first three sections, but they seem worth investigating and in some cases link to related areas of mathematics, such as geometric group theory and the study of lattices in rank-one Lie groups.

1. Commensurability

Manifolds M and M' are *commensurable* if another manifold N finitely covers each of M and M'. The *commensurability class* of M is the set of manifolds commensurable to M. We will say that groups G and G' are commensurable if they contain isomorphic subgroups of finite index. Mostow's rigidity theorem implies for instance that closed hyperbolic 3-manifolds are commensurable if and only if their fundamental groups are. This section I contains fundamental questions concerning the commensurability relation among 3-manifolds, and concerning commensurabilities between their fundamental groups and other classes of groups, and a program for answering them.

Many desirable properties of 3-manifolds pass to finite covers; for instance, having an embedded, π_1-injective surface that is not a 2-sphere bounding a ball (such a manifold is said to be *Haken*) or positive first Betti number, or fibering over S^1. If M has a finite cover with some property (P), M is said to *virtually* have (P). As long as (P) passes to finite covers then M has virtual (P) if and only if all members of its commensurability class do as well. A class of hyperbolic 3-manifolds with all of the virtual properties described above are the *virtually special* manifolds introduced by Haglund–Wise [33].

A compact, non-positively curved cube complex is defined in [33] to be *special* if it satisfies certain requirements on the intersections of its hyperplanes. We will say a group is special if it is isomorphic to $\pi_1 K$ for some special cube complex K, and that a hyperbolic 3-manifold M is special if $\pi_1 M$ is special. Haglund–Wise showed that special groups embed as quasi-convex subgroups of right-angled Coxeter groups, thus inheriting many nice properties. In particular, work of Agol [3] shows that if a hyperbolic 3-manifold M is virtually special, then it is virtually fibered as well.

Many hyperbolic manifolds are known to be virtually special, and in fact it seems reasonable to conjecture that they all are. Among the examples currently in the literature are those commensurable with a reflection orbifold [32], arithmetic hyperbolic manifolds obtained from a quadratic form construction [9], and the following examples from joint work with Eric Chesebro and Henry Wilton [15].

Theorem 1 ([15], Theorem 1.1)**.** *Suppose M is a complete hyperbolic 3-manifold with finite volume that admits a decomposition into right-angled ideal polyhedra. Then M is virtually special.*

A large class of manifolds to which Theorem 1 applies are the *augmented link* complements. A link L in S^3 can be described by a *projection*, a 4-valent planar graph Γ together with crossing information at each vertex. Given such a projection, L is augmented by adding unkotted components enclosing each *twist region*, a maximal string of bigon regions of Γ. In an appendix to [41], Agol–D. Thurston described a decomposition of an augmented link complement into right-angled ideal polyhedra. In [15] we classify the hyperbolic augmented links of low complexity (measured by the number of twist regions). We also describe an infinite family of augmented links whose complements are not commensurable with any 3-dimensional right-angled reflection orbifold.

Although we do not make an explicit connection in [15], our construction is related to the following fundamental result of Sageev [60]: a hyperbolic 3-manifold M contains a π_1-injective immersed surface if and only if $\pi_1 M$ acts "essentially" on a CAT(0) cube complex. Given a sufficiently nice collection of surfaces, Sageev's construction produces an action that is faithful and cocompact. If M admits a decomposition into right-angled polyhedra as above, such a collection is formed by "face" surfaces: totally geodesic surfaces that are unions of faces. I believe that the same result of Sageev may be used to produce a nice action of $\pi_1(S^3 - L)$ on a cube complex for many hyperbolic links L, using the associated augmented links as a sort of bootstrap.

We remark that a recent theorem of Kahn–Markovic asserts that every hyperbolic 3-manifold contains a π_1-injective immersed closed surface [38]. In fact this likely implies, in combination with Sageev's theorem, that for every hyperbolic 3-manifold M, $\pi_1 M$ is isomorphic to $\pi_1 K$ for some compact, non-positively curved cube complex K. (There is not currently a reference for this assertion, however.) The main advantage of the construction outlined below is that it is sufficiently explicit that one has a reasonable hope of establishing the virtually special criteria.

If L is a hyperbolic link, and L_a is the augmented link obtained from L, the cubulation of $M = S^3 - L_a$ associated to the collection of face surfaces is a square complex C that embeds in M as a spine dual to the right-angled decomposition. An immersion of a surface into C is determined by a formal linear combination of its faces that satisfies certain matching conditions. Such surfaces are not necessarily π_1-injective in $S^3 - L_a$, and even if they are, they do not necessarily π_1-inject into $S^3 - L$ after Dehn filling. However, I believe that methods similar to those of Cooper–Long [16] may ensure π_1-injectivity of those satisfying certain additional geometric constraints, which again may be interpreted in terms of the matching data.

