Multiple bumping of components of deformation spaces of hyperbolic 3-manifolds

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Abstract

Let M be a hyperbolizable 3-manifold with non-empty incompressible boundary of negative Euler characteristic. Suppose that $B_1 \dots, B_k$ is a collection of components of the interior of the space of complete, marked hyperbolic 3-manifolds homotopy equivalent to M, such that for any i, j, $\overline{B}_i \cap \overline{B}_j \neq \emptyset$. We prove that there is a geometrically finite hyperbolic structure on $int(M)$ which is in the closure of each B_i. Moreover, we show that this structure can be constructed so as to admit quasiconformal deformations which also lie in the closure of every B_i.

1 Introduction

If a 3-manifold M admits a complete hyperbolic structure on its interior it is natural to seek to understand all the possible structures. In particular, one might want to understand what structures one can deform to from a given structure. The natural approach is to consider the set of all isometry classes of marked hyperbolic 3-manifolds homotopy equivalent to the interior of M, to topologize this set in a suitable fashion, and then to study its topology. The suitable topology turns out to be the algebraic topology: two marked hyperbolic 3-manifolds are close in the algebraic topology if, up to conjugation by an element of $Isom(\mathbb{H}^3)$, the two maps on fundamental group induced by the two marking maps take any element of a generating set for $\pi_1(M)$ to nearby isometries of $\mathbb{H}^3$. The resulting space is generally denoted $AH(\pi_1(M))$.

The algebraic topology does not give total insight as to how close the two hyperbolic manifolds might be geometrically - two marked hyperbolic 3-manifolds might be close in the algebraic topology but not homeomorphic. For this reason we consider the notion of geometric convergence.

Two hyperbolic 3-manifolds with base-frame are close geometrically provided that there is a biLipshitz diffeomorphism, with small biLipshitz constant, between a large portion of one manifold and a large portion of the other, taking base-frame to base-frame. This notion of closeness gives a topology, not

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on $AH(\pi_1(M))$ but on the set of all hyperbolic 3-manifolds with base-frame, or equivalently the set of all (conjugacy classes of) Kleinian groups. In the setting of Kleinian groups the geometric topology translates into the Hausdorff topology on the space of closed subsets of $Isom^+(\mathbb{H}^3) \simeq PSL_2(\mathbb{C})$.

It is often convenient to consider an element of $AH(\pi_1(M))$ not as a marked hyperbolic 3-manifold (representing an isometry class of hyperbolic manifolds), but instead as a faithful representation of $\pi_1(M)$ into $PSL_2(\mathbb{C})$ with discrete image (representing a conjugacy class of such representations). In this context, for an algebraically convergent sequence $\rho_n \to \rho$ we can consider the associated geometric limit $\hat{\Gamma}$ of the Kleinian groups $\Gamma_n = \rho_n(\pi_1(M))$, should it exist.

Jørgensen and Marden were the first to systematically explore the relationship between an algebraic limit and its geometric limits ([17]). In particular they showed that, up to subsequence, every algebraically convergent sequence has a geometric limit. It is clear that the geometric limit contains the image of the algebraic limit, and so there is a cover between the associated quotient hyperbolic 3-manifolds.

Since the geometric limit gives geometric information about the limiting quotient hyperbolic manifolds $N_n = \mathbb{H}^3/\Gamma_n$ it is natural to try to study this covering map in order to extract information about how the marked, oriented homeomorphism type of the approximates differs from that of the algebraic limit. This approach was first used in [1] to show that under certain natural conditions the marked, oriented homeomorphism type of N_n is always eventually the same as that of the algebraic limit. When these conditions do not hold however, this result is false, as shown by [2]. In [3], Anderson, Canary and McCullough, amongst other things, completed the analysis, classifying all the possible homeomorphism types of the manifolds N_n, as ρ_n converges to ρ algebraically, in the case that M has incompressible boundary.

In [13] we showed that for the family of manifolds studied in [2], all of which are examples of books of I-bundles, there exists a representation ρ with the property that for each of the possible marked, oriented homeomorphism types, as above, there is a sequence ρ_n converging to ρ so that, for large n, $N_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ has the prescribed marked, oriented homeomorphism type. Moreover, given $K \geq 1$, ρ can be constructed so that the set of K-quasiconformal deformations of ρ shares this property.

In this work we extend the analysis of [13] to the general case of a hyperbolizable 3-manifold with incompressible boundary. Hence [13] becomes a particular case of the theorem proven here.

In particular we prove the following theorem.

Theorem 3.13

Let M be a compact, hyperbolizable 3-manifold with non-empty, incompressible boundary. Suppose that $B_1, \dots, B_k$ are components of the interior of $AH(\pi_1(M))$ so that $\overline{B}_i \cap \overline{B}_j \neq \emptyset$, for each i, j.

Then

$$\bigcap_i \overline{B}_i \neq \emptyset.$$

Moreoever, the intersection contains a geometrically finite uniformization of $int(M)$.

The requirement that we consider only a finite set of components is not a restriction. Anderson, Canary and McCullough show that the only finitely many components of the interior of $AH(\pi_1(M))$ can bump. It should be noted that it can occur that there are infinitely many components to the interior of $AH(\pi_1(M))$; Canary and McCullough [9] prove that this happens if and only if M has a topological property known as *double trouble.* (A compact manifold M has double trouble provided that there two curves in ∂M that are not homotopic in ∂M but are homotopic in M to a curve lying in a torus component of ∂M.)

