Do Topological Spaces Form an Elementary Class?

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Notes: The following work was completed in discussion with Belal Mutabagani Al-Madani, Rishi Banerjee, Michael Lange, Professor Joel David hamkins, Professor Ronnie Nagloo, Professor Caroline Terry, and many others.

I do not claim to be the originator of these proofs. However, I have attempted to trace and credit the sources I have used.

1 How to Read These Notes

- If you are looking for a quick reference for knowing how to prove that topologies do not form an elementary class, see Theorems 7.2 and 7.5. If you wish to reference the stronger claim about power set algebras, see Proposition 7.4 and the proof of Theorem 7.2.
- Note that some of the prose is more for sharing my experience working through this problem with others, some of which provides insight into both ideas we had as well as nuances to posed questions.
- Section 5 contains a very cool Compactness proof by Rishi for compact topological spaces, but is not necessary for a streamlined walkthrough to the final results. Similarly, sections 3 and 6 are also technically unnecessary for the final results but showcase the nuances of a Compactness approach to the problem and why it (may have) failed.
- These notes are provided for assisting others in understanding how to prove these claims. I am merely a graduate student, and these proofs have not been thoroughly double-checked by others. Please email me at jesolak2@uic.edu if you notice any mistakes and I'll be happy to credit you.

2 The Problem Statement

It has been said that Topological spaces are not axiomatizable or "not first order". In some sense, this entails first order Model Theory not being able to describe "being a topological space" perfectly. In order to formalize this, I pose the following question: do Topological spaces form an elementary class?

This is still a little imprecise, since we have not specified a language. Indeed, it seems the claim is meant to be made about any possible language; a powerful claim, but seemingly hard to prove. So, instead, we may consider whether they form an elementary class in any "reasonable" language. What I take this to mean, and I leave up to the reader to decide to agree or disagree with, will become clear in the final proof statements.

In these notes, I will demonstrate some preliminary information and approaches to solving this problem, show their nuances and why the strategies fail, and then provide a solid proof that Topological spaces do not form an elementary class (in any "reasonable" language) and more than that.

3 Compactness and How it Works

My initial intuition for solving such a problem was inspired by practice for the Master's Exam, which notably included Compactness proofs such as the one below. First, I will state the Compactness Theorem, and then the example application as a proposition.

Theorem 3.1(Compactness) Let T be any (possibly finite, countable, uncountable, etc.) set of first order sentences in a language $\mathcal{L}$. If, for every finite subset $\Delta \subseteq T$, we have an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta$, then there exists an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \models T$.

Definition 3.2 In a language $\mathcal{L}$, a class of $\mathcal{L}$-structures, $\mathcal{C}$, is called an *elementary class* if and only if there exists a theory T such that
$\mathcal{C} = \{\mathcal{L}\text{-structures } \mathcal{M} : \mathcal{M} \models T\}$.

Notably, if a familiar class of objects is used, then for the class to be considered elementary, every such object must be "naturally" interpreted as an $\mathcal{L}$-structure.

Proposition 3.3 The class of (loopless, undirected) connected graphs is not elementary in the language of graphs, $\mathcal{L} = \{E\}$

Proof. Suppose toward contradiction that there was an $\mathcal{L}-$theory T witnessing that the class of connected graphs, $\mathcal{C}$ is elementary i.e. $\mathcal{C} = \{\mathcal{M} : \mathcal{M} \models T\}$. Then, expand the language by adding two new constants, c_1, c_2, obtaining $\mathcal{L}'$. Next, consider the theory

$$T' := T \cup \{\neg E(c_1, c_2)\} \cup \{\neg\exists x_1 ... \exists x_n E(c_1, x_1) \wedge ... \wedge E(x_n, c_2) : n \in \mathbb{N}\}$$