One wishes to produce a collection which, when filled out with Seifert surfaces and those supplied by the Culler–Shalen character variety technology, satisfies Sageev's conditions for obtaining an isomorphism between $\pi_1(S^3 - L)$ and $\pi_1 K$ for some cube complex K. The final step is to use the fact that C is virtually special to remove pathological intersections between hyperplanes of K.

The conjecture below, due to Reid–Walsh, concerns the commensurability relation among a special class of hyperbolic 3-manifolds. We say a knot K is *hyperbolic* if its complement $S^3 - K$ admits a complete hyperbolic structure with finite volume.

Conjecture ([57], Conjecture 5.2(i))**.** Let K be a hyperbolic knot. The commensurability class of $S^3 - K$ contains at most three knot complements.

This conjecture is one of many assertions that pick out hyperbolic knot complements as a distinguished class among all hyperbolic 3-manifolds. It is related to an older result of Reid along these lines, that the figure-8 knot complement is the unique arithmetic knot complement in S^3 [56]. Recently, Boileau–Boyer–Walsh considerably narrowed the range of possible counterexamples to the conjecture above [11]. They showed that if K is a hyperbolic knot such that $S^3 - K$ does not admit *hidden symmetries*, then the conjecture holds.

A manifold M has hidden symmetries if there is a finite-degree cover $N \to M$ such that N has a self-isometry that does not lift a self-isometry of M. We will say that a knot or link K has hidden symmetries if $S^3 - K$ does. There are many reasons to believe that knots with hidden symmetries are rare. In fact, only three knots are known to have hidden symmetries, and they are quite special: the figure-8 and the "dodecahedral knots" of Aitchison–Rubinstein [6]. (These are so-named because their complements decompose into two copies of an ideal dodecahedron.)

Further evidence that knots with hidden symmetries should be unusual is provided by another result of Reid, that any knot with hidden symmetries has cusp field $\mathbb{Q}(i)$ or $\mathbb{Q}(i\sqrt{3})$ [56, Theorem 6]. To put this result in context, a computer search by Nathan Dunfield revealed exactly four hyperbolic knots with this property among those with up to 14 crossings, over 59,000 knots (cf. [35]). The fourth knot is $12n706$, an example with nontrivial symmetries but no hidden symmetries.

In seeking distinct but commensurable knots, it is natural to consider families related by *mutation* along 4-punctured spheres. A mutation is a mapping class with order 2 that acts as an even permutation of the punctures. Mutation is well known to leave many invariants of knots and links unchanged, including such commensurability invariants as the invariant trace field [51] and most likely the Bloch invariant (see the discussion in [14, §6]).

E. Chesebro and I study the commensurability relation among certain families of link complements related by mutation in [14]. We describe arbitrarily large finite families of 2-component links related by mutation along 4-punctured spheres with complements that are commensurable but not isometric. These links have hidden symmetries. In obtaining such links, we first construct a 4-string tangle T in $S^2 \times I$ such that $S^2 \times I - T$ has a special kind of hidden symmetry: there is a mutation of $S^2 \times \partial I - \partial T$ that does not extend over $S^2 \times I - T$, but lifts to a cover over which it does extend. We will call such a homeomorphism a *boundary symmetry with hidden extension.*

Question 2. How widespread are tangle complements that admit a boundary symmetry with hidden extension?

There is a well known tangle S in B^3 such that $B^3 - S$ admits a hyperbolic structure with totally geodesic boundary is isometric to the totally geodesic boundary of $S^2 \times I - T$. As it happens, $B^3 - S$ has a cover matching the cover of $S^2 \times I - T$ with the hidden extension above. Adjoining (B^3, S) to finite collections of $(S^2 \times I, T)$ strung together in sequence thus produces infinitely many tangles that admit a boundary symmetry with hidden extension. The links of [14] are produced by capping off such tangles with a copy of the mirror image of S. Each such link is commensurable with its mutants by the symmetry with hidden extension.

On the other hand, in [14] we also describe arbitrarily large families of incommensurable 2-components, obtained from the links described above using a different mutation isometry. We establish incommensurability by showing that this mutation changes *cusp parameters*, a commensurability invariant of cusp cross-sections. This computation in turn depends on the fact that the cusp cross-sections of S, say, which are Euclidean annuli with geodesic boundary, fall into two different similarity classes; in particular, a mutation that exchanges their boundaries does not have a hidden extension. We suspect that this behavior predominates among tangle complements.

It seems plausible that a variation of the construction above could produce *knot* complements with hidden symmetries, and families of commensurable knot complements related by mutation. In determining whether it does, it would be useful to have a table of, say, two- and three-string tangles in B^3, with few crossings, that admit hyperbolic structures with totally geodesic boundary. This is analogous to a knot table, e.g. in [59]. Similar questions were addressed in [34], although they are not quite germane to our considerations. I intend to study whether the techniques outlined there and in [35] can be adapted to produce such a table. This would also help answer:

Question 3. Does there exist a hyperbolic knot in S^3 whose complement contains a totally geodesic Conway sphere?

