Bumping and self-bumping of deformation spaces

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Introduction

One of the central problems in the study of hyperbolic structures on 3-manifolds is that of classification. More precisely, if M is a compact manifold whose interior admits a complete hyperbolic structure, then the problem is to classify and parameterize the set of all possible structures, up to isometry. To expand the problem further, can we classify and parameterize the set of orientation-preserving isometry classes of (marked) hyperbolic 3-manifolds homotopy equivalent to M? If the manifold is closed or has only tori in its boundary then Mostow [27] and Prasad [29] show that this set is easily classified, since it consists of two points (only interval bundles over surfaces admit orientation reversing self-homeomorphisms homotopic to the identity). However, when the boundary of M contains a surface of higher genus there is a rich deformation theory.

To give a more concrete footing to our discussion we introduce the main object of study, $AH(M)$. Let $AH(M)$ denote the space of orientation-preserving isometry classes of marked complete hyperbolic 3-manifolds homotopy equivalent to M. $AH(M)$ is the *deformation space* associated to M. In this paper we will survey the recent results concerning the topological structure of $AH(M)$ due to Anderson and Canary [2], Anderson, Canary and McCullough [3], Bromberg and Holt [10], [11], Ito [18], Holt [17], [16] and McMullen [24]. There will be some unavoidable overlap with R. Canary's article in this same volume, but in the interest of being as self-contained as possible we will refrain from attempting to minimize this overlap.

Recent work of Minsky [25] and Brock, Canary and Minsky [7] establishes the truth of Thurston's Ending Lamination Conjecture for manifolds with incompressible boundary. Thus, any hyperbolic 3-manifold whose fundamental group does not split non-trivially as a free product is uniquely determined up to isometry by a well-defined collection of topological and analytical data. It is not our intention to go into a detailed discussion of this data, but simply to note that this data does not, in general, vary continuously within the deformation space. Thus, while the problem of classifying the elements of $AH(M)$ is settled when M has incompressible boundary, the problem of finding a parametrization of $AH(M)$ which varies continuously with respect to the topology on $AH(M)$ is still very much open. The work surveyed here is but a first step towards a solution to this problem and a small step at that; currently there is not even a conjectural parameterization of $AH(M)$, in contrast to $AH(M)$'s analogue in complex dynamics, the Mandelbrot set.

In order to avoid stating theorems about the empty set, or a point set, we will always assume that M admits a hyperbolic structure on its interior and has non-empty boundary of non-zero euler characteristic. For expository reasons we will assume that ∂M contains no tori. Thus if M is atoroidal, irreducible and has non-empty boundary, Thurston's hyperbolization theorem guarantees that the interior of M admits a hyperbolic metric, so that $AH(M)$ is non-empty. We will also restrict our attention to orientable manifolds M for which ∂M is incompressible. Partly this is for ease of exposition, and partly because many of the results are not known to hold in the compressible case. When the assumption of incompressible boundary is not needed we will make note of it.

The deformation space

The space $AH(M)$ can be viewed as the space of equivalence classes of maps $h : M \longrightarrow N$, where N is a hyperbolic 3-manifold and h is a homotopy equivalence, and where $h : M \longrightarrow N$ and $h' : M \longrightarrow N'$ are equivalent provided that $h' \circ h^{-1}$ is homotopic to an orientation-preserving isometry between N and N'. Note that a representative $h : M \longrightarrow N$ gives rise to a faithful representation h_* of $\pi_1(M)$ in $PSL_2(\mathbb{C}) \simeq Isom^+(\mathbb{H}^3)$ with discrete image, and because M (and N) is a $K(\pi_1(M), 1)$, the converse also holds: for any discrete, faithful representation ρ of $\pi_1(M)$ into $PSL_2(\mathbb{C})$ there is a homotopy equivalence $h : M \longrightarrow \mathbb{H}^3/\rho(\pi_1(M))$ so that $h_* = \rho$. Representatives of the same class in $AH(M)$ give rise to conjugate representations, so that we may consider $AH(M)$ as the space of conjugacy classes of discrete, faithful representations in $PSL_2(\mathbb{C})$. It is from this point of view that $AH(M)$ obtains its topology: ρ is close to ρ' provided that there are representatives which are close on every element of $\pi_1(M)$. $AH(M)$ embeds in the character variety associated to representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$ (see [26]), and we denote the interior of $AH(M)$ by $MP(M)$.