This is an $\mathcal{L}'$-theory which, besides requiring that our structure be a connected graph (because it contains T), also requires that it has two points who do not have a path between them of length 1, of length 2, of length 3, and so on. Therefore, T' is inconsistent since any model of it will have two points which are unconnected. However, we may see that any finite subset $\Delta \subseteq T'$ only requires that there is no path of length m or less for some $m \in \mathbb{N}$. Any such Δ is satisfiable by the graph on $m+2$ vertices, $v_1, ..., v_{m+2}$ consisting only of the edge relations $E(v_1, v_2), E(v_2, v_3), ..., E(v_{m+1}, v_{m+2})$ (and their symmetric counterparts), call it $\mathcal{N}$, where the constants may be chosen to be $c_1^{\mathcal{N}} = v_1$ and $c_2^{\mathcal{N}} = v_{m+2}$, respectively, so that the only path between them is of length $m+1$. Thus, by the Compactness Theorem, T' is consistent, a contradiction. □

4 Finding a Suitable Language

If the proof that Topological spaces do not form an elementary class is going to mean anything, then the language we choose has to be "reasonable". The primary language I propose is $\mathcal{L} := \{\subseteq, T\}$ where $\subseteq$ is a binary relation (thought of as "is a subset of"), and T is a unary predicate ("is in the topology"). In this language a "topological space" would be some $\mathcal{M} := (\mathcal{P}(X), \subseteq, T)$ where $\subseteq$ acts like it should, and the predicate exactly identifies which subsets are open and which ones are not. The ambient space is X.

Here are some results.

Remarks 4.1 Realtive to the class $\mathcal{C}$ of topological spaces viewed as $\mathcal{L}$-structures,

1. X is definable, and so is $\emptyset$.
2. Given $A, B \in \mathcal{P}(X)$, then $A \cup B$ is definable, as is $A \cap B$
3. Given $A, B \in \mathcal{P}(X)$, then $A \setminus B$ is definable. Thus, the complement is definable
4. Being open is definable. Being closed is definable.

Proof. I will prove unions are definable, and maybe set subtraction if there is time.
Let $x, y \in M$, two subsets of the ambient space X. Then, we define their union, z, over the empty set with the following sentence:

$$\forall u, x \subseteq u \wedge y \subseteq u \rightarrow z \leq u$$

That is to say, the finite union is the least upper bound of the two sets. □

After some consideration, hopefully the reader can prove that the correspondence between topological spaces in their usual notation and $\mathcal{L}$-structures of the form mentioned in the first paragraph of this section is one-to-one and surjective. A further observation of particular note is that we might expect some kind of correspondence between isomorphism in this language (when applied to topologies) and homeomorphism.

This will require the following lemma: the predicate "is a singleton" is definable

Lemma 4.2 In the language $\mathcal{L} = \{\subseteq, T\}$, being a singleton is definable in the class of topological spaces

Proof. It is defined by the sentence asserting that the only subsets of this given set, x, are the empty set and itself, and requiring that this set is not itself the empty set:

$$[\forall y \; y \leq x \rightarrow (\forall z \; y \leq z \vee y = x)] \wedge [\neg \forall z \; x \leq z]$$

$\square$

Proposition 4.3 Given two models $\mathcal{X} := (\mathcal{P}(X), \subseteq, T_X)$ and $\mathcal{Y} := (\mathcal{P}(Y), \subseteq, T_Y)$, we have that $\mathcal{X} \cong \mathcal{Y}$ if and only if $(X, T_X) \cong (Y, T_Y)$.

Proof. Note the abuse of notation: I will say $\mathcal{X} \models T_X(z)$ if and only if $z \in T_X$.

For the ($\Rightarrow$) direction, suppose that $\mathcal{X} \cong \mathcal{Y}$ is witnessed by the isomorphism $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$. Then, define the map g as,

$$\begin{aligned} g : X &\rightarrow Y \\ x &\mapsto y \qquad \text{where } y \in f(\{x\}) \end{aligned}$$