A *Conway sphere* in a knot complement is a properly embedded, incompressible 4-punctured sphere with meridional boundary. A positive answer to the question above would be a useful technical tool, as the geometry of such a knot complement is directly related to the geometries of the tangle complements obtained by cutting along the Conway sphere. On the other hand, a negative answer would be another way of picking out knot complements as a special class among all hyperbolic 3-manifolds. This is similar in spirit to the "Menasco–Reid conjecture" [47]: no hyperbolic knot complement in S^3 contains a closed, embedded totally geodesic surface.

2. Volume

A persistent theme in the study of hyperbolic 3-manifolds is recorded in the final sentence of Ch. 6 of Thurston's notes [66]: "One gets the feeling that volume is a very good measure of the complexity of a link *complement*, and that the ordinal structure is really inherent in 3-manifolds." A less-precise paraphrase of this assertion is that volume measures the complexity of a hyperbolic 3-manifold well. This section contains an outline of results that I proved with M. Culler and P. Shalen supporting this theme, and a program for proving more along the same lines.

One practical consequence of the principle above is that one expects the volumes of hyperbolic 3-manifolds to grow with other measures of complexity, for instance the rank of homology or cohomology. Work of Marc Culler, Peter Shalen, and various other authors (among others) supports this expectation. For example [21, Theorem 1.2] asserts that if M is a closed orientable hyperbolic 3-manifold and $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 6$ then vol $M > 3.08$. Under the stronger hypothesis $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 8$, [20, Theorem 1.6] asserts that vol $M > 3.44$.

As some context for these results, the closed, orientable hyperbolic 3-manifold of smallest volume is now known to be the "Weeks manifold", with approximate volume .9427 (see [30]; more on this below). Furthermore, eight different numbers less than 3.08 are known to be volumes of one-cusped hyperbolic 3-manifolds [67]; each is approached from below by an infinite sequence of volumes of closed hyperbolic 3-manifolds (this is the "ordinal structure" mentioned above, a consequence of the hyperbolic Dehn surgery theorem [66, Ch. 5]).

A joint paper due to Culler, Shalen, and me extends [21, Theorem 1.2] as follows:

Theorem 4 ([19], Theorem 6.3)**.** *Let M be a closed, orientable hyperbolic 3-manifold, and suppose that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 4$ and the cup product map $H^1(M;\mathbb{Z}_2)\otimes H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$ has image of dimension at most 1. Then* vol $M > 3.08$.

The proof begins with the same observation as that of [21, Theorem 1.2]: that [7, Corollary 9.3] implies the desired conclusion if there is no 3-generator subgroup $G < \pi_1 M$ that is not free (that is, if $\pi_1 M$ is 3-*free*). If such a subgroup does exist, then the homological hypotheses imply that there is a twofold cover $\widetilde{M} \to M$ with $\dim_{\mathbb{Z}_2} H_1(\widetilde{M};\mathbb{Z}_2) \geq 6$, to which G lifts. Now [21, Proposition 7.1] implies that $\widetilde{M}$ contains a closed incompressible surface of genus 2, and the conclusion follows upon applying the theorem below to $\widetilde{M}$.

Theorem 5 ([19], Theorem 6.1)**.** *Let M be a closed, orientable hyperbolic 3-manifold containing a closed connected incompressible surface of genus 2, and suppose that $\mathrm{Hg}(M) \geq 6$. Then* vol $M > 6.45$.

Above, $\mathrm{Hg}(M)$ refers to the *Heegaard genus* of M, which is the minimal genus of a closed surface that divides M into a disjoint union of two handlebodies. The key new idea in the proof of Theorem 5 is an organizing principle for cylinders properly embedded in the complement of a separating, closed incompressible surface S in M: roughly, since the complexity of M is large (as measured by the Heegaard genus), there must exist such a surface S, of genus 2, with the property that a complementary component is acylindrical, or that each has nonempty "kishkes" (or "guts"). Under these circumstances, combining work of Agol–Storm–Thurston [5] with estimates due to Kojima–Miyamoto [40] and Miyamoto [48] implies the desired volume bound.

In Theorem 1.2 of [25], Shalen and I extended [20, Theorem 1.6] in much the same way as Theorem 4 extends [21, Theorem 1.2]. The main new ingredient is an extension of the main theorem of [40] reflecting the principle that volumes of manifolds with totally geodesic boundary track their topological complexity.

Theorem 6 ([25], Theorem 1.1)**.** *Let N be a compact, orientable hyperbolic 3-manifold with ∂N a connected totally geodesic surface of genus 2. If $\mathrm{Hg}(N) \geq 5$, then* vol $N > 6.89$.

