RESEARCH STATEMENT

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1. Introduction

My research interests are in the area of dynamical systems, specifically group actions on topological spaces. I focus on classification questions about these systems.

Classical dynamics is the study of iterating a single function f on a topological space X, and of the behavior of the orbits $\{f^n(x)\}_n \in I$ for points $x \in X$, where I is $\mathbb{N}$ or $\mathbb{Z}$. If I is $\mathbb{Z}$, f is a homeomorphism, i.e. a function that is continuous and has a continuous inverse, but may not be smooth or have a derivative. We can expand this idea by considering actions on a topological space X by a more general but still finitely generated group Γ. A group action of Γ on X is a map $\phi : \Gamma \to Homeo(X)$, taking $\gamma \to \phi_\gamma$, in a way that respects the Γ group operation. Thus, this map is no longer generated by a single homeomorphism iterated in a single direction, but instead may be generated by several homeomorphisms iterated in a way that can be viewed as multi-dimensional or multidirectional.

Classification problems deal with finding invariants to classify dynamical systems up to various levels of equivalence. There are several different types of equivalence that may result in stronger or weaker classification theorems. The strongest equivalence relation between systems is topological conjugacy; (X, Γ, ϕ) and (Y, Δ, ψ) are topologically conjugate if there is a homeomorphism h that conjugates the two actions. We consider a weaker notion of equivalence known as orbit equivalence. The systems (X, Γ, ϕ) and (Y, Δ, ψ) are orbit equivalent if there is a homeomorphism which maps orbits of ϕ into orbits of ψ, without necessarily preserving the time parametrization of orbits. Giordano, Putnam, Matui, and Skau have classified $\mathbb{Z}^2$ ([8]) and $\mathbb{Z}^n$ ([9]) actions on Cantor sets up to orbit equivalence. My work deals with classifying, up to orbit-equivalence, dynamical systems that come from minimal equicontinuous group actions by non-Abelian groups on Cantor sets.

There are many natural examples of dynamical systems given by non-Abelian group actions on Cantor sets. One of the basic examples, which is also studied in my work, is a weak solenoid. A weak solenoid is the inverse limit of compact manifolds which are finite-to-one covering spaces of each other. The projection onto the first manifold (also called the base space) in the sequence is a fiber bundle, and the fiber over a point is a Cantor set, naturally acted upon by the fundamental group of the base space. This gives a rich type of dynamical system that fits our above criteria, a finitely generated group acting topologically on a Cantor set.

One classic example of a weak solenoid is the Schori solenoid, given by 3-1 coverings of a genus 2 surface. There are also examples coming from the discrete Heisenberg group, and other non-Abelian groups. My work aims to classify these examples up to orbit equivalence.

In my thesis, I approach the classification problem based on expressing Cantor dynamical systems as group chains or AF presentations, and develop algebraic invariants of the AF presentations.

2. Background

We now introduce some needed definitions. Let X be a topological space with metric d. Let Γ be a group that acts on X by homeomorphisms, that is, there is a homomorphism $\phi : \Gamma \to \text{Homeo}(X)$. For each $x \in X$, denote by $\Gamma(x) = \{\phi_\gamma(x) | \gamma \in \Gamma\}$ the *orbit* of x under the action of Γ. Two actions on X, $\phi : \Gamma \to \text{Homeo}(X)$ and $\psi : \Gamma \to \text{Homeo}(X)$, are *orbit equivalent* if there is a homeomorphism h mapping orbits of ϕ continuously onto orbits of ψ. The action of Γ on X is called *minimal* if the closure $\overline{\Gamma(x)} = X$ for all $x \in X$. The action of Γ on X is called *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that for all $g \in \Gamma$ and any $x, y \in X$, if $d(x, y) < \delta_\varepsilon$ for $y \in X$, then $d\left(\phi(g)(x), \phi(g)(y)\right) < \varepsilon$. A set X is called a *Cantor set* if it is perfect, metrizable, compact, and totally disconnected (or, equivalently, if it is homeomorphic to the standard middle thirds Cantor set).

From now on, we assume X is a Cantor set, Γ a finitely generated group, and Γ acts on X minimally and equicontinuously.

DEFINITION 2.1. A *Bratteli diagram* (P, E) is an infinite directed (connected) graph with vertex set P and edge set E satisfying the following conditions:

(1) The vertex set $P = P_0 \cup P_1 \cup ...$ consists of disjoint levels $P_0, P_1, ...$ such that P_0 is a single point and each P_n is a finite, non-empty set. We refer to P_n as the vertex set at level n.
(2) The edge set $E = E_1 \cup E_2 \cup ...$ consists of disjoint levels $E_1, E_2, ...$ such that E_n is the set of edges with source in P_{n-1} and range in P_n.
(3) Let r be the range map and s the source map on this directed graph. Then $s^{-1}(p) \neq \emptyset$ for all $p \in P$, and $r^{-1}(p) \neq \emptyset$ for all $p \in P \setminus P_0$.