Classical work of Ahlfors, Bers, Maskit, Kra, Marden, Sullivan and Thurston shows that $MP(M)$ is a collection of topological balls so that $h : M \longrightarrow N$ and $h' : M \longrightarrow N'$ lie in the same component of $MP(M)$ if and only if $h' \circ h^{-1}$ is homotopic to an orientation preserving bi-Lipschitz diffeomorphism between N and N'; the set of possible homotopy classes of marked oriented homeomorphism types indexes the set of components.

We can phrase this more precisely once we have introduced some terminology.

A hyperbolic manifold N with finitely generated fundamental group is *geometrically finite* if it has a finite volume convex core. (The convex core of a hyperbolic 3-manifold N is the smallest, closed, convex submanifold for which the inclusion into N is a homotopy equivalence). It is *minimally parabolic* provided that the only maximal parabolic subgroups of its fundamental group are rank 2 free-abelian. In our setting, where our manifolds contain no torus boundary components, minimally parabolic and geometrically finite is the same as *convex cocompact.* Marden [22] showed that the minimally parabolic, geometrically finite elements of $AH(M)$ lie in the interior of $AH(M)$ and Sullivan [30] established the reverse inclusion, so that the interior of $AH(M)$ is the space $MP(M)$ of geometrically finite, minimally parabolic hyperbolic manifolds. The solution of the ending lamination conjecture and, independently (in the minimally parabolic case), the work of Bromberg [8] and Brock and Bromberg [6] establishes the truth of the Density Conjecture of Bers, Sullivan and Thurston in the case the M has incompressible boundary, allowing us to conclude that $AH(M) = \overline{MP(M)}$.

There is a map $\Theta : AH(M) \longrightarrow \mathcal{A}(M)$ from $AH(M)$ to the set $\mathcal{A}(M)$ of homeomorphism classes of marked, oriented, compact, irreducible 3-manifolds homotopy equivalent to M (when ∂M contains tori we would need to add "atoroidal" to this list of adjectives). Suppose that $\rho = (h : M \longrightarrow N)$ represents a point in $AH(M)$. Then N contains a compact submanifold M_ρ so that the inclusion map $M_\rho \longrightarrow N$ is a homotopy equivalence - a compact core. If $j : N \longrightarrow M_\rho$ is a homotopy inverse to inclusion then the map $j \circ h : M \longrightarrow M_\rho$ is a homotopy equivalence between M and M_ρ. Thus Θ is defined by $\Theta(\rho) = [(M_\rho, j \circ h)]$. By McCullough, Miller and Swarup [23], this map is well-defined; by Marden [22], Θ is continuous on $MP(M)$, mapping each component to a point; by Thurston [28], Θ is surjective.

We say that two components of $MP(M)$ *bump* provided that their closures intersect. Saying that a component B of $AH(M)$ *self-bumps* at a point $\rho \in \overline{B}$ is to say that for any sufficiently small neighbourhood V of ρ in $AH(M)$, the intersection $V \cap B$ is disconnected. The results we will survey here are on when and how bumping and self-bumping has been shown to occur.

Bumping

Points of bumping are precisely those points in $AH(M)$ where Θ fails to be continuous. Thus to study bumping one must study sequences $\rho_n \in MP(M)$ converging to $\rho \in AH(M)$, say, where Θ is constant throughout the sequence, but $\Theta(\rho) \neq \Theta(\rho_n)$. That is to say, we need to establish how the marked oriented homeomorphism type of the sequence can change in the limit.

In some sense atoroidal, irreducible 3-manifolds with incompressible boundary of higher genus are simple to understand. The very powerful theory of the characteristic submanifold, due to Jaco and Shalen [19] and Johannson [20] gives an almost block-like picture of such a manifold (see also [12]).