The proof uses tools introduced in [40], but is informed by the philosophy used by Gabai–Meyerhoff–Milley ([29], [30]) in identifying the minimum-volume closed orientable hyperbolic 3-manifold. The following dichotomy is fundamental: if $\ell_1 = \ell(\lambda_1)$ satisfies $\cosh \ell_1 \geq 1.215$, then vol $N > 6.89$; if not, there is a $(1,1,1)$-hexagon C whose interior embeds in N.

We refer above to *return paths* in N by $\lambda_1, \lambda_2, \ldots$, indexed in order of increasing length; each is a properly embedded geodesic arc perpendicular to ∂N at its endpoints. In particular, $\ell_1/2$ is the depth of a maximal embedded collar of ∂N. Kojima–Miyamoto's volume estimates begin with this observation, and ours extend theirs slightly to produce the bound of 6.89 above. An (i,j,k) hexagon is a totally geodesic right-angled hexagon embedded in the universal cover $\partial \widetilde{N}$ so that its edges alternately lie in components of $\partial \widetilde{N}$ and on lifts of the return paths λ_i, λ_j, λ_k.

In the second case of the fundamental dichotomy above, the regular neighborhood $X = \mathcal{N}(\partial N \cup \lambda_1 \cup C) \subset N$ is an embedded codimension-0 submanifold with a well-understood cell decomposition. In particular, we prove that X has incompressible boundary and nonempty kishkes. Applying the estimates of [5] and [48] again, the philosophy that informs [19, Theorem 6.1] yields the alternative $\mathrm{Hg}(N) < 4$ or vol $N > 7.32$. The theorem follows.

Theorem 1.1 of [25] is likely not sharp. Indeed, [28] contains a census of hyperbolic 3-manifolds with totally geodesic boundary and low Matve'ev complexity. This suggests that after the manifolds with minimum volume $6.452\ldots$ described in [40], the next-smallest compact hyperbolic 3-manifolds with totally geodesic boundary have volume approximately 7.10. I believe it should be possible to sharpen the results of [25] via classification, analogous to the classification of low-volume cusped hyperbolic 3-manifolds in [30]. The approach is to classify the manifolds containing two $(1,1,1)$ hexagons on the one hand, and on the other take advantage of their absence to prove better volume bounds for the remaining manifolds.

Additionally, it seems possible that a classification scheme along the lines above could work for the following families of hyperbolic manifolds with totally geodesic boundary: noncompact hyperbolic 3-manifolds with totally geodesic boundary (by [48], the minimal-volume such examples are obtained by gluing faces of a single octahedron), cusped hyperbolic 3-manifolds with compact totally geodesic boundary (a candidate minimal-volume example is identified in [28, Figure 1]), and hyperbolic 3-manifolds with totally geodesic boundary and a closed cusp (I described a candidate minimal-volume example in my thesis, see [23, §4.3]). The technical obstacles here are more formidable—for instance, the shortest "return path" in has length 0 in some classes above—but not insurmountable.

A classification of low-volume hyperbolic 3-manifolds with totally geodesic boundary along the lines proposed above should help in understanding the volumes of certain hyperbolic 3-manifolds without boundary. In particular, the additional information from such classifications could prove useful in determining the minimal-volume 3-cusped hyperbolic 3-manifold, using an approach similar to the one used by Agol in determining the 2-cusped manifolds of minimum volume [4].

Furthermore, such a classification may provide a means for identifying the closed hyperbolic Haken manifold M_0 of lowest volume. There exists a Haken manifold with approximate volume 2.2077, obtained by 5/3-Dehn surgery on the cusped census manifold $m007$ (this seems to be the smallest-volume example known, see [31, Remark 10.11]). The results of [5] thus imply that every incompressible surface in M_0 has empty "guts" to either side. Drilling out a carefully-chosen curve that intersects the surface should result in a one-cusped hyperbolic 3-manifold with small volume (cf. [5, appendix]) and a separating incompressible surface whose complement has topology that can be easily understood.

3. Rank

The *rank* of a group G, $\mathrm{rk}(G)$, is the minimal cardinality of a generating set for G. The rank of abelian groups is easy to discern, and among classes of groups such as free and surface groups, rank is equal to the rank of the abelianization. Until recently however, the ranks of, for example, hyperbolic 3-manifold groups have been largely mysterious, and despite its seemingly fundamental nature, the relation of rank to other invariants of interest has remained unclear.

A classical question, first asked by W. Haken, is whether the rank of $\pi_1 M$ is equal to the Heegaard genus of M when M is a 3-manifold. If S is a Heegaard surface for a 3-manifold M, and V is a complementary handlebody to M, the inclusion $V \to M$ determines a surjection at the level of π_1. Thus the Heegaard genus bounds the rank above; Haken asked if this inequality is ever strict. The answer is "yes": Boileau–Zieschang first described Seifert fibered spaces M with $\mathrm{Hg}(M) = 3$ and $\mathrm{rk}(\pi_1 M) = 2$ [12]. More recently, Schultens–Weidmann exhibited a family of graph manifolds $\{M_n\}_{n\geq 2}$ such that $\mathrm{Hg}(M_n) = 4n$ and $\mathrm{rk}(\pi_1 M_n) \leq 3n$ for each n [61].