We draw this diagram so that P_0 is a single point at the top. Then the definition implies that all edges are directed downwards from one level to the subsequent level. If $P_n = \{p_{n,1}, ..., p_{n,k}\}$, we draw our diagram with the vertices ordered from left to right in numerical order. It follows from the definition that the space of infinite paths in a Bratteli diagram is a Cantor set.

An order can also be added to a Bratteli diagram to carry information about the dynamics of the system. An *ordered* Bratteli diagram is a Bratteli diagram equipped with an order on each set of edges sharing a range vertex. This then induces a (reverse) lexicographic partial order on the infinite path space. The *Vershik map* is a map on the infinite path space of a Bratteli diagram, mapping each path to its successor based on this order. The Vershik map makes an ordered Bratteli diagram into a dynamical system.

The work of Durand, Host, Skau ([1]), Giordano, Putnam, and Skau ([7]), and Herman, Putnam, Skau ([10]) shows how to build a Bratteli diagram for a $\mathbb{Z}$ action using a refining nested sequence of Kakutani-Rokhlin partitions. The work of Forrest ([6]) and Giordano, Matui, Putnam, and Skau ([9]) extends this idea to $\mathbb{Z}^2$ and $\mathbb{Z}^d$ actions, which are multidimensional but still abelian. Our work consists of further extending a Bratteli-type model to the case of non-abelian finitely generated group actions; for example, the discrete Heisenberg group, or the fundamental group of a (genus ≥ 2) compact manifold.

3. Main Results

In my thesis, I aim to classify equicontinuous group actions on Cantor sets up to orbit equivalence. For that, I use group chains and almost finite presentations to represent a dynamical system.

Based on work by Thomas ([13]), Clark and Hurder ([2]), and others, we build an model for a finitely generated discrete group that acts minimally and equicontinuously on a Cantor set, which we call an Almost Finite (AF) Presentation for the system. Under certain conditions, we use this model to associate to the action an unordered Bratteli diagram. We are in the process of building an order into the model, analogous to an ordered Bratteli diagram.

DEFINITION 3.1. An *AF* presentation of (X, Γ, ϕ) is an infinite nested sequence of finite clopen partitions V^i of X with the following properties:

(1) The first partition $V^0 = X$, and each V^i refines V^{i-1}.
(2) For each partition V^i there is an alphabet A^i and a map $\phi_i : X \to A^i$ that numbers the pieces of the ith partition.
(3) The diameters of the clopen sets in the partitions decrease to 0. There are inclusion maps $k_i : A^i \to A^{i-1}$ which are compatible with ϕ_i, and a homomoprhism $\psi_i : \Gamma \to Perm(A^i)$ which factors through the representation $\phi : \Gamma \to \text{Homeo}(X)$.

Then we say $\{V^i, \phi_i, A^i, k_i, \psi_i\}$ is an AF Presentation for the system (X, Γ, ϕ).

We have shown constructively how to build such an AF Presentation for any minimal equicontinuous action of a finitely generated group on a Cantor set:

THEOREM 3.2. Let Γ be a finitely generated group, and suppose Γ acts on a Cantor set X minimally and equicontinuously. Then there exists an almost finite presentation of Γ as in Definition 3.1.

This presentation is constructed using a process of partitions and coding functions, roughly analogous to the Kakutani-Rokhlin partitions used in the $\mathbb{Z}$-action case. We have the following properties:

(1) There is a homeomorphism $f : X \to A_\infty$, where $A_\infty = \lim_{\leftarrow}\{A^i, k_i\}$,
(2) Γ acts on A_∞,
(3) f gives a conjugacy between the Γ-action on X with the Γ-action on A_∞.

Thus, A_∞ is a Bratteli-type representation of the system, and A_∞ and X are orbit equivalent.

We can also look at these systems algebraically instead of topologically, to build an algebraic model for the action of Γ on A_∞, by means of group chains.

A *group chain* is an infinite nested sequence of finite index proper subgroups $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > ....$ Let $X_i = \Gamma/\Gamma_i$, which is a finite non-trivial set, and let $X = \lim_{\leftarrow} X_i$ be the inverse limit, which is a Cantor set. Then Γ acts on X on the left.

Given an AF presentation $\{V^i, \phi_i, A^i, k_i, \psi_i\}$ for a system, we obtain a group chain by setting Γ_i as the isotropy subgroup of V_i. We can also start with a group chain and then $A^i = \Gamma/\Gamma_i$ yields an AF Presentation of the action.

Let $N_i = N_\Gamma(\Gamma_i)$, the normalizer of Γ_i. Then N_i/Γ_i acts on X_i on the right, and the action commutes with the left Γ-action. A group chain is called normal if each of the subgroups is normal in Γ_0. We have defined a slight weakening of this condition. Let $N_* = \cap N_i$.

DEFINITION 3.3. A group chain is called *subnormal* if the chain of normalizers stabilizes after some point, i.e. $\exists k\ \forall i > k, N_i = N_{i+1}$, and if N_* is nontrivial.

We assume N_* is nontrivial. Then the inverse limit $\lim_{\leftarrow} N_*/\Gamma_i$ acts on each X_i on the right, and thus on X. So the left Γ action commutes with the right N_*/Γ_i-action.