The *characteristic submanifold* $\Sigma(M)$ of M is defined up to ambient isotopy and consists of a collection of interval bundles over surfaces (*I-bundles*) and Seifert fibred manifolds embedded in M, so that any essential annulus or torus is properly isotopic into some component of $\Sigma(M)$. The fact that M admits a hyperbolic structure on its interior greatly constrains which Seifert fibred manifolds can be components of $\Sigma(M)$: any Seifert fibred component of $\Sigma(M)$ must be a solid torus with at least three (annulus) components to its frontier, or a thickened torus (a torus crossed with an interval) with at least three annulus components in its frontier (the solid tori with only two frontier components can be given an I-fibering, so we don't consider them as Seifert fibred). In our situation these thickened tori will not arise. The complement in M of $\Sigma(M)$ consists of acylindrical manifolds with boundary.

Johannson's deformation theorem [20] states that any homotopy equivalence $h : M \longrightarrow M'$ can be homotoped to a homotopy equivalence $g : M \longrightarrow M'$ so that g restricts to a homeomorphism between $\overline{M - \Sigma(M)}$ and $\overline{M' - \Sigma(M')}$ and restricts to a homotopy equivalence between $\Sigma(M)$ and $\Sigma(M')$. For each I-bundle X attached to a Seifert fibred solid torus in M we can homotope $g|_X$ rel $Fr(X)$ to a homeomorphism. Thus the only obstructions to deforming a homotopy equivalence to become a homeomorphism arise from I-bundles attached to acylindrical pieces and Seifert fibred solid torus components of $\Sigma(M)$, and acylindrical pieces attached to Seifert fibred solid tori.

The frontier of a Seifert fibred solid torus component of $\Sigma(M)$ is a collection of at least three properly embedded essential annuli in M. Form the union of pairwise disjoint regular neighbourhoods of all these annuli and remove it from M. Let $\mathcal{V}$ be the collection of Seifert fibred solid torus components in this complement. Certain essential solid tori in M deserve a special name: we say that a solid torus V is *primitive* if the inclusion into V of any component of its frontier is a homotopy equivalence.

Anderson, Canary and McCullough define a *primitive shuffle* between two oriented manifolds M_1 and M_2 with incompressible boundary to be a homotopy equivalence $h : M_1 \longrightarrow M_2$ with the following properties:

1. There is a collection $\mathcal{V}_i$ of solid tori in $\Sigma(M_i)$, $i = 1, 2$, so that $h : \overline{M_1 - \mathcal{V}_1} \longrightarrow \overline{M_2 - \mathcal{V}_2}$ is an orientation preserving homeomorphism;

2. $h^{-1}(\mathcal{V}_2) = \mathcal{V}_1$;

3. Each solid torus in $\mathcal{V}_1$ is primitive.

The collection $\mathcal{V}_1$ is called the *support* of h. By an abuse of language we often identify $\mathcal{V}_1$ and $\mathcal{V}_2$ with a set of solid tori $\mathcal{V}$ and refer to $\mathcal{V}$ as the support of h.

The examples constructed by Anderson and Canary provide relatively simple examples of primitive shuffle equivalences; see [2] and [16].

Note that a homotopy equivalence is homotopic to a primitive shuffle if the I-bundles attached to acylindrical pieces of the complement of $\Sigma(M)$ present no obstruction to being homotoped to a homeomorphism.

A primitive shuffle is a particular sort of homotopy equivalence called a *shuffle homotopy equivalence*. A shuffle with respect to compact submanifolds K_1 of M_1 and K_2 of M_2 is a homotopy equivalence $s : M_1 \longrightarrow M_2$ such that $Fr(K_i)$ is incompressible in M_i, $i = 1, 2$, $s^{-1}(K_2) = K_1$, and $s|_{\overline{M_1 - K_1}}$ is a homeomorphism onto $\overline{M_2 - K_2}$.

We say that two points $[(M_1, h_1)]$ and $[(M_2, h_2)]$ of $\mathcal{A}(M)$ are *primitive shuffle equivalent* provided that there is a primitive shuffle $h : M_1 \longrightarrow M_2$ so that $[(M_2, h_2)] = [(M_2, h \circ h_1)]$. In [3] it is shown that primitive shuffle equivalence is an equivalence relation on $\mathcal{A}(M)$. Let $\hat{\mathcal{A}}(M)$ be the set of equivalence classes.

The systematic study of the topology of $AH(M)$ began with the following seminal theorems.