The answer to Haken's question is still not known for *hyperbolic* 3-manifolds. Recently, however, some evidence has accumulated supporting the following broad principle: among many families of hyperbolic manifolds, geometric reasons ensure that a "generic" member will have closely related rank and Heegaard genus. Below we outline some further evidence for this principle, some questions and conjectures, and a strategy for answering them.

Our particular focus will be the *rank gradient* among covers of a fixed hyperbolic manifold. Given a finitely-generated group G, M. Lackenby [42] defined the rank gradient of a family of finite–index subgroups $\{G_n\}$ as:

$$\mathrm{rg}\,\{G_n\} = \liminf_{n\to\infty} \frac{\mathrm{rk}\,G_n - 1}{[G : G_n]}.$$

The rank gradient is positive if and only if $\mathrm{rk}\,G_n$ grows linearly with its index in G. If G is the fundamental group of a 3-manifold M, it is easy to see that the Heegaard genus of the covers $M_n \to M$ corresponding to the G_n grows at most linearly with the index. Thus if some family of covers of M has positive rank gradient, the ratio of rank to Heegaard genus among these covers is at least bounded away from 0.

I proved the following result about the rank gradient of "cyclic covers" in [24].

Theorem 7 ([24], Theorem 0.1)**.** *Let G be the fundamental group of a hyperbolic 3-manifold of finite volume, and suppose there is a homomorphism $\phi\colon G \twoheadrightarrow \mathbb{Z}$. Then $\mathrm{rg}\,\{\phi^{-1}(n\mathbb{Z})\} = 0$ if and only if* $\ker\ \phi$ *is finitely generated.*

This generalizes [43, Theorem 1.11], which proved the identical alternative for the *Heegaard gradient* of cyclic covers. (The Heegaard gradient of $\{M_n \to M\}$ is a ratio analogous to rank gradient, where the numerator is replaced by the Heegaard genus and the denominator by the covering degree.)

In the situation of Theorem 7, if $\ker\phi$ is finitely generated then the rank of $\{\phi^{-1}(n\mathbb{Z})\}$ is in fact universally bounded above, by $\mathrm{rk}\,(\ker\phi) + 1$, since $\phi^{-1}(n\mathbb{Z})$ is generated by a set of generators for $\ker\ \phi$ and an element mapping to the generator of $n\mathbb{Z}$. Stallings' fibration theorem implies that in this case $\ker\ \phi$ is the subgroup corresponding to the fiber surface of a fibration $M \to S^1$. Thus Theorem 7 describes a stark dichotomy: either the rank of $\{M_n\}$ grows linearly with the degree of the covering, or it does not grow at all.

I intend to study further what other families of finite covers $\{M_n \to M\}$ enjoy positive rank gradient. Recent work of Lackenby [43], Abert–Nikolov [1], and Long–Lubotzky–Reid [44] among others has drawn attention to the following:

Question 8. Under which circumstances does a *co-final* family of finite covers (ie, $\{M_n \to M\}$ such that $\bigcap \pi_1 M_n = \{1\} \subset \pi_1 M$) have positive rank gradient?

Abert–Nikolov showed that a virtually fibered hyperbolic 3-manifold M has a co-final tower of finite regular covers $\{M \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots\}$ with the property that $\mathrm{rg}\,\{\pi_1 M_n\} = 0$; and furthermore, that the "fixed price" conjecture from topological dynamics implies every co-final tower of regular covers has the same rank gradient (see [1, Theorem 2]). On the other hand, Lackenby showed that families of covers with property τ have positive Heegaard gradient [43, Theorem 1.5]. Many virtually fibered hyperbolic 3-manifolds have such towers with property τ (see e.g. [44]); thus the fixed price conjecture would imply that among such covers the ratio of rank to Heegaard genus tends to 0.

Some standard tools of geometric group theory apply to Question 8 when the 3-manifold M in question contains an embedded incompressible surface. A family of such surfaces in M determines a *splitting* of M; an action of $\pi_1 M$ on a tree whose quotient graph has an edge for each surface and a vertex for each complementary component. Given a splitting, *acylindrical accessibility* bounds the rank of $\pi_1 M$ below using the combinatorics of the quotient graph, and *Grushko's theorem* bounds the rank below in terms of groups associated to its edges and vertices.