In my thesis, I produce examples of actions represented by normal, subnormal and non-normal group chains. One important class of examples are weak solenoids.

Let X_0 be a compact connected manifold, and choose a basepoint x_0. Inductively, for $i \geq 1$ let $p^i_{i-1} : X_i \to X_{i-1}$ be a finite-to-one covering map of X_{i-1} by another compact connected manifold of degree at least 2, and choose $x_i \in X_i$ such that $p^i_{i-1}(x_i) = x_{i-1}$. The inverse limit

$$X_\infty = \varprojlim \{p^i_{i-1} : X_i \to X_{i-1}\} \tag{1}$$

is a compact connected metrizable space called a *weak solenoid* ([12, 5, 16]).

A weak solenoid can be represented by a group chain in the following way.

Denote by $p^i_0 = p^i_{i-1} \circ \cdots \circ p^1_0 : X_i \to X_0$ the composition of covering maps. Let $\Gamma_0 = \pi_1(X_0, x_0)$ be the fundamental group of X_0, and for any $i \geq 1$ denote by

$$\Gamma_i = (p^i_0)_* \pi_1(X_i, x_i)$$

the image of the fundamental group of X_i onto G_0 under the injective homomorphism $(p^i_0)_*$ induced by the projection p^i_0. Then subgroups $\Gamma_0 > \Gamma_1 > \Gamma_2 > \ldots$ form a group chain.

Denote by $p_0 : X_\infty \to X_i : (x_k)_{k \in \mathbb{N}_0} \mapsto x_0$ the projection onto the first factor in the inverse sequence. The fiber $F = p_0^{-1}(x_0)$ is a Cantor set ([12]), and is homeomorphic to the inverse limit

$$F \cong \varprojlim \{\imath : \Gamma_0/\Gamma_i \to \Gamma_0/\Gamma_{i-1}\}$$

of maps of cosets spaces of groups Γ_i given by inclusion. The action of Γ_i on F is given by lifting of paths, and is well known to be minimal and equicontinuous.

A weak solenoid is called normal or respectively subnormal if its associated group chain is normal or respectively subnormal. We study weak solenoids which are not normal, examples of which are given by Schori ([16]) and Rogers and Tollefson ([15]). The Schori solenoid is given by 3-1 coverings of a genus 2 surface.

THEOREM 3.4. (1) The Schori solenoid is neither normal nor subnormal.
(2) The Rogers and Tollefson solenoid is neither normal nor subnormal.

We can also consider examples of group chains that come from the algebraic structure of a group. For example, let H be the discrete Heisenberg group, i.e. $H \cong \mathbb{Z}^3$ with the group operation $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. Note that this is standard addition in the first two coordinates, but addition with a twist in the last coordinate. Hence, we think about H as $\mathbb{Z}^2 \times \mathbb{Z}$, where the group operation is commutative in the first two coordinates, and not commutative (given by a specified twist) in the third coordinate.

Lightwood, Şahin, and Ugarcovici studied the structure of subgroups of the Heisenberg group ([11]). They showed that subgroups of H can be written in the form $\Gamma = M\mathbb{Z}^2 \times m\mathbb{Z}$ where $M \in GL_2(\mathbb{Z})$, $m \in \mathbb{Z}$, and m divides both entries of one row of M. They presented examples of normal group chains in the Heisenberg group. We can also use subgroups of this form to generate group chains that are subnormal.

THEOREM 3.5. The following examples of group chains in the Heisenberg group are classified as normal or subnormal:

(1) $M_n = \begin{pmatrix} p^n & 0 \\ 0 & p^n \end{pmatrix}$, $m = p$, is a normal chain.
(2) For distinct primes p, q, $M_n = \begin{pmatrix} p^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix}$, $m = p$. Then we have $N(\Gamma_n) = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for all n, so the chain is subnormal.
(3) For distinct primes p, q, r, $M_n = \begin{pmatrix} rp^n & rpq^n \\ p^{n+1} & rq^{n+1} \end{pmatrix}$, $m = p$.
Then we have $p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ contained in $N(\Gamma_n)$, but the normalizer is larger and not the whole group. This chain is subnormal.

We then use our above work to associate a Bratteli-type model to these examples.

The main theorem of my thesis is the following:

THEOREM 3.6. *If a dynamical system has an AF presentation that is normal or subnormal, then we can associate a Bratteli diagram to the system.*

4. Future Work

The classification invariants for my thesis apply mainly to equicontinuous actions. In the next stage of my research, I plan to extend these ideas to non-equicontinuous actions, for example, to Toeplitz actions, which are almost 1-1 extensions of equicontinuous actions. This will be carried out in the continuation of our seminar on Cantor actions, in collaboration with Professors Hurder and Lukina.

I also plan to explore the connection between the methods of this work, and the dynamics of substitution systems and tilings, which are not equicontinuous, but do admit special algebraic presentations. Since tilings are very visual, undergraduates could be involved in projects based on this work.

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