Bratteli Diagrams For Weak Solenoids

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Questions/Problems

• We study dynamical systems given by group actions on Cantor sets. The goal is to develop invariants for these systems which reflect the structure of the group, and the dynamics of the action.

• Classification up to orbit equivalence for $\mathbb{Z}$ and $\mathbb{Z}^n$-actions on Cantor Sets was done in a series of works by Giordano, Herman, Matui, Putnam, Skau, Forrest, and others.

• We study how well we can implement their methods for non-abelian actions.

• Let Γ be a finitely generated non-abelian group acting on a Cantor set X. We construct a Bratteli-type model for this system.

• Theorem (work in progress): For Γ a finitely generated group, acting minimally and equicontinuously on a Cantor set X, there exists a Bratteli model that represents the group action dynamics.

• We also study the role of normalizers in classifying the action.

• This work is joint with Steve Hurder and Olga Lukina (UIC).

Dynamics

- The action of Γ on the Cantor set X defines the homomorphism $\phi : \Gamma \to \mathrm{Homeo}(X)$.

- For each $x \in X$, denote by $\Gamma(x) = \{\phi(\gamma)(x) | \gamma \in \Gamma\}$ the *orbit* of x under the action of Γ.

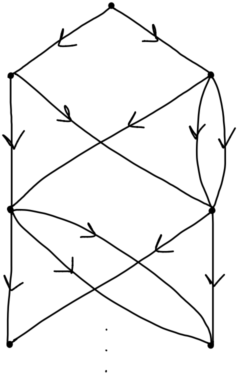
- The action of Γ on X is called *minimal* if every orbit is dense in X.

- The action of Γ on X is called *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that for all $g \in \Gamma$ and any $x, y \in X$, if $d(x, y) < \delta_\varepsilon$, then $d\left(\phi(g)(x), \phi(g)(y)\right) < \varepsilon$.

- We study the special case where Γ is finitely generated, and the action ϕ is minimal and equicontinuous.

Bratteli Diagrams

- A *Bratteli Diagram* $B = (V, E)$ is an infinite directed graph with one vertex at the top level, a finite number of vertices at each subsequent level, and edges all directed downwards.



• The points of X are represented by infinite paths in the diagram. The infinite path space is homeomorphic to X.

• An ordered Bratteli diagram comes equipped with an order on the edges ending at each vertex. This induces a reverse lexicographic order on the entire space of infinite paths.

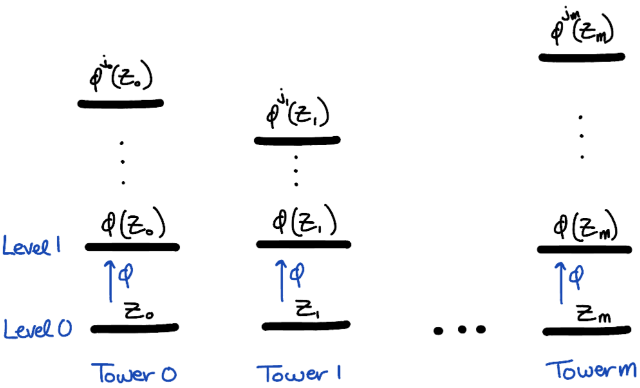
• In the $\mathbb{Z}$ or $\mathbb{Z}^n$ case, an order and associated Vershik map are added to represent the dynamics. We want to extend our model in a similar way.

• The question is, for non-abelian groups, how to model the dynamics on the path space?

Kakutani-Rokhlin Partitions

- In the case of a $\mathbb{Z}$-action, the Bratteli diagram is constructed using Kakutani-Rokhlin Partitions.

- Let (X, ϕ) be a Cantor system, $x \in X$ a basepoint. Let $Z \in X$ be a clopen set containing x. Let $\{Z_k\}_{k=1,\ldots,m}$ be a finite clopen partition of Z. One Rokhlin partition for the action:



• A Bratteli diagram is then constructed using an infinite nested sequence of refining Rokhlin partitions.

• For abelian left-orderable groups such as $\mathbb{Z}^n$, a similar technique can be employed, as in work by Forrest.

• For a more general group Γ, we use a method developed in the paper of Clark and Hurder, *Homogeneous matchbox manifolds*, Trans. AMS, 365 (2013).

• We introduce an analogous sequence of refining partitions representing our system.

AF Presentation

- An Almost Finite (AF) Presentation of (X, Γ, ϕ) is an infinite nested sequence of partitions $V^{(i)}$ of X with the following properties:
- Each partition $V^{(i)}$ is clopen and finite.
- $V^{(0)} = X$, and $V^{(i)}$ refines $V^{(i-1)}$.

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- For each i, there is a finite set (alphabet) $\mathcal{B}^{(i)}$ and a map $\phi_i : X \to \mathcal{B}^{(i)}$ such that, for each $b \in \mathcal{B}^{(i)}$, $\phi_i^{-1}(b) = V_k^{(i)}$ for some k. That is, the ith alphabet numbers the pieces of the ith partition.