Theorem 1 *(Anderson, Canary and McCullough [3])*

Suppose $\rho_n \in AH(M)$ converges to ρ. Then, up to subsequence, $\Theta(\rho_n)$ is primitive shuffle equivalent to $\Theta(\rho)$.

Theorem 2 *(Anderson, Canary and McCullough [3])*
Suppose B and B' are two components of $MP(M)$ so that $\Theta(B)$ is primitive shuffle equivalent to $\Theta(B')$. Then B and B' bump.

Thus the components of $AH(M)$ are in a one-to-one correspondence with the set $\hat{\mathcal{A}}(M)$. Moreover, the map Θ quotients to a map $\hat{\Theta} : AH(M) \longrightarrow \hat{\mathcal{A}}(M)$, and $\hat{\Theta}$ is *continuous.*

How bumping can occur

We will sketch a proof of Theorem 1 to give some idea of how these primitive shuffle equivalences arise, and in order to do so we need to discuss the geometric topology. The geometric topology is a topology on the set of hyperbolic 3-manifolds with base-frame, or equivalently, the set of Kleinian groups. We say that two Kleinian groups are close in the geometric topology provided that they are close with respect to the Hausdorff topology on the set of closed subsets of $Isom(\mathbb{H}^3)$. More geometrically, we have the following formulation of geometric convergence.

We work in the ball model of $\mathbb{H}^3$ and let $B_R(0)$ denote the ball in $\mathbb{H}^3$ of radius R centered at 0.

Proposition 3 *(Theorem E.1.13 in Benedetti-Petronio [4].)*
A sequence of torsion-free Kleinian groups $\{\Gamma_i\}$ converges geometrically to a torsion-free Kleinian group $\hat{\Gamma}$ if and only if there exists a sequence $\{(R_n, K_n)\}$ and a sequence of orientation-preserving maps $\tilde{f}_n : B_{R_n}(0) \longrightarrow \mathbb{H}^3$ such that the following hold:

1. *$R_n \to \infty$ and $K_n \to 1$ as $n \to \infty$;*
2. *the map $\tilde{f}_n$ is a K_n-biLipschitz diffeomorphism onto its image, $\tilde{f}_n(0) = 0$, and $\{\tilde{f}_n(A)\}$ converges, in the C^∞-topology, to the identity on any compact subset A of $\mathbb{H}^3$; and*
3. *$\tilde{f}_n$ descends to a map $f_n : Z_n \longrightarrow N_n = \mathbb{H}^3/\Gamma_n$, where $Z_n = B_{R_n}(0)/\hat{\Gamma} \subset \mathbb{H}^3/\hat{\Gamma}$. Moreover, f_n is an orientation-preserving K_n-biLipschitz diffeomorphism onto its image.*

In the case that the groups Γ_i arise as the images of an sequence ρ_i converging algebraically to ρ the next lemma gives information about how the algebraic limit and the geometric limit are related. Let $N = \mathbb{H}^3/\rho(\pi_1(M))$ and let $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ for some geometric limit $\hat{\Gamma}$ of $\Gamma_i = \rho_i(\pi_1(M))$. Since $\rho(\pi_1(M)) \subset \hat{\Gamma}$ there is a covering map $\pi : N \longrightarrow \hat{N}$. If M_0 is a compact core for N then we can restrict the covering map to obtain an immersion of M_0 into $\hat{N}$, and for n sufficiently large, the map $f_n \circ \pi|_{M_0}$ maps M_0 into $N_n = \mathbb{H}^3/\Gamma_n$. The next lemma tracks the homotopy class of $f_n \circ \pi|_{M_0}$.

Lemma 4 *(Canary-Minsky [14])*
Suppose that M is a compact, hyperbolizable 3-manifold and $\{\rho_n\}$ is a sequence in $AH(M)$ converging to ρ, and that $\{\rho_n(\pi_1(M))\}$ converges geometrically to $\hat{\Gamma}$. Set $N = \mathbb{H}^3/\rho(\pi_1(M))$, $N_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ and $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$, let $\pi : N \longrightarrow \hat{N}$ be the covering map, and let $\{f_n : Z_n \longrightarrow N_n\}$ be the sequence of biLipschitz diffeomorphisms produced by Proposition 3. Suppose that K is a compact subset of N such that $\pi_1(K)$ injects into $\pi_1(N)$. Then for all sufficiently large n, $(f_n \circ \pi|_K)_*$ agrees with the restriction of $\rho_n \circ \rho^{-1}$ to $\pi_1(K)$, where we regard both as giving maps of $\pi_1(K)$ into $\rho_n(\pi_1(M))$.