Now, to show that this is well-defined, we need to know that $f(\{x\})$ is a singleton. Indeed, isomorphsims preserve the truth value of formulae (in both directions) for any parameter from the domain's universe, and so $f(\{x\})$ is a singleton. Further, this map is injective since for any $x_1, x_2 \in X$, if $g(x_1) = g(x_2)$ then $f(\{x_1\}) = f(\{x_2\})$. Since f is injective, this implies $\{x_1\} = \{x_2\}$ i.e. $x_1 = x_2$. Surjectivity also follows from f since for any $y \in Y$, $\{y\} \in \mathcal{P}(Y)$, and so there exists $A \subseteq X$ such that $f(A) = \{y\}$. Again, preservation of "is a singleton" guarentees $A = \{a\}$ for some $a \in X$, and hence $g(a) = y$. Now, for continuity of g, we have that for any open set $U \subseteq Y$, that $g(g^{-1}(U)) = U$. Thus, we have that $\mathcal{Y} \models T_Y(U)$ if and only if $\mathcal{Y} \models T_Y(g(g^{-1}(U)))$ if and only if $\mathcal{X} \models T_X(g^{-1}(U))$. Similarly, for any $W \subseteq X$ open, we have that $\mathcal{Y} \models T_Y(W)$.

For the ($\Leftarrow$) direction, let $g : X \rightarrow Y$ be a homeomorphism. Define,

$$\begin{aligned} f : \mathcal{P}(X) &\rightarrow \mathcal{P}(Y) \\ A &\mapsto g(A) \end{aligned}$$

For any $A, B \subseteq X$, we have that $A \subseteq B$ implies that $g(A) \subseteq g(B)$. And, if for any $A, B \subseteq X$ we have that $g(A) \subseteq g(B)$, then for every $a \in A$, we have that $g(a) \in g(B)$ which implies $g(a) = g(b)$ for some $b \in B$. By the injectivity of g, this implies that $a = b \in B$, so $A \subseteq B$. For any $A, B \in \mathcal{P}(X)$, we have that $f(A) = f(B)$ implies $g(A) = g(B)$, and so by the previous analysis, $A = B$, so f is injective. Further, f is surjective since for every $C \in \mathcal{P}(Y), f(g^{-1}(C)) = g(g^{-1}(C)) = C$. We also have that for any $W \in \mathcal{P}(X)$, if W is open i.e. $\mathcal{X} \models T_X(W)$, then $f(W) = g(W)$ is open in Y since g is a homeomorphism. And similarly, for any $W \in \mathcal{P}(X)$, if $\mathcal{Y} \models T_Y(f(W))$ i.e. $g(W)$ is open n Y, then $g^{-1}(g(W)) = W$ is open in X. Hence, f is a surjective embedding, and thus, an isomorphism. $\square$

5 Compact Topologies

For a small but related tangent, we explore proving that compact topological spaces do not form an elementary class. The following proof was only slightly adapted (small change of language) based on one shown to me by a colleague, Rishi Banerjee. One particularly neat aspect is the use of a predicate in order to discuss a cover and the union of its elements. It is also nice to see that one can prove this claim about compact topological spaces using the compactness theorem.

Theorem 5.1 In the language $\mathcal{L} := \{\subseteq, T\}$, the compact topological spaces do not form an elementary class.

Proof. Suppose toward contradiction that they do form an elementary class, witnessed by T. Extend the language, $\mathcal{L}' := \mathcal{L} \cup \{R\}$ with a single unary relation, "in the cover". Let $P(v)$ denote the predicate defined by "is the ambient space" i.e. "is maximal". Then, the sentence ϕ_R given by $\forall x[\forall y(R(y) \to y \subseteq x) \to P(x)]$, says that the only set which contains everything in R is the ambient space i.e. in particular, their union is the whole space, so R is a cover. So, consider the theory

$$T' := T \cup \{\phi_R\} \cup \{\neg \exists x_1, ..., x_n R(x_1) \wedge ... \wedge R(x_n) \wedge \forall y[(x_1 \subseteq y \wedge ... \wedge x_n \subseteq y) \to P(y)] : n \in \mathbb{N}\}$$

It is obtained by taking T together with the sentence saying that R is a cover, but that there do not exist any subcovers of size 1, 2, 3, ... Hence, since we assumed all models of T were compact topologies, then T' is inconsistent. However, we may see that any finite subset of T' only requires that there are no subcovers of size less than or equal to m for some $m \in \mathbb{N}$, and that is satisfiable by a compact topological space and a cover chosen so that any finite subcover must be of at least size $m + 1$. For example, by the Heine-Borel theorem, any collection of $m + 1$ disjoint discs in $\mathbb{R}^2$ will be compact, and if we take as our cover the $m + 1$ many disks, then there will be no finite subcover of size less than or equal to m. So, T' is finitely consistent and hence, consistent, by the Compactness Theorem, a contradiction. $\square$

6 Compactness and How it Failed

Given the results in section 3, it may seem that $\mathcal{L} = \{\subseteq T\}$ would be a suitable language for topological spaces. We can simply throw in the predicate T where appropriate to say that T ("being an open set") applies to the empty set and the ambient space, is preserved under finite intersections, and even is preserved under finite unions. It looks like the only axiom that remains is preservation of T under arbitrary unions. There is a second possible issue, but I will leave that for later.