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• For each $i \geq 1$ and each $b \in \mathcal{B}^{(i)}$ and a sequence ϵ_i such that $\lim_{i\to\infty} \epsilon_i = 0$, we have $\mathrm{diam}(\varphi_i^{-1}(b)) < \epsilon_i$. That is, the diameters of the clopen sets in the partition decrease to 0.

• There are inclusion maps $k_i : \mathcal{B}^{(i)} \to \mathcal{B}^{(i-1)}$ which are compatible with φ_i, that is, for any $x \in X$, $k_i \circ \varphi_i(x) = \varphi_{i-1}(x)$.

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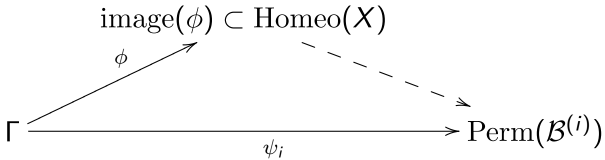
• There exists a homomorphism $\psi^{(i)} : \Gamma \to \mathrm{Perm}(\mathcal{B}^{(i)})$ which factors through the representation $\phi : \Gamma \to \mathrm{Homeo}(X)$. that is, the following diagram commutes.



and

$$\varphi_i\left(\varphi(g)(x)\right) = \psi_i(g)\left(\varphi_i(x)\right). \tag{1}$$

• Inclusions k_i are compatible with homomorphisms ψ_i in the following sense:
if $b \in \mathcal{B}^{(i)}$, then for any $g \in \Gamma$ $k_i(\psi_i(g)(b)) = \psi_{i-1}(g)(k_i(b))$.

• Then $\{V^{(i)}, \varphi_i, \mathcal{B}^{(i)}, k_i, \psi_i\}$ is an AF Presentation for (Γ, ϕ, X).

• How do we build an AF system for (X, Γ, ϕ)?

• We start with an arbitrary clopen partition $W^{(1)} = \{W_1^{(1)}, ..., W_{n_i}^{(1)}\}$ of X, and then use coding to obtain a partition $V^{(1)} = \{V_1^{(1)}, ..., V_{n_i}^{(1)}\}$ that is permuted by Γ.

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• For each $x \in X$, define a coding function $C_x^{(1)} : \Gamma \to \{1, ..., n\}$ as follows:

$$C_x^{(1)}(\gamma) = i, \quad \text{if } \gamma(x) \in W_i^{(1)}$$

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• We create our new partition out of such sets and renumber them, so $V^{(1)} = \{V_1^{(1)}, ..., V_{n_1}^{(1)}\}$ consists of sets on which the coding function (with respect to $W^{(1)}$) is constant.

• Inductively, we again arbitrarily partition $V_1^{(i-1)}$ and again use coding to obtain the *ith* partition $V^{(i)} = \{V_1^{(i)}, ..., V_{n_i}^{(i)}\}$, where the elements of Γ permute the partition elements.

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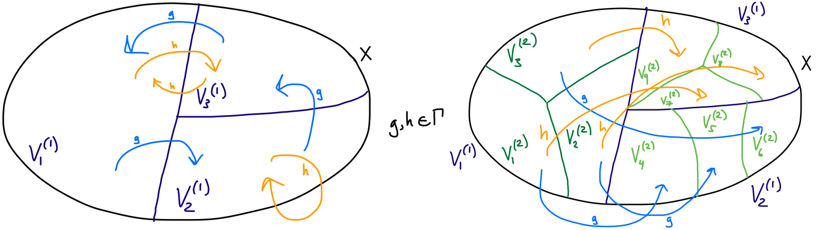
- Thus we obtain an AF presentation for the group action.



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- Theorem: Given such an AF Presentation, then:

  1. There is a homeomorphism $f : X \to \mathcal{B}_\infty$, where $\mathcal{B}_\infty = \varprojlim \left\{ \mathcal{B}^{(i)}, k_i \right\}$ is the inverse limit.
  2. There is a well-defined action of Γ on $\mathcal{B}_\infty$.
  3. The homeomorphism f conjugates the Γ-action on X with the Γ action on $\mathcal{B}_\infty$.

- $\mathcal{B}_\infty$ is a Bratteli-type representation of the system.

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• Consider an infinite chain of nested finite index subgroups

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- Let $X_i = \Gamma/\Gamma_i$, which is a finite set, and not a singleton.
- Then the inverse limit $X = X_\infty = \varprojlim X_i$ is a Cantor set.
- Γ acts on X on the left.

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- $N_* \subset N_i\ \forall i$, so $\lim_{\leftarrow} N_*/\Gamma_i$ acts on each X_i on the right, and so on X.