An acylindrical accessibility theorem proved by R. Weidmann [69] is one of the fundamental tools used in the proof of Theorem 7. It implies in particular that when a family of surfaces in M is acylindrical, the rank of $\pi_1 M$ is bounded below by $(v+1)/2$, where v is the number of vertices of the quotient graph. The classical Grushko theorem asserts that the rank of a free product of groups is equal to the sum of the ranks of the factors. Weidmann proved a version for amalgamated products $G = A *_C B$ with malnormal amalgam [68] (this occurs for example when M is split by a closed, embedded, separating totally geodesic surface), showing:

$$\text{rk}\, G \geq \frac{1}{3}(\text{rk}\, A + \text{rk}\, B - 2\,\text{rk}\, C + 5). \tag{1}$$

More generally, we will say a "Grushko-type" inequality for a family of groups is one of the form $\text{rk}\, G \geq \alpha(\text{rk}\, A + \text{rk}\, B - \gamma\,\text{rk}\, C + \beta)$ for constants α, β, and γ.

It is unavoidable that the rank of the amalgam C will impose some penalty on the rank of the amalgamated product. For instance, if minimal-rank generating sets for A and B each contained a minimal-rank generating set for C, then there would be a generating set for G with rank $\text{rk}\, A + \text{rk}\, B - \text{rk}\, C$. In fact, there exist Fuchsian groups with this property for which the resulting generating set is not even of minimal rank (see e.g. [52]).

On the other hand, if a 3-manifold M is split by S into manifolds-with-boundary N_A and N_B, it frequently happens that $A = \pi_1 N_A$ and $B = \pi_1 N_B$ have smaller rank than $C = \pi_1 S$. However if g_A and g_B are the Heegaard genera of N_A and N_B, respectively, then an "amalgamated" Heegaard splitting may be constructed for M with genus $g_A + g_B - g_S$, where g_S is the genus of S. The generating set for $\pi_1 M$ that corresponds to this splitting thus has rank $r_A + r_B - \frac{1}{2}\text{rk}\, C$, where $r_A = g_A$ and $r_B = g_B$ are the ranks of generating sets for $\pi_1 N_A$ and $\pi_1 N_B$, respectively, corresponding to minimal-genus splittings.

The observation above suggests that among any family of 3-manifolds that split, the optimal coefficient of $\text{rk}\, C$ in a Grushko-type inequality occurs in the range $[1/2, 2]$. In fact I believe that among those split along totally geodesic surfaces, the "generic" coefficient should be $1/2$:

Conjecture 9. For each $g \geq 2$ there exists $L(g) < \infty$ such that if a closed 3-manifold M is split by a closed, embedded, separating totally geodesic surface S of genus g into manifolds with boundary N_A and N_B, and the shortest return paths of N_A and N_B have length at least $L(g)$, then $\text{rk}\, \pi_1 M = \text{rk}\, \pi_1 N_A + \text{rk}\, \pi_1 N_B - g$.

This conjecture is similar in spirit to a result due to Namazi–Souto [50], which describes an analogous conclusion for a manifold with a Heegaard surface that has a deep collar. Below I will describe the philosophy that motivates my conjecture and the result of Namazi–Souto; for now let us observe that the conclusion of the conjecture does not hold in general, as the following example shows.

Let M be a fibered hyperbolic 3-manifold containing a separating, closed embedded totally geodesic surface S (certain covers of the figure-8 knot complement have this property). Under the map $\phi\colon \pi_1 M \to \mathbb{Z}$ determined by the fibering, $\pi_1 S$ must surject $k\mathbb{Z}$ for some $k \in \mathbb{N}$. Thus S has connected preimage in the covers $M_{nk} \to M$ corresponding to $\phi^{-1}(nk\mathbb{Z})$, $n \in \mathbb{N}$, each itself a separating totally geodesic surface. The genus of this surface is $nk\cdot(g-1)+1$, where g is the genus of S. However, the rank of $\pi_1 M_{nk}$ is universally bounded.

For a 3-manifold N with boundary, the "half lives, half dies" lemma implies that with coefficients in any fixed field, the rank of $H_1 N$ is at least half that of $H_1 \partial N$. Hence $\mathrm{rk}\,\pi_1 N \geq \frac{1}{2}\mathrm{rk}\,H_1\partial N$, and among any family of hyperbolic 3-manifolds for which the coefficient of $\mathrm{rk}\,C$ is less than 1 in an optimal Grushko-type inequality, the rank of $\pi_1 M$ increases with the genus of a separating acylindrical surface S. Among the covers described above, the optimal such constant is thus at least 1.

Given Grushko's theorem and "half lives, half dies," a second approach to understanding rank gradient through splittings lies in answering the following question.