By Jørgensen and Marden [21], if $\rho_n = (h_n : M \longrightarrow N_n)$ converges to ρ then up to subsequence, $\rho_n(\pi_1(M))$ converges geometrically. Let $\hat{\Gamma}$ be the geometric limit, and set $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$. If $\rho = (h : M \longrightarrow N)$, then there is a covering map $\pi : N \longrightarrow \hat{N}$. If there is a compact core M_0 for N which embeds in $\hat{N}$ under π, then for large enough n, $f_n \circ \pi|_{M_0}$ is an embedding of M_0 into N_n, and $f_n \circ \pi(M_0)$ is a compact core for N_n. Moreover, $\rho_n = (f_n \circ \pi)_* \circ \rho$, and we may assume that $h(M)$ contains M_0.

Thus, in this case, $\Theta(\rho_n) = [(f_n \circ \pi(M_0), f_n \circ \pi|_{M_0} \circ h)] = [(M_0, h)] = \Theta(\rho)$, so that there has been no change in the marked oriented homeomorphism type in the limit.

In general, for a compact core M_0 of N, the composition $f_n \circ \pi|_{M_0}$ is not homotopic to a homeomorphism onto its image, for any n. It is a homotopy equivalence, however, and so there is a compact core M_n for N_n and a homotopy equivalence $g_n : M_0 \longrightarrow M_n$ so that $(g_n)_*$ is conjugate to $\rho_n \circ \rho^{-1}$. We may assume that g_n restricts to a homeomorphism between $\overline{M_0 - \Sigma(M_0)}$ and $\overline{M_n - \Sigma(M_n)}$.

The results of Anderson and Canary in [1] show that g_n is homotopic to a shuffle with respect to a collection $\mathcal{V}_0$ of Seifert fibred solid tori in M, and a corresponding collection in M_n. Furthermore they show that $\mathcal{V}_0$ doesn't necessarily contain all of the Seifert fibred solid torus components of $\Sigma(M_0)$, but rather is a subset of those whose core curve is parabolic in N (see pages 724-726 of [3]). Since any solid torus in M_0 whose core curve is parabolic must be a primitive solid torus, we find that $f_n \circ \pi|_{M_0}$ is a primitive shuffle.

Further, it turns out that it is only the primitive solid tori V for which $\pi_1(V)$ is parabolic and $\pi_*(\pi_1(V))$ is contained in a rank 2 parabolic subgroup of $\pi_1(\hat{N})$ which form any obstruction to embedding a compact core for N in $\hat{N}$ with the covering map (again, see [3]).

Suppose that $\rho \in AH(M)$ is geometrically finite, $\Theta(\rho) = [(M_0, h_0)]$, and $[(M_1, s \circ h_0)] \in \mathcal{A}(M)$ is primitive shuffle equivalent to $[(M_0, h_0)]$ via a primitive shulfle s. If s is supported on $\mathcal{V}_0$ and maps $\mathcal{V}_0$ into $\mathcal{V}_1$, say, then we may assume that s maps M_0 into $\hat{M}_1 = M_1 - \mathcal{N}(\Delta)$, where Δ is the set of core curves of elements in $\mathcal{V}_1$, and $\mathcal{N}(\Delta)$ denotes a small regular neighbourhood of Δ. There is an "inclusion" $i : M_1 \longrightarrow int(\hat{M}_1)$, obtained by first pushing M_1 into its interior and then choosing the curves in Δ to be in the complement. For a structure $\hat{\rho} \in MP(\hat{M}_1)$ we say the pair $(\hat{\rho}, s)$ *induces* ρ provided that $\rho = \hat{\rho} \circ i_* \circ s_*$. (Note that it is really the homotopy class of s that matters, but we suppress this). When ρ is induced by $(\hat{\rho}, s)$, it is possible to perform hyperbolic Dehn surgery on the torus cusps of $int(\hat{M}_1)$ (with the $\hat{\rho}$ structure) to produce a sequence of structures in the component of $MP(M)$ corresponding to $[(M_1, s \circ h_0)]$ and converging to ρ. (See [9], [5], [4], [15], and [31] for more on hyperbolic Dehn surgery). Since ρ is also in the closure of the component indexed by $[(M_0, h_0)]$ we see that these two components bump. By constructing a set $\{(\hat{\rho}, s)\}$ of pairs inducing ρ, Holt proved the following theorem.