Further to the above theorem, we show that the bumping locus is fairly large. The next theorem makes this statement more precise.

Theorem 3.15

Assume the assumptions and notation of theorem 3.13. Let $K \geq 1$ *be given. Then there exists a geometrically finite representation* ρ *so that any* K*-quasiconformal deformation of* ρ *is an element of*

$$\bigcap_i \overline{B}_i.$$

In final section we show the existence of shuffle immersions with particularly nice properties, which can be used to study self-bumping (see [25], [8]), and bumping of components of the space of complex projective structures with quasi-fuchsian holonomy ([7], [14], [25]).

The proofs of the main theorems are by construction. Our main tools in the construction will be the Klein-Maskit combination theorems and the theory of the characteristic submanifold due to Jaco-Shalen and Johannson. These topics are covered in section 2. In section 3 we begin by introducing some technical machinery that will be needed when finally we address the proofs of Theorems 3.13 and 3.15. It is suggested that the reader first read [13] to get an idea of the proofs in a particular case. Then skim read section 2 for definitions and notation before reading the outline of proof in section 3.

1.1 Acknowledgements

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2 Preliminaries

2.1 Elements of Kleinian group theory

Definition 2.1 A *Kleinian group* is a discrete subgroup of the group of orientation preserving isometries of $\mathbb{H}^3$.

The upper half space model of $\mathbb{H}^3$ identifies the orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL_2(\mathbb{C})$. A Kleinian group has a natural action on the Riemann sphere $\widehat{\mathbb{C}}$ as conformal automorphisms, and this action is a continuous extension of its action on $\mathbb{H}^3$. For an element $\delta \in PSL_2(\mathbb{C})$ we denote by $\overline{\delta}$ the extension of δ to an isometry of $\mathbb{H}^3$.

Definition 2.2 The *domain of discontinuity* of a Kleinian group Γ is the largest open subset $\Omega(\Gamma)$ of $\widehat{\mathbb{C}}$ such that Γ acts properly (freely) discontinuously on $\Omega(\Gamma)$. The *limit set* $\Lambda(\Gamma)$ of Γ is the complement of $\Omega(\Gamma)$ in $\widehat{\mathbb{C}}$.

The limit set is either a perfect set (and hence uncountable) or it consists of at most 2 points, in which case we call it *elementary*. An elementary Kleinian group is a virtually free abelian group of rank 1 or 2. **We will always assume our Kleinian groups to be non-elementary and torsion free**. When Γ is non-elementary the limit set is the smallest, closed, non-empty Γ-invariant subset of $\widehat{\mathbb{C}}$.

Definition 2.3 The quotient $\Omega(\Gamma)/\Gamma$ naturally has the structure of a Riemann surface. By Ahlfor's finiteness theorem if Γ is finitely generated then $\Omega(\Gamma)/\Gamma$ is of finite analytic type. $\Omega(\Gamma)/\Gamma$ is called the *conformal boundary* of the hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma$ and the *conformal extension* $N(\Gamma)$ is $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

Notation: If $\delta\Gamma_0\delta^{-1} = \Gamma_1$ for some quasi-conformal homeomorphism δ of $\widehat{\mathbb{C}}$ (including the case that δ is conformal) then δ extends to an equivariant map $\mathbb{H}^3 \longrightarrow \mathbb{H}^3$ (see [28] or [11]) which descends to a map $\overline{\delta} : N(\Gamma_0) \longrightarrow N(\Gamma_1)$.

2.2 Geometric finiteness

Definition 2.4 Let M be a compact, orientable, irreducible, atoroidal 3-manifold and let $P \subset \partial M \neq \emptyset$ consist of annuli and tori. The pair (M, P) is a *pared* 3-manifold if the following holds:

1. if P_1 is a component of P then $\pi_1(P_1)$ injects into $\pi_1(M)$ and $\pi_1(P_1)$ is a maximal abelian subgroup of $\pi_1(M)$;

2. every non-cyclic abelian subgroup of $\pi_1(M)$ is conjugate into the fundamental group of some component of P;

3. every proper map $\phi : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, P)$ that induces an injection from $\pi_1(S^1 \times I)$ into $\pi_1(M)$ is properly homotopic to a map $\phi' : (S^1 \times I, S^1 \times \partial I) \longrightarrow (M, P)$ such that $\phi'(S^1 \times I) \subset P$.

Definition 2.5 Let Γ be a finitely generated, torsion free Kleinian group. If there is a pared 3-manifold (M, P) so that $N(\Gamma)$ is homeomorphic to $M - P$, then we say that Γ is *geometrically finite*, and further, if P contains no annuli then we say that Γ is *minimally parabolic*. More precisely, a geometrically finite Kleinian group is minimally parabolic if every maximal parabolic subgroup is of rank 2.

By Thurston's hyperbolization theorem [26] if (M, P) is an oriented pared 3-manifold then there is a torsion free geometrically finite Kleinian group Γ so that $M - P$ is orientation-preserving homeomorphic to $N(\Gamma)$. In particular there is a faithful representation of $\pi_1(M)$ in $PSL_2(\mathbb{C})$ with discrete image.

Definition 2.6 Let J be a subgroup of a Kleinian group Γ. A set $A \subset \widehat{\mathbb{C}} \cup \mathbb{H}^3$ is *precisely J-invariant in* Γ provided that it is J-invariant and that if $g \in \Gamma - J$ then $g(A) \cap A = \emptyset$.