Question 10. Let $\{p_n\colon N_n \to N\}$ be a sequence of connected finite covers of a manifold N with boundary. What is the *relative rank gradient*

$$\liminf_{n\to\infty} \frac{\mathrm{rk}\,\pi_1 N_n - \frac{1}{2}\mathrm{rk}\,H_1(\partial N_n)}{\deg\, p_n}?$$

By a standard argument, $\mathrm{rk}\,\pi_1 N_n > \frac{1}{2}\mathrm{rk}\,H_1(\partial N)$ for any manifold N with boundary, so all terms are positive in any sequence as above. On the other hand, multiplicativity of the Euler characteristic implies that in any such family of covers, $\mathrm{rk}\,H_1(\partial N_n) \geq \deg\, p_n\cdot(\mathrm{rk}\,H_1(\partial N)-2))$. The Reidemeister–Schreier process thus implies that the relative rank gradient is bounded above by $\mathrm{rk}\,\pi_1 N - \frac{1}{2}\mathrm{rk}\,H_1(\partial N)$. Examples show this difference may be arbitrarily large.

Geometric methods may yield insight into Conjecture 9 and Question 10. The motivating insight for this approach is due to M. White [70]: when M is a closed hyperbolic 3-manifold, any set of generators for $\pi_1 M$ determines a homotopy class of *carrier graphs* in M with a representative with minimal length. The first fruit of this insight was White's theorem that the injectivity radius of a closed hyperbolic 3-manifold M is universally bounded above in terms of $\mathrm{rk}(\pi_1 M)$. This implies for instance, that for every cofinal tower of covers $\{M_n \to M\}$,

$$\liminf\ \mathrm{rk}(\pi_1 M_n) = \infty.$$

White's ideas were refined by Agol, in proving that for all but finitely many hyperbolic 3-manifolds M with $\mathrm{rk}(\pi_1 M) = 2$, the Heegaard genus of M is also equal to 2 (cf. [64, §9]). J. Souto has taken these ideas still further, in solo work and with others (e.g. [65], [50], [10]), using the following "decomposition theorem": a carrier graph divides into an ascending chain of subgraphs, each of which does not stray far outside the convex hull of the preceding subgraph (cf. [64, Prop. 7.3]).

The idea in approaching Conjecture 9 and Question 10 is to play off Souto's decomposition of a carrier graph against the geometry of hyperbolic 3-manifolds with totally geodesic boundary. If N is such a manifold, the *cut locus* $\mathcal{C}(N)$ is the set of points in N with more than one closest point on ∂N. This is a spine for N, with a canonical cellular decomposition whose 2-cells consist of points with exactly 2 nearest points on ∂N. Each such is dual to a unique return path of N.

The cut locus determines a "geometric" Heegaard splitting of N as follows. Let H_1 be the union of a collar of the boundary with small regular neighborhoods of the return paths dual to the 2-cells of $\mathcal{C}(n)$, and let $H_2 = \overline{N - H_1}$. Then H_1 and H_2 are handlebodies, and the one-skeleton of $\mathcal{C}(N)$ is a spine for H_2. The idea in proving Conjecture 9 is to show that in a 3-manifold split along a totally geodesic surface into manifolds with very long return paths, a minimal-genus Heegaard splitting must be obtained by amalgamating geometric Heegaard splittings on either side, and using Souto's decomposition theorem, that a minimal-rank carrier graph is a spine for this splitting.

I believe that similar ideas may have something to say about the rank gradient of covers of the following form: let M be a hyperbolic 3-manifold containing an embedded, separating totally geodesic surface S, and let $\{M_n \to M\}$ be a cofinal tower of regular covers that inherits property τ from the associated family of covers $\{S_n \to S\}$. The existence of such covers was first pointed out in work of Long–Lubotzky–Reid [44, Prop. 4.1].

By the work of Abert–Nikolov discussed above, the fixed price conjecture would imply that if M were virtually fibered, then any such family of covers would have rank gradient 0. Some evidence for the fixed price conjecture combines work of Lück [46] and Lott–Lück [45], which together imply that the "Betti number gradient" of any cofinal tower of regular covers must be 0. This evidence must be considered very weak however, as the example of knot complements shows that among hyperbolic 3-manifolds the rank of H_1 exerts very little control over the rank of π_1. Indeed, I suspect that among many families supplied by [44], the rank gradient is positive.

It also seems that the approach outlined above may shed light on a conjecture of Shalen, that the rank of a hyperbolic 3-manifold decreases by at most 1 when passing to a finite-degree cover [63, Conjecture 4.2]. The broad theme is that the rank of manifolds that split along incompressible surfaces is well-controlled by the genera of these surfaces. Since the sum of these genera increases in finite-sheeted covers of such manifolds, one also expects the rank of these manifolds to increase.

4. Tangential Directions

This section collects a number of questions and issues not directly relevant to the investigations above, but that seem worth pursuing. Many of them arose in the course of work discussed above.

Question 11. Let G be a finitely generated group and $H < G$ a finitely generated subgroup such that G splits over H, with associated tree T. Give a bound on the cylindricity of the action $G \times T \to T$ that depends only on the rank of H.