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- If the chain stabilizes, i.e. $\exists k \forall i \; N_i = N_{i+1}$, and N_* is nontrivial, then we call the chain subnormal.
- $N_* \subset N_i \; \forall i$, so $\lim_{\leftarrow} N_*/\Gamma_i$ acts on each X_i on the right, and so on X.
- Thus the left Γ-action commutes with the right N_*/Γ_i-action.

Main Result

Theorem: Let Γ_i be a subnormal group chain.

- Then there is a Bratteli-type model for the action of Γ on the Cantor set X;
- A dynamical system defined by the right action of N_* on each level of the diagram;
- If N_* is a left-orderable group, then the subrelation defined by the action of N_* admits a Vershik-type transformation.

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- Work in progress: use the result above to classify actions of non-abelian groups on a Cantor set up to orbit equivalence.

Example: Schori Solenoid

- The Schori Solenoid is a particular example of a weak solenoid.

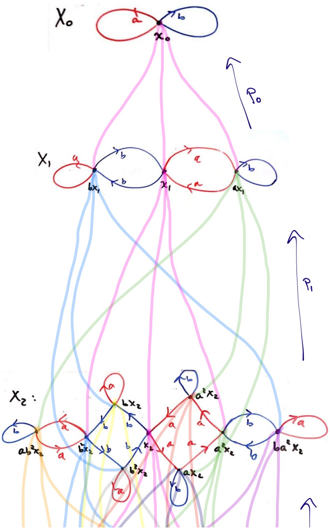
- It is given by 3-1 covering maps with the base space X_0 being a surface of genus 2.

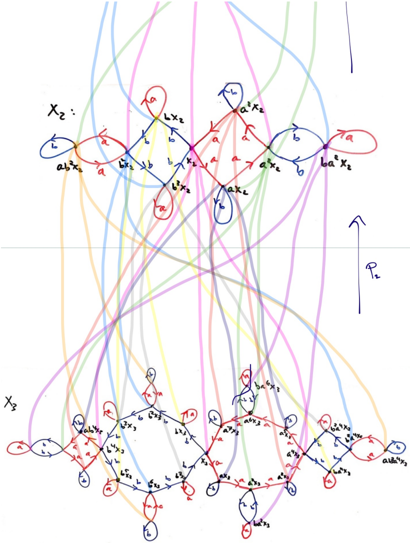
- It is constructed by taking three copies of X_i and cutting and gluing them according to a particular pattern to obtain X_{i+1}.

- In the following diagrams, the Schrier diagram for each of the first four levels is shown, as well as the inclusion maps between levels.

- In the following schematic diagrams, each edge can be "thickened up" into a tube to visualize the surface.

Schori Solenoid





Schori Solenoid

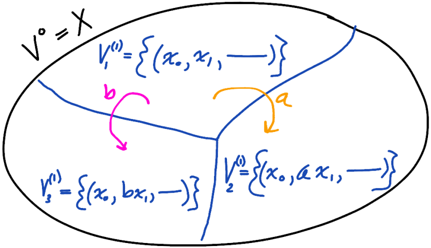
• The Cantor set here is given by the fiber F over a basepoint x_0. The fundamental group $\Gamma = \pi_i(X_0)$ (non-abelian) acts on F minimally and equicontinuously.

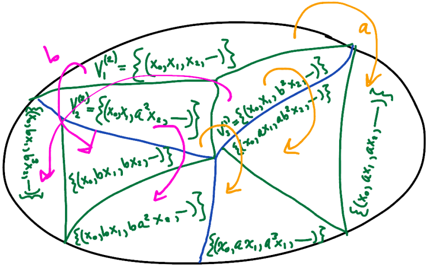
• This type of diagram can be extended inductively, but will get increasingly more complicated. So, we want to represent its properties in a simpler model.

• Letting $\Gamma_i = \pi_1(X_i)$, we have a group chain $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \ldots$.

Schori Solenoid

- The partitions in this case can be given by cylinder sets.





Semi-Direct Product Example

• In the weak solenoid case, the group chain came from a chain of covering spaces, but we can follow the same technique with any group chain, regardless of covering spaces.

• Let $\Gamma = \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}$, where θ is a homomorphism $\theta : \mathbb{Z} \to Aut(\mathbb{Z}^2)$.

• Let $H_i = \mathbb{Z}^2 \rtimes_\theta p^i\mathbb{Z}$, for some integer p. So we have a group chain $\Gamma = H_0 > H_1 > H_2 > \ldots$

• Each $H_i \trianglelefteq \Gamma$ (finite index), but Γ is not abelian.

• We obtain a Cantor set: $K = \lim_{\leftarrow}\{H_0/H_{i-1} \leftarrow H_0/H_i\}$

• Γ acts on K, giving a dynamical system.

• We can use a method of partitions similar to Rokhlin partitions to build a Bratteli model for this dynamical system.

• Here, N_* is nontrivial and the chain is subnormal.

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