Similar to the case of connected graphs, my hope was to throw in witnesses showing that the arbitrary union fails. Because of the finite nature of first order logic, it seems that any cardinality of infinity will do, so why not try the one we likely have the best chance of describing with infinitely many sentences: countable union?

Theorem Topological spaces do not form an elementary class in the language $\mathcal{L} = \{\subseteq, T\}$

proof attempt. Suppose toward contradiction that topological spaces do form an elementary class, witnessed by the theory T. Then, extend our language to $\mathcal{L}' := \mathcal{L} \cup \{c_n : n] \in \mathbb{N}\} \cup \{u\}$ by adding constants. We will want to require u to be the union of the other constants. Extend T to $T' := T \cup \{c_n \subseteq u : n \in \mathbb{N}\} \cup \{T(c_n) : n \in \mathbb{N}\} \cup \{\neg T(u)\}$.

Currently, we now have that all of the constants c_n, for any choice of $n \in \mathbb{N}$, are open, and yet u is not open. However, a careful analysis of the above sentences reveals that u need only be a super set of the union. When we defined the finite union, it was relatively easy to say that u is the *least* such upper bound, but here, with infinitely many sets to contain, it is less clear how to require that. Even the complement seems unhelpful since the relationship between a set and its complement will flip the direction of $\subseteq$ e.g. $A \subseteq B$ is logically equivalent to $B^c \subseteq A^c$, and so we won't get an upper bound. Furthermore, I tried to adapt Rishi's proof of compact topologies not being an elementary class to address this issue, but unfortunately, it seems to fall into a similar problem: the cover R might contain more than just the constants c_n, and hence we won't be able to use it to limit the size of u.

7 Lowenheim-Skolem Saves the Day

At this point, it is not clear whether, somehow, we can in fact define the countable union of our constants with infinitely many sentences or not. However, I mentioned there being *two* possible issues. The other possible issue has to do with the format we expect our topological spaces to end up in, $(\mathcal{P}(X), \subseteq, T_X)$.

To exploit this, we will need another powerful theorem: Lowenheim-Skolem.

Theorem 7.1 (Lowenheim-Skolem) Let T be a (consistent) first order theory in $\mathcal{L}$. Then, for any cardinal κ where $\kappa \geq |\mathcal{L}|$, and any structure $\mathcal{M} \models T$, there exists a structure $\mathcal{N}$ with $|\mathcal{N}| = \kappa$ such that

$$\text{if } \kappa \leq |\mathcal{M}|, \text{ then } \mathcal{N} \preceq \mathcal{M}$$
$$\text{if } \kappa \geq |\mathcal{M}|, \text{ then } \mathcal{N} \succeq \mathcal{M}$$

Or, to use more of the vocabulary we've developed and simplify the theorem: every consistent first order theory, T, has for each cardinal, κ, at least one model, $\mathcal{N}$, where kappa is greater than or equal to the cardinality of the language, $|\mathcal{L}|$.

Now, let's see if we can tackle the problem in a new way. The following proof was inspired by a similar one sketched by Prof. Joel David Hamkins. With this proof technque, we can even strengthen our claim.