When G is the fundamental group of a closed hyperbolic 3-manifold M and H corresponds to a connected incompressible surface S that is not a fiber or semi-fiber of M, I proved a bound of $7r - 12$ in [24, Corollary 1.6], where $r = \text{rk } H$. Using Stallings' fibration theorem, the requirement that S not be a fiber or semi-fiber can be rephrased as G is not an extension of H by $\mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, respectively.

The proof of Corollary 1.6 is a combinatorial argument that relies on the characteristic submanifold theory for 3-manifolds with boundary [36], [37]. It is possible that its conclusion may be extended to general hyperbolic groups using the characteristic submanifold theory developed by Scott–Swarup [62].

Question 12. Let Γ_m be a lattice in $\text{Isom}(\mathbb{H}^m)$ and Γ_n a lattice in $\text{Isom}(\mathbb{H}^n)$, where $n > m$. When does Γ_m embed as a quasi-convex subgroup of Γ_n?

The motivation for this question comes from [15], where we describe hyperbolic 3-manifold groups that fall under the purview of Theorem 1 but are not commensurable with any 3-dimensional hyperbolic right-angled reflection group. It seems likely, however, that some of the groups that we describe are commensurable with quasi-convex subgroups of $PO_0(4, 1; \mathbb{Z})$. This is a 4-dimensional arithmetic lattice commensurable with a right-angled reflection group, so nice properties of its quasi-convex subgroups are implied by work of Agol–Long–Reid [2].

The quasi-convex representations that we refer to above are produced by a "bending" construction. This has a long history going back to Thurston's notes, and more recently addressed in [13] and [8], for example. Say a representation $\rho\colon G \to \text{Isom}(\mathbb{H}^n)$ is of *bending type* if G has a graph of groups decomposition such that for each vertex group H, ρ maps H into the stabilizer of a geodesic subspace of $\mathbb{H}^n$ with positive codimension. Then a question that bears on Question 12 is:

Question 13. Let Γ be a lattice in $\text{Isom}(\mathbb{H}^n)$ for $n \geq 4$. Does there exist a lattice $\Gamma' < \text{Isom}(\mathbb{H}^m)$, $3 \leq m < n$ and a faithful representation $\Gamma' \to \Gamma$ that is not of bending type?

Let M be a hyperbolic 3-manifold containing distinct embedded, totally geodesic surfaces that intersect. These determine independent families of bending-type representations into $\text{Isom}(\mathbb{H}^4)$; thus the $\text{Isom}(\mathbb{H}^4)$-character variety of $\pi_1 M$ should have dimension at least 2 near the totally geodesic representation. Since discrete, faithful representations occupy an open subset of this variety, but the bending type representations determine at most countably many lines, there exist discrete, faithful representations that are not of bending type. It would be interesting to have an explicit description of such a representation, and to know if any higher-dimensional lattice could contain its image.

One could equally well ask Questions 12 and 13 for representations into other rank-one Lie groups. This problem is studied for $PU(3,1)$ in recent work of Cooper–Long–Thistlethwaite [17], [18]. They describe 3-dimensional (real) hyperbolic lattices with large families of discrete, faithful representations into $PU(3,1)$, including some that are not of bending type. In fact, a non-bending type representation of $\pi_1(vol3)$ into a lattice is displayed ("$vol3$" is a closed hyperbolic 3-manifold whose volume is that of the regular ideal tetrahedron in $\mathbb{H}^3$). Although this representation is not known to be faithful, the authors give evidence that it most likely is (see the end of §2 in [17]).

Question 14. Let M be a hyperbolic 3-manifold, and let g be a Riemannian metric on M with negative sectional curvatures. Is there a smooth one-parameter family of metrics on M that interpolate between g and the hyperbolic metric?

This was a motivating question for [22]. The answer to the corresponding question in higher dimensions is an emphatic "no", by work of Farrell–Ontaneda [27]. In three dimensions, though, the proof of geometrization suggests that geometric flows may provide a positive answer. Besides, there is a nice stability theorem [39] showing that with respect to Ricci flow, the 3-dimensional situation is different from higher dimensions [26].

Question 15. Does there exist a hyperbolic 3-manifold M with finite volume and one cusp, such that a minimal-rank generating set for $\pi_1 M$ contains a pair of commuting parabolic elements?

This question arose during conversations with Ian Biringer. The circumstances under which the Heegaard genus of a one-cusped hyperbolic manifold is greater than that of its surgeries is well-understood; (see [49] and [58], for example). In particular, outside of a compact set in the Dehn surgery space, the surgeries that reduce Heegaard genus occur only along a finite set of "lines." Thus if the question above had a positive answer, then surgery on such a manifold M would yield infinitely many closed manifolds with different rank and Heegaard genus, since each surgery on M would reduce rank. If the answer was negative, this would provide evidence for a conjecture of Shalen, that every one-cusped hyperbolic 3-manifold has an infinite family of hyperbolic Dehn surgeries under which the rank does not decrease [63, Conjecture 5.1].

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