Definition 2.7 Suppose that Γ is a geometrically finite Kleinian group and F an infinite cyclic maximal parabolic subgroup of Γ with fixed point p. There are two open disks B_1 and B_2 in $\widehat{\mathbb{C}}$ such that $B_1 \cup B_2$ is precisely F-invariant in Γ (see Prop A.10 in [21]). Let B_i^3 be the totally geodesic open half-space in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ which meets $\widehat{\mathbb{C}}$ in B_i, $i = 1, 2$. By Prop A.5 of [21] there exists an open horoball B_0^3 centered at the fixed point p, precisely F-invariant in Γ, where now we consider Γ acting on $\mathbb{H}^3$. Set $\tilde{T} = B_0^3 \cup B_1^3 \cup B_2^3$. Then $\tilde{T}$ is precisely F-invariant and simply connected, so that $T = \tilde{T}/F$ is an embedded solid torus in $N(\Gamma)$. T is called a *Marden tube* for F in $N(\Gamma)$. See Marden [20] for details on the existence of Marden tubes.

We say that a collection of sets $\{T_1, \dots, T_k\}$ is a *disjoint* set if they are pairwise disjoint. We make the trivial observation that if $\{T_1, \dots, T_k\}$ is a disjoint set of Marden tubes for $N(\Gamma)$ then the union of all the lifts to $\mathbb{H}^3 \cup \Omega(\Gamma)$ of elements of $\{T_1, \dots, T_k\}$ is disjoint. Note that when Γ is geometrically finite a maximal collection of disjoint Marden tubes can always be found (see Prop A.15 in [21].)

Definition 2.8 For a geometrically finite Kleinian group Γ we define a particular class of submanifold of $N(\Gamma)$ as follows. Let $\mathcal{H}$ be a Γ-invariant collection of disjoint, open horoballs in $\mathbb{H}^3$ centered at the fixed points of rank-2 parabolic subgroups, such that for any $H \in \mathcal{H}$, H is precisely invariant under $Stab_\Gamma(H)$ in Γ. Choose a maximal disjoint set of Marden tubes for $N(\Gamma)$, $\mathcal{T}$. By Prop A.15 in [22] we can choose $\mathcal{H}$ so that $\mathcal{T} \cap \mathcal{H}/\Gamma = \emptyset$.

Let $\pi : \mathbb{H}^3 \cup \Omega(\Gamma) \longrightarrow N(\Gamma)$ be the quotient map. Let $\tilde{X}_\Gamma$ be the complement in $\mathbb{H}^3 \cup \Omega(\Gamma)$ of the union $\pi^{-1}(\mathcal{T}) \cup \mathcal{H}$, and set $X_\Gamma = \tilde{X}_\Gamma/\Gamma$. Let $\tilde{Q}_\Gamma = \overline{\partial \tilde{X}_\Gamma \cap \mathbb{H}^3}$

and let $Q_\Gamma = \tilde{Q}_\Gamma/\Gamma$. The submanifold X_Γ is a compact core for $N(\Gamma)$, since Γ is geometrically finite.

We call X_Γ a *Marden core* for $N(\Gamma)$.

It is clear from the construction that a Marden core for $N(\Gamma)$ is not unique, but that any two are isotopic in $N(\Gamma)$. It will be necessary to keep track of the Marden tubes defining a particular Marden core. Denote by $\mathcal{T}(X_\Gamma)$ the set of Marden tubes defining X_Γ.

If $N(\Gamma)$ is orientation preserving homeomorphic to $M - P$, for some pared manifold (M, P), via a map $\tilde{\phi}$, then (M, P) and (X_Γ, Q_Γ) are orientation preserving homeomorphic as pairs via a map ϕ such that the composition of ϕ with the inclusion of X_Γ into $N(\Gamma)$ is homotopic to $\tilde{\phi}$. (See [9] for details.)

Definition 2.9 For a pared 3-manifold (M, P) a *pared uniformization* of (M, P) is an orientation preserving map of pairs $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$, for some geometrically finite Kleinian group Γ.

Definition 2.10 Suppose that Γ is geometrically finite and F is an infinite cyclic maximal parabolic subgroup of Γ. A *knuckle* S for F and X_Γ is a closed solid torus in $N(\Gamma)$ such that, if $T \in \mathcal{T}(X_\Gamma)$ denotes the Marden tube for F, then there is a Marden tube $T' \subset T$ so that $\overline{T'} \subset T$ with $S = \overline{T - T'}$; thus $S \cap \partial X_\Gamma = T \cap \partial X_\Gamma \subset Q_\Gamma$, and $\emptyset \neq S \cap \widehat{\mathbb{C}} \subset T \cap \widehat{\mathbb{C}}$. By $\tilde{S}$ we will mean that component of the lift of S to $\mathbb{H}^3 \cup \Omega(\Gamma)$ which is stabilized by F.

There is always a knuckle for such an X_Γ and F.

Definition 2.11 Let $\Gamma_1, \ldots, \Gamma_n$ be subgroups of a geometrically finite group Γ and let F be a maximal cyclic parabolic subgroup of Γ such that $F = \Gamma_i \cap \Gamma_j$ for all $i \neq j$. Let $\pi_j : \mathbb{H}^3/\Gamma_j \longrightarrow \mathbb{H}^3/\Gamma$ be the covering map.