Theorem 5 *(Holt [17])*

Let $B_1, \ldots, B_n$ be a collection of components of $MP(M)$ so that $\Theta(B_i)$ is primitive shuffle equivalent to $\Theta(B_j)$, for each i, j. Then there is a point $\rho \in AH(M)$ so that $\rho \in \overline{B_i}$, for $i = 1, \ldots, n$.

We have seen that solid torus components of $\Sigma(M)$ which are parabolic in N and which map into a neighbourhood of a torus cusp in the geometric limit are responsible for $\rho = (h : M \longrightarrow N)$ being a bumping point. What about the other infinite cyclic parabolic subgroups of $\pi_1(N)$ which cover rank 2 subgroups of $\pi_1(\hat{N})$? It turns out that these subgroups can cause *self-bumping* to occur.

Self-bumping

As stated in the introduction, a component B of $MP(M)$ self-bumps, provided that there is a point $\rho \in \overline{B}$ so that for any sufficiently small neighbourhood V of ρ, the intersection $V \cap B$ is disconnected.

The phenomenon of self-bumping was first discovered by McMullen [24] using techniques of Anderson and Canary and the theory of complex projective structures. When M is a trivial I-bundle over a closed surface S, $MP(M)$ has a single component, often referred to as $QF(S)$, the space of quasi-fuchsian structures. The full deformation space is $AH(S) := AH(S \times [-1,1])$. McMullen proved the following theorem.

Theorem 6 *(McMullen [24])*
$QF(S)$ self-bumps.

Ito [18] extended McMullen's analysis to show that for any integer n there is a representation $\rho \in \overline{QF(S)}$ so that for any sufficiently small neighbourhood V of ρ, $V \cap QF(S)$ has at least n components. As in the bumping case, this count is related to the number of pairs $(\hat{\rho}, f)$ which induce ρ - more will be said about this later.

It turns out that self-bumping if fairly common, as expressed by the following theorem of Bromberg and Holt. Note that for this theorem the manifold M need not have incompressible boundary.

Theorem 7 *(Bromberg-Holt [11]*
Suppose that M contains a primitive, essential annulus in M. Then every component of $MP(M)$ self-bumps.

We conjecture that when M does not contain a primitive, essential annulus then self-bumping does not occur. This is a weaker form of a conjecture of Thurston, which states that when M is acylindrical every component of $AH(M)$ is a closed ball.

How self-bumping can occur

Suppose that B is a component of $MP(M)$ so that $\Theta(B) = [(M_0, s]$, where $s : M \longrightarrow M_0$ is a primitive shuffle equivalence. We wish to give an idea of how B might be shown to self-bump. To this end we want to demonstrate the existence of a structure $\rho \in \overline{B}$ so that for all sufficiently small neighbourhoods V of ρ, $V \cap B$ is disconnected.

Recall that a collar of M, $Collar(M)$, is an open three-manifold containing M, so that $Collar(M) - M \simeq \partial M \times (0,1))$.

We define a *shuffle immersion* $f : M \longrightarrow \hat{N}$ to be an immersion of M into an open, hyperbolizable 3-manifold $\hat{N}$ with the following properties.

- There is a collection $\mathcal{V}$ of pair-wise disjoint, pair-wise non-homotopic, embedded primitive solid tori in M, so that for each V in $\mathcal{V}$ the frontier FrV consists of two essential annuli in M ($\mathcal{V}$ is called the *support* of f);
- $\hat{N} = int(M_0) - \Delta$, where Δ is the collection of core curves of elements of $\mathcal{V}$. (We can consider $\mathcal{V}$ as a subset of M_0 since these solid tori are characteristic);
- The cover of $\hat{N}$ corresponding to $f_*(\pi_1(M))$ is homeomorphic to $Collar(M)$ and f factors up to homotopy as $f = i \circ s$ where $i : M_0 \longrightarrow \hat{N}$ is inclusion and $s : M \longrightarrow M_0$ is the homotopy equivalence above.;
- f is an embedding restricted to $\overline{M - \mathcal{V}}$.