Theorem 7.2 Topological spaces do not form an elementary class in any language extension of $\mathcal{L} = \{\subseteq\}$

Proof. Suppose toward contradiction we have a theory T witnessing that topologies form an elementary class, in a "natural" way like the one described before: $\mathcal{C} := \{(\mathcal{P}(X), \subseteq, ...)\} = \{\mathcal{M} : \mathcal{M} \models T\}$. Consider the language extension $\mathcal{L}' := \mathcal{L} \cup \{c_l : l \in \mathcal{L}\} \cup \{S\}$ where S is a unary predicate, and every c_l is a constant. Next, consider the theory

$$T' := T \cup \{\neg c_m = c_n : m, n \in \mathbb{N}\} \cup \{S(c_n) : n \in \mathbb{N}\} \cup \{S \text{ is equivalent to "is a singleton"}\}$$

This theory is consistent (no need for Compactness, consider $\mathcal{L}$ with the discrete topology), and so by (Downward) Lowenheim-Skolem, it has a model of cardinality $|\mathcal{L}|$. But let's observe what this tells us: this theory, which stipulates that any model of it is a topological space with $|\mathcal{L}|$ many distinct singletons i.e. $|\mathcal{L}|$ many points. That implies that the power set has cardinality at least $2^{|\mathcal{L}|}$. Yet, Lowenheim-Skolem tells us there is a model whose universe is a power set of cardinality $|\mathcal{L}|$, a contradiction. Thus, Topological spaces do not form an elementary class in any language extension of $\mathcal{L} = \{\subseteq\}$. $\square$

Remarks 7.3

- It is enough for $\{(A, B) : A \subseteq B\}$ to be definable in our language.
- Something to notice is that we never actually used the fact that our models were topologies except to claim that T' was consistent, and nothing in T', besides T, made reference to a topology. So, I think this allows us to make the further claim:

Proposition 7.4 Any class of L'-structures, where $\{\subseteq\} = \mathcal{L} \subseteq \mathcal{L}'$, of the form $(\mathcal{P}(X), \subseteq, ...)$, are not elementary.

Before, I set aside this problem, there is one more language we might consider. Indeed, topological structures are most often thought of as many-sorted structures, having universes (X, T_X) where $T_X \subseteq \mathcal{P}(X)$. It is not immediately clear that, for example, in the language $\mathcal{L} := \{\in_{1,2}\}$ where '$\in_{1,2}$' is a binary relation whose first coordinate is a point, from the first sort, and whose second coordinate is an open set, from the second sort, that the class of topological spaces is not elementary.
This language is surprisingly strong since, for example, in the class of topological

spaces $\mathcal{C}$, the closed sets are definable with parameters from the model e..g given $U \in T_X$, we have $X \setminus U = \{x \in X : \neg x \in_{1,2} U\}$. And further, the subset relation is definable on open sets via $U_1 \subseteq U_2$ if and only if $\forall x x \in_{1,2} U_1 \to x \in_{1,2} U_2$. However, it doesn't seem likely that our previous argument will go through since, a priori, the actual topology may be countable. Thankfully this is not an issue.

Proposition 7.5 The class of topological spaces is not elementary in any extension of the two-sorted language $\mathcal{L} \supseteq \{\in_{1,2}\}$.

Proof. Suppose toward contradiction that the class is elementary, witnessed by T, an $\mathcal{L}$-theory. Then, consider the language extension $\mathcal{L}' := \mathcal{L} \cup \{c_l : l \in \mathcal{L}\}$. Let

$$T' := T \cup \{\phi\} \cup \{\neg c_{l_1} = c_{l_2} : l_1, l_2 \in \mathcal{L}\}$$

where ϕ is the sentence asserting that the topology is discrete by asserting that the singletons are open, given by $\forall x \exists U\ [x \in_{1,2} U \wedge \forall y\ (y \in_{1,2} U \to x = y)]$. Then, T' is consistent (e.g. $\mathcal{L}$ with the discrete topology), but has no models of cardinality $|\mathcal{L}|$, a contradiction with Lowenheim-Skolem. $\square$

8 Conclusion

At this point, many (possibly all) seemingly "reasonable" languages have been covered. Essentially, topological spaces do not form an elementary class in a 1-sorted language as soon as you hope to have your models be power sets and have $\subseteq$ be definable, nor do they form an elementary class in two-sorted languages where you can define the relation "is an element of" and where your topological spaces are of the form (X, T_X). In theory, there are plenty of other ways to attempt to give a language in which they are elementary, but it is unclear to me that any of these would be "reasonable". If you have an idea, please feel free to try it and/or reach out!