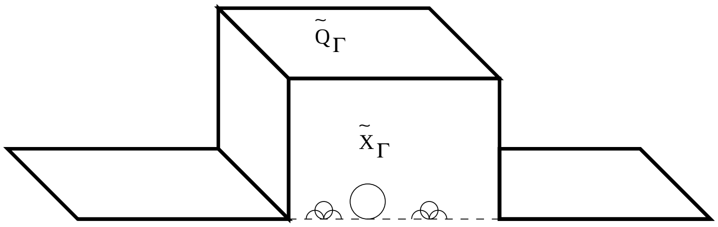
Suppose that X_{Γ_j} is a Marden core for $N(\Gamma_j)$, $j = 1, \ldots, n$. Suppose that there is a Marden core X_Γ for $N(\Gamma)$ such that π_j restricted to $int(X_{\Gamma_j})$ is an embedding into $int(X_\Gamma)$, $j = 1, \ldots, n$, so that $\{\overline{\pi_j(int(X_{\Gamma_j}))}\}_{j=1}^n$ is disjoint, and

$$S = \overline{X_\Gamma - \bigcup_j \overline{\pi_j(int(X_{\Gamma_j}))}}$$

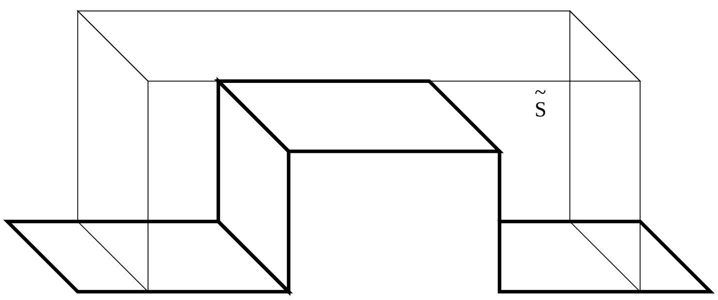
is a solid torus with $\pi_1(S) = F$. Then S is called a *spine* for F and $\{X_{\Gamma_1}, \ldots, X_{\Gamma_n}\}$ in X_Γ. Clearly the existence of such a spine depends on the groups Γ, Γ_j, $j = 1, \ldots, n$, and the choices of Marden cores.

2.3 Klein-Maskit Combination Theorems

The Klein-Maskit combination theorems are tools with which we can build hyperbolic manifolds by "sewing together" hyperbolic manifolds. They have been proven in great generality, but for our purposes the following special cases are sufficient. See Maskit [21] for a more general treatment.

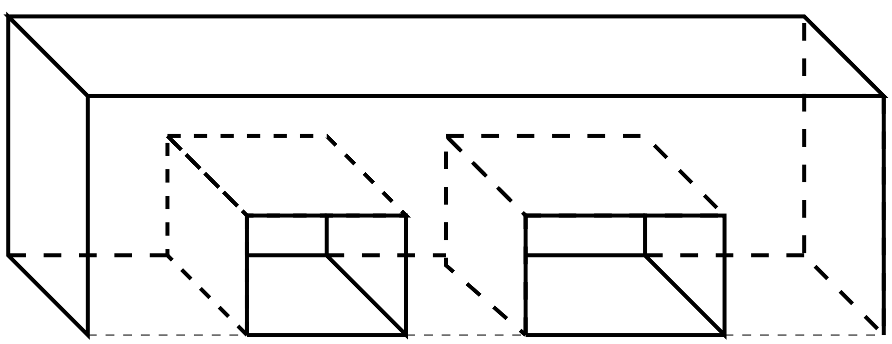


A lift of a Marden core



A lift of a knuckle

Figure 1: Marden cores and knuckles



A lift of a spine

Figure 2: A spine

Theorem 2.12 *Klein-Maskit Combination I*

Let $J =< z \mapsto z+1 >$ be a maximal parabolic subgroup of geometrically finite groups Γ_1 and Γ_2. Let (M_i, P_i) be pared 3-manifolds and $\phi_i : (M_i, P_i) \longrightarrow (X_{\Gamma_i}, Q_{\Gamma_i})$ be orientation preserving homeomorphisms of pairs, $i = 1, 2$. Suppose there is a circle W dividing $\widehat{\mathbb{C}}$ into two closed disks U_1 and U_2 such that U_i is precisely J-invariant in Γ_i, for each i, and $U_i \cap \Lambda(\Gamma_i) = \infty$. Further suppose that $U_i \cap \tilde{X}_{\Gamma_i} = \emptyset$. Set $\Gamma =< \Gamma_1, \Gamma_2 >$.

Then the following hold:

1. $\Gamma = \Gamma_1 *_J \Gamma_2$;
2. Γ *is discrete and geometrically finite;*
3. *Let S_i be a knuckle for J and X_{Γ_i}, $i = 1, 2$, so that the intersection of $\partial\tilde{S}_1$ with the geodesic plane bounded by W is equal to the intersection of the same plane with $\partial\tilde{S}_2$. Let $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$. There is a Marden tube T so that $\tilde{\tilde{T}} \cap \tilde{S} = \overline{\partial\tilde{T} \cap \mathbb{H}^3}$;*
4. *Let $T_i \in \mathcal{T}(X_{\Gamma_i})$ be such that its closure contains S_i, $i = 1, 2$, and let $\pi_i : \mathbb{H}^3/\Gamma_i \longrightarrow \mathbb{H}^3/\Gamma$ be the covering map. Then π_i embeds the union of the interiors of the elements of $\mathcal{T}(X_{\Gamma_i}) - \{T_i\}$ in $\mathbb{H}^3/\Gamma$, $i = 1, 2$, and $\mathcal{T} = \overline{\pi_1(\mathcal{T}(X_{\Gamma_1}) - \{T_1\})} \cup \overline{\pi_2(\mathcal{T}(X_{\Gamma_2}) - \{T_2\})} \cup \{T\}$ is a maximal, disjoint collection of Marden tubes. Hence there is a Marden core X_Γ with $\mathcal{T}(X_\Gamma) = \mathcal{T}$ and $\tilde{S}$ descends to a spine for J and $\{X_{\Gamma_1}, X_{\Gamma_2}\}$ in X_Γ.*
5. *Let V be an oriented solid torus and let A_0, A_1 and A_2 be disjoint, parallel longitudinal annuli in ∂V. Let S be a spine for J and $\{X_{\Gamma_1}, X_{\Gamma_2}\}$ and set $D_i = \overline{\mathbb{H}^3 \cap \tilde{X}_{\Gamma_i} \cap \tilde{S}}/J$, $i = 1, 2$. D_1 and D_2 are parallel, longitudinal annuli in ∂S. Let $D_0 = (\partial\tilde{S} \cap \mathbb{H}^3 - \tilde{D}_1 \cup \tilde{D}_2)/J$. Let $\phi_0 : V \longrightarrow S$ be an orientation preserving homeomorphism so that $\phi_0(A_i) = D_i$. Form a manifold M from the disjoint union of M_1, M_2 and V by identifying $B_i = \phi_i^{-1}(D_i)$ with A_i with the orientation reversing homeomorphism $\phi_0^{-1} \circ \phi_i$, $i = 1, 2$. Let P be $(P_1 - B_1) \cup (P_2 - B_2) \cup A_0$.*

   Then there is an orientation preserving homeomorphism $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ such that $\phi \circ i_j$ is equal to $\iota_j \circ \phi_j$ where $\iota_j : X_{\Gamma_j} \longrightarrow X_\Gamma$ and $i_j : M_j \longrightarrow M$ are inclusions.

Remark 1 Note that the inclusion $\iota_j : X_{\Gamma_j} \longrightarrow X_\Gamma$ is not strictly an inclusion. We regard it as an inclusion, but more correctly it is the extension of the restriction of the covering map $\pi_j|_{int(X_{\Gamma_j})} : int(X_{\Gamma_j}) \longrightarrow int(X_\Gamma)$ to an injective map $X_{\Gamma_j} \longrightarrow X_\Gamma$.

Remark 2 There is no mention in the version of the first Klein-Maskit combination theorem cited in [21] to the existence of spines, as stated above. But it

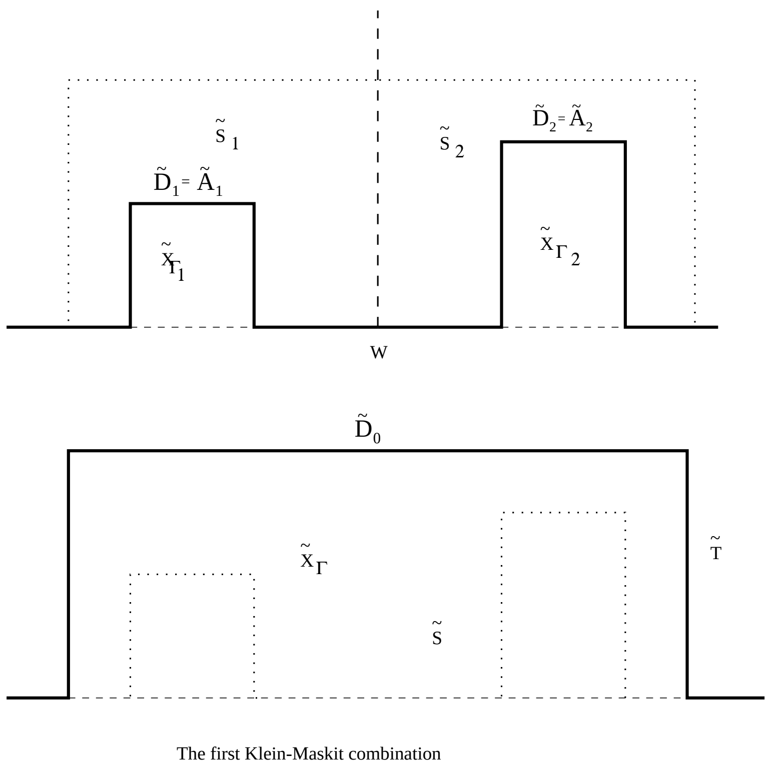


Figure 3: Klein-Maskit I

is a consequence of the combination theorem in its standard form, when Γ_1 and Γ_2 are geometrically finite and are being amalgamated over an infinite cyclic subgroup. In its standard form (see [21]) the combination theorem implies that, if $\pi_j : \mathbb{H}^3/\Gamma_j \longrightarrow \mathbb{H}^3/\Gamma_j$ is the covering map, $i = 1, 2$, then π_j is an embedding when restricted to $int(S_j \cup X_{\Gamma_j})$, and that $\pi_1|_{int(\tilde{S}_1 \cap \tilde{S}_2)/J} = \pi_2|_{int(\tilde{S}_1 \cap \tilde{S}_2)/J}$, so that $\overline{\pi_1(int(S_1)) \cup \pi_2 int((S_2))} = S$ is a spine for J and $\{X_{\Gamma_1}, X_{\Gamma_2}\}$.