Note that $\mathcal{V}$ need not be homotopic into the support of a primitive shuffle equivalence. For example, M_0 might be equal to M so that s is a homeomorphism.

We say that a shuffle immersion is *non-trivial* if it is not homotopic to an embedding. We say that two shuffle immersions $f_1 : M \longrightarrow \hat{N}$ and $f_2 : M \longrightarrow \hat{N}$ are *equivalent* provided that $\hat{N}_1 = \hat{N}_2$ and f_1 is homotopic to f_2.

If $f : M \longrightarrow \hat{N}$ is a shuffle immersion and $\hat{\rho}$ is a minimally parabolic uniformization of $\hat{N}$ then we obtain a point in $\overline{B}$ by setting $\rho = \hat{\rho} \circ f_*$ - the properties that define a shuffle immersion ensure that $\Theta(\rho) = [(M_0, s]$. We say that the pair $(\hat{\rho}, f)$ *induces* ρ. We say that two pairs $(\hat{\rho}_0, f_0)$ and $(\hat{\rho}_1, f_1)$ inducing ρ are *distinct* provided that f_0 is not equivalent to f_1.

The composition of f with the uniformizing homeomorphism between $\hat{N}$ and $\mathbb{H}^3/\hat{\rho}(\pi_1(\hat{N}))$ lifts to a developing map for M. By an abuse of notation we call this $\tilde{f} : \tilde{M} \longrightarrow \mathbb{H}^3$, where $\tilde{M}$ is the universal cover of M. If we perform a sequence of Dehn surgeries on $\hat{\rho}$ to obtain structures on $int(M_0)$ then this produces a sequence of hyperbolic 3-manifolds converging geometrically to

$\mathbb{H}^3/\hat{\rho}(\hat{N})$), so that sufficiently far along the sequence we obtain developing maps from $\tilde{f}_n \circ \tilde{f}$, where $\tilde{f}_n$ is the sequence of approximate isometries arising from geometric convergence.

It is a theorem of Thurston's [13] that a small neighbourhood of $\tilde{f}$ in the space of developing maps on $\tilde{M}$ splits as a product $\mathcal{I} \times V$, where $\mathcal{I}$ is a neighbourhood of the inclusion map of M into $Collar(M)$ in the space of locally flat (locally non-self-linking) embeddings, and V is a neighbourhood of ρ in the character variety of $\pi_1(M)$.

Thus for a representation $\rho_0 \in V \cap B$, there is a developing map corresponding to (i, ρ_0), where i is inclusion. This developing map descends to an immersion $f_0 : M \longrightarrow N$, where $\rho = (h : M \longrightarrow N)$. If we have a continuous path ρ_t of representations in $V \cap B$ one can find a continuous path in the space of developing maps D_t on M with the holonomy of D_t equal to ρ_t.

In the case that the representation ρ_0 is obtained as a Dehn filling of $\hat{\rho}$, we can say still more. Since the topology on the space of developing maps is the geometric topology, by shrinking the neighbourhood V sufficiently we may assume that $f_0(M)$ lies in the complement of the geodesic representatives in N of the core curves of $\mathcal{V}$; since these curves are getting very short, curves that pass through them become too long. Thus we consider f_0 as a map into $\hat{N}$, where $\hat{N}$ is the result of removing from N the geodesic representatives of the curves in Δ.

Suppose that $(\hat{\rho}_0, f_0)$ and $(\hat{\rho}_1, f_1)$ both induce $\rho \in \overline{B}$. If ρ_0 is obtained as a Dehn filling of $\hat{\rho}_0$ and ρ_1 is obtained as a Dehn filling of $\hat{\rho}_1$ and both ρ_0 and ρ_1 are contained in the same component of $V \cap B$, then there is a path between them that can be "lifted" to a path in the space of developing maps on M and this path in turn can be realized as a homotopy between f_0 and f_1 in $\hat{N}$. Thus if f_0 and f_1 are not equivalent we obtain a contradiction.

This reasoning, made precise, gives the following local picture of $\overline{B}$ at ρ.