Theorem 2.13 *Klein-Maskit Combination II*

Let J_1 and J_2 be non-conjugate, cyclic, maximal parabolic subgroups of a geometrically finite group Γ_0. Let (M_0, P_0) be an oriented pared 3-manifold and $\phi_0 : (M_0, P_0) \longrightarrow (X_{\Gamma_0}, Q_{\Gamma_0})$ an orientation preserving homeomorphism. Suppose that U_i is a closed, precisely J_i-invariant subset of $\widehat{\mathbb{C}}$, bounded by circles, so that $U_i \cap \Lambda(\Gamma_0) = \partial U_i \cap \Lambda(\Gamma_0) = Fix(J_i)$. Suppose further that $U_i \cap \tilde{X}_{\Gamma_0} = \emptyset$. Let δ be a transformation conjugating J_1 to J_2 so that $\delta(\widehat{\mathbb{C}} - U_1) = int(U_2)$. Set $\Gamma =< \Gamma_0, \delta >$.

Then the following statements hold:

1. $\Gamma = \Gamma_0 *_\delta$;

2. Γ *is geometrically finite;*

3. *Let S_i be a knuckle for J_i and X_{Γ_0} such that $\tilde{S}_i \cap U_i = \partial U_i$, $i = 1, 2$. Let $\tilde{S}$ be the $\overline{\text{union of } \tilde{S}_2} \cup \delta(\tilde{S}_1)$. There is a Marden tube T in $N(\Gamma)$ so that $\bar{\tilde{T}} \cap \tilde{S} = \overline{\partial \tilde{T} \cap \mathbb{H}^3}$. Let T_i be the Marden tube in $\mathcal{T}(X_{\Gamma_0})$ whose closure contains S_i, $i = 1, 2$. Let $\pi : \mathbb{H}^3/\Gamma_0 \longrightarrow \mathbb{H}^3/\Gamma$ be the covering map. Then*

$$\pi : \bigcup_{T \in \mathcal{T}(X_{\Gamma_0}) - \{T_1, T_2\}} int(T) \longrightarrow \mathbb{H}^3/\Gamma$$

   extends to an embedding of $\bigcup_{T \in \mathcal{T}(X_{\Gamma_0}) - \{T_1, T_2\}} T$ into $N(\Gamma)$. The set $\mathcal{T} = \overline{\pi(int(\mathcal{T}(X_{\Gamma_0}) - \{T_1, T_2\}))} \cup \{T\}$ is a maximal set of disjoint Marden tubes for Γ and hence there is a Marden core X_Γ with $\mathcal{T}(X_\Gamma) = \mathcal{T}$.

4. *Let C_i be the totally geodesic plane in $\mathbb{H}^3$ bounded by ∂U_i, $i = 1, 2$, so that $\delta(\overline{C_1} = \overline{C_2}$. With $\tilde{S}$ as above, $\tilde{S}$ descends to a solid torus S in X_Γ. Identify S with an oriented solid torus V by an orientation preserving homeomorphism $\phi_1 : V \longrightarrow S$. Set $A_1 = \phi_1^{-1}(S_2 \cap X_{\Gamma_0}))$, and $A_2 = \phi_1^{-1}(\bar{\delta}(S_2 \cap X_{\Gamma_0}))$. Let D_1 be that portion of the frontier of $\bar{\delta}(S_1)$ in $\bar{\delta}(N(\Gamma_0))$ not intersecting $\bar{\gamma}(X_{\Gamma_0})$ or C_2/J_2. Let D_2 be that portion of the frontier of S_2 in $N(\Gamma_0)$ not intersecting X_{Γ_0} or C_2/J_2. Let A_0 be obtained from the disjoint union*

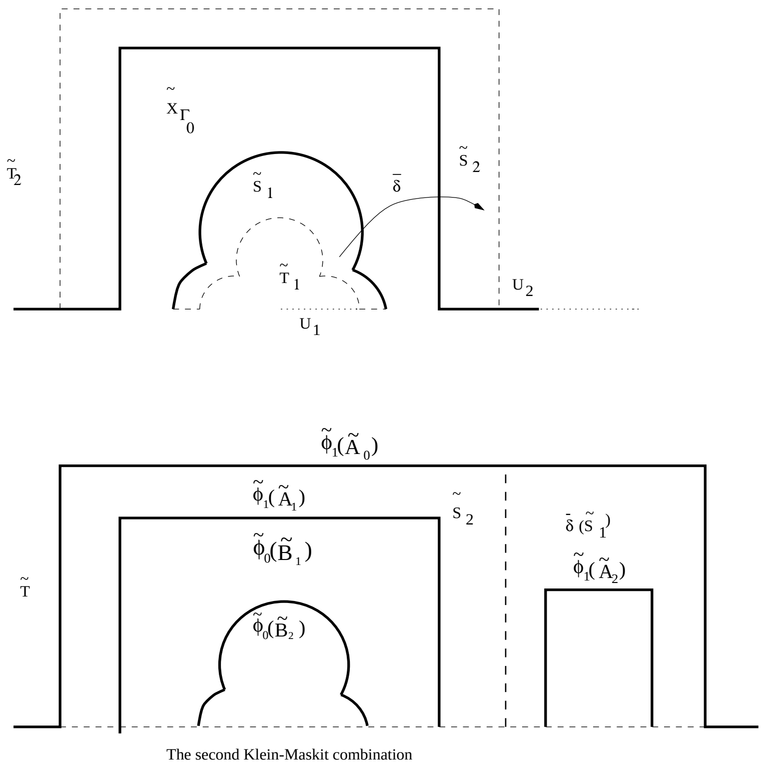


Figure 4: Klein-Maskit II

of D_1 and D_2 by identifying $\partial D_1 \cap (C_2/J_2)$ with $\partial D_2 \cap (C_2/J_2)$. Finally, let $B_i = \phi_0^{-1}(S_{3-i} \cap Q_{\Gamma_0})$, $i = 1, 2$.

Form a manifold homeomorphic to M_0 from the disjoint union of M_0 and V by identifying A_1 with B_1. Obtain M from this manifold by identifying A_2 with B_2 via the map $\phi_0^{-1} \circ \bar{\delta}^{-1} \circ \phi_1$, and set $P = P - \{B_1, B_2\} \cup \{A_0\}$. Then there is an orientation preserving homeomorphism $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that, with $i : M_0 \longrightarrow M$ and $j : X_{\Gamma_0} \longrightarrow X_\Gamma$ being inclusion, we have $\phi \circ i = j \circ \phi_0$.

Remark: Note that once again, and not for the last time, we abuse language and refer to the extension of the covering map restricted to $int(X_{\Gamma_0})$ to an injective map of X_{Γ_0} into X_Γ as an inclusion.

2.4 Deformation spaces

Thurston's hyperbolization theorem guarantees the existence of a geometrically finite hyperbolic structure on the interior of any compact, orientable, irreducible, atoroidal 3-manifold M with boundary. We are interested in the set of all possible complete hyperbolic structures on the interior of M. To this end let $H(\pi_1(M))$ denote the set of marked hyperbolic 3-manifolds homotopy equivalent to M, modulo an equivalence relation in which (N, h) is equivalent to (N', h') if and only if there is an orientation preserving isometry $\phi : N \longrightarrow N'$ so that $\phi \circ h$ is homotopic to h'.

There is another way to think of $H(\pi_1(M))$, which is as the set of $PSL_2(\mathbb{C})$-equivalence classes of discrete, faithful representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$. The action of $PSL_2(\mathbb{C})$ on the space of representations is by conjugation. The correspondence is given by $[(N, h)] \mapsto [h_*]$. In this way we can consider $H(\pi_1(M))$ as the set of conjugacy classes of discrete, faithful representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$.

We define the *algebraic topology* on $H(\pi_1(M))$ as follows. In the algebraic topology, a sequence $[\rho_i]$ of conjugacy classes of discrete faithful representations into $PSL_2(\mathbb{C})$ converges to a class $[\rho]$ if and only if there are representatives ρ_i for $[\rho_i]$ which converge to a representative ρ for $[\rho]$. We denote $H(\pi_1(M))$ with the algebraic topology by $AH(\pi_1(M))$. By [16] $AH(\pi_1(M))$ is a closed subset of the space of all representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$, and as such has an interior.

For $\rho \in AH(\pi_1(M))$ let N_ρ denote the quotient manifold $\mathbb{H}^3/\rho(\pi_1(M))$.

Definition 2.14 Let $MP(\pi_1(M))$ denote the subspace of $AH(\pi_1(M))$ consisting of those representations with minimally parabolic image. Marden [20] proved that $MP(\pi_1(M))$ is open in $AH(\pi_1(M))$ and Sullivan [27] proved that it contains the interior of $AH(\pi_1(M))$: $MP(\pi_1(M))$ is the interior of $AH(\pi_1(M))$.

It is a conjecture of Bers and Thurston that $AH(\pi_1(M))$ is the closure of $MP(\pi_1(M))$. This is generally referred to as the *Bers Conjecture*. We remark that recent work of Brock-Canary-Minsky, Brock-Bromberg and Bromberg establishes the Bers Conjecture for topologically tame manifolds - that is to say, if N_ρ admits a manifold compactification then it is in the closure of $MP(\pi_1(M))$. Bonahon's result [5] implies that if M has incompressible boundary then N_ρ is topologically tame for any $\rho \in AH(\pi_1(M))$, so in our restricted setting $\overline{MP(\pi_1(M))} = AH(\pi_1(M))$.

Let ρ be an element of $MP(\pi_1(M))$ and suppose that B is the component of $MP(\pi_1(M))$ containing ρ. Marden and Sullivan show that B consists of all the minimally parabolic representations of $\pi_1(M)$ quasiconformally conjugate to ρ. This is to say that if ρ' is in B then there is a quasiconformal map ϕ of the Riemann sphere which conjugates the action to ρ to the action of ρ'; for any element g of $\pi_1(M)$ we have $\rho(g)(z) = \phi \circ \rho'(g) \circ \phi^{-1}$. This quasiconformal map extends to an equivariant diffeomorphism of $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ which descends to diffeomorphism $\bar{\phi}$ between N_ρ and $N_{\rho'}$, the associated quotient manifolds. In particular $\bar{\phi}$ can be homotoped to give a homeomorphism between a Marden core for N_ρ and a Marden core for $N_{\rho'}$ (see the proof of the Isomorphism theorem (theorem 8.1) in [20] for details). For a geometrically finite manifold a Marden core is a compact core, so we see that N_ρ and $N_{\rho'}$ have orientation preserving homeomorphic compact cores.

Definition 2.15 For M a compact, orientable, irreducible, atoroidal 3-manifold let $\mathcal{A}(M)$ denote the set of homeomorphism classes of marked, oriented, irreducible, atoroidal, compact 3-manifolds homotopy equivalent to M. That is, $\mathcal{A}(M)$ consists of equivalence classes of pairs (M', h') where $h' : M \longrightarrow M'$ is a homotopy equivalence between M and an oriented, compact, irreducible, atoroidal 3-manifold M', and where two pairs (M', h') and (M'', h'') are equivalent provided that there is an orientation preserving homeomorphism $j : M' \longrightarrow M''$ such that $j \circ h'$ is homotopic to h''.

There is a map Θ from $AH(\pi_1(M))$ onto $\mathcal{A}(M)$ defined as follows. If ρ is an element of $AH(\pi_1(M))$ then there is a homotopy equivalence h_ρ between M and N_ρ such that $(h_\rho)_* = \rho$. Let M_ρ be a choice of compact core for N_ρ and let $j_\rho : N_\rho \longrightarrow M_\rho$ be the homotopy inverse to inclusion; it is a homotopy equivalence. Define

$$\Theta(\rho) = [(M_\rho, j_\rho \circ h_\rho)].$$

This map is well-defined. By McCullough, Miller and Swarup [23] any two compact cores for M are orientation preserving homeomorphic by a homeomorphism homotopic to the identity. That is, if M_1 and M_2 are compact cores for N_ρ and $\iota_i : M_i \longrightarrow N_\rho$ is inclusion, with homotopy inverse j_i, $i = 1, 2$, then $j_2 \circ \iota_1$ is homotopic to an orientation preserving homeomorphism. Hence $[(M_2, j_2 \circ h_\rho)] = [(j_2 \circ \iota_1(M_1), j_2 \circ h_\rho)] = [(M_1, j_1 \circ h_\rho)]$. Hence $\Theta(\rho)$ is independent of the choice of compact core. If h_1 and h_2 are two homotopy equivalences between M and N_ρ with $(h_i)_* = \rho$, $i = 1, 2$, then $h_2^{-1} \circ h_1$ is homotopic to the identity map, which is certainly an orientation preserving homeomorphism. Hence $\Theta(\rho)$ is independent of the choice of homotopy equivalence h_ρ.

Thurston's hyperbolization theorem implies that Θ is surjective. By the above remarks Θ is continuous on $MP(\pi_1(M))$ (endowing $\mathcal{A}(M)$ with the discrete topology), mapping each connected component to an element of $\mathcal{A}(M)$.

Thus the components of $MP(\pi_1(M))$ are indexed by the elements of $\mathcal{A}(M)$. Θ is not continuous on all of $AH(\pi_1(M))$ however, and so a coarser notion of equivalence is needed to index the components of $\overline{MP(\pi_1(M))}$. For this we need the notion of a primitive shuffle equivalence.

2.5 The characteristic submanifold for a pared 3-manifold with incompressible boundary

Definition 2.16 An irreducible 3-manifold M has *incompressible boundary* if the fundamental group of every component of ∂M injects into $\pi_1(M)$.

From this point onwards we will assume that our manifold M has incompressible boundary.

Definition 2.17 A map $(S^1, \emptyset)$ or $(I, \partial I)$ into an 3-manifold pair (M, T) is *essential* if it is not homotopic as a map of pairs to a constant map. In the case of $(S^1, \emptyset)$ this defines an essential loop, in the case of $(I, \partial I)$ an essential path.

Suppose (M, T) is a 3-manifold pair and (X, Y) a connected 3-manifold pair. A map of pairs $f : (X, Y) \longrightarrow (M, T)$ is *essential* provided f maps essential paths and loops in (X, Y) to essential paths and loops in (M, T).

Definition 2.18 An I-bundle R embedded in M is *admissibly embedded* if $R \cap \partial M$ is the associated ∂I-bundle of R. An embedded Seifert fibred space R in M is *admissibly embedded* if $R \cap \partial M$ is a union of fibres in ∂R. An admissibly embedded I-bundle or Seifert fibred space R defines a 3-manifold pair $(R, \partial_0 R)$, where $\partial_0 R = R \cap \partial M$.

Definition 2.19 For a pared manifold (M, P), a map h of an I-bundle or Seifert fibred manifold $(R, \partial_0 R)$ into $(M, \partial M)$ is *admissible* provided

$$\partial_0 R = \coprod_{T \subset \overline{\partial M - P}} \{components\ of\ h^{-1}(T)\} \cup \coprod_{T \subset P} \{components\ of\ h^{-1}(T)\}$$

(In the unions above, T is always a connected component of whichever set it belongs to.) In particular, if R is an I-bundle over a surface that is not an annulus or a torus then an admissible map of R into (M, P) maps $\partial_0 R$ into $M - P$.

Definition 2.20 Let (M, P) be a pared 3-manifold. A disjoint collection Σ of essential admissible I-bundles and Seifert fibred spaces is a *characteristic sub-manifold* for (M, P) if

1. Σ is *full*, i.e. the union of Σ with any of the complementary components of M is not a disjoint union of essential, admissible I-bundles and Seifert fibred spaces.

2. (Engulfing property) every essential admissible I-bundle and Seifert fibred space $(R, \partial_0 R)$ in $(M, \partial M)$ is admissibly isotopic (i.e. isotopic through admissible maps) into Σ;

3. (Enclosing property) every essential map