Lemma 8 *(Bromberg-Holt [11])*
Let $f_0 : M \longrightarrow \hat{N}$ and $f_1 : M \longrightarrow \hat{N}$ be non-trivial, inequivalent shuffle immersions and let $\hat{\rho}_i$, $i = 0, 1$ be minimally parabolic structure in $AH(\hat{N})$ so that $\rho = \hat{\rho}_0 \circ (f_0)_* = \hat{\rho}_1 \circ (f_1)_*$. Let V be the neighbourhood of ρ defined above. Then for every $U \subset V$, $B \cap U$ contains at least two components.

In fact, if $f_1, \ldots, f_n$ is a collection of pair-wise inequivalent non-trivial shuffle immersions and $\hat{\rho}_1, \ldots, \hat{\rho}_n$ is a set of minimally parabolic structures in $AH(\hat{M})$ so that $\hat{\rho}_1 \circ (f_1)_* = \hat{\rho}_2 \circ (f_2)_* = \cdots = \hat{\rho}_n \circ (f_n)_* = \rho$, and V

is the neighbourhood of ρ defined above, then $B \cap V$ contains at least n components. In the special case that M_0 is homeomorphic to M, there are at least $n+1$ components, since the trivial shuffle immersion can also be used to induce ρ.

The next theorem combines the theory of primitive shuffles and shuffle immersions to show that bumping and self-bumping can occur simultaneously.

Theorem 9 *(Holt [17])*

Let M be compact, atoroidal and irreducible, with non-empty and incompressible boundary. Assume that there exists a solid torus component or an I-bundle component to the characteristic submanifold of M.

Let $B_1, \ldots, B_k$ be a collection of distinct components of $MP(M)$ so that for each i and j, $\overline{B_i} \cap \overline{B_j} \neq \emptyset$. Moreover, let $n_1, \ldots, n_k$ be any collection of positive integers. Set $\Theta(B_i) = [(M_i, s_i \circ h)]$, for a primitive shuffle s_i.

Then there exists a geometrically finite representation $\rho \in \bigcap \overline{B}_i$ so that the following holds.

For each i there are n_i distinct pairs $(\widehat{\rho}_j, f_j)$, $j = 1, \ldots, n_i$, inducing ρ. That is $\rho = \widehat{\rho}_j \circ (f_j)_*$, $j = 1, \ldots, n_i$, and each f_j is a shuffle immersion between $M = M_1$ and $\hat{N}_i = int(\hat{M}_i)$, so that no f_j is equivalent to any f_k, $j \neq k$. Moreover, if $\iota : \hat{N}_i \longrightarrow M_i$ is inclusion, then $\iota \circ f_j \circ s_i^{-1}$ is homotopic in M_i to an orientation preserving homeomorphism.

We obtain the following theorem as a corollary.

Theorem 10 *(Bromberg-H)*

Let ρ be the structure given by the above theorem. Then for any sufficiently small neighbourhood V of ρ, $V \cap B_i$ contains at least n_i components, $i = 2, \ldots, k$, and $V \cap B_1$ contains at least $n_1 + 1$ components.

For a given structure $\rho \in \overline{B}$, there are at most finitely many pairs $(\hat{\rho}, f)$ inducing ρ, where $\hat{\rho} \in MP(\hat{M})$ and f is a homotopy class of non-trivial shuffle immersion. Say that there are t_ρ such pairs.

Using the theory of complex projective structures on surfaces Bromberg and Holt (and independently, Ito) have refined the picture of the self-bumping locus in the case that M is an I-bundle over a surface. In this case, each pair $(\hat{\rho}, f)$ is shown to induce at least two components of $QF(S) \cap V$. In particular we have the following theorem.

Theorem 11 *(Bromberg-Holt [10])*
For a geometrically finite self-bumping point $\rho \in AH(S)$, for every sufficiently small neighbourhood V of ρ, $V \cap QF(S)$ contains at least $2t_\rho + 1$ components.

One would be tempted to conjecture that there should be exactly $2t_\rho + 1$ such components; however, there is evidence that $AH(S)$ need not be locally connected. (A discussion of this is beyond the scope of this article, however).

But we can make the following conjecture.

Conjecture 1 *If $t_\rho = 0$ then ρ is not a self-bumping point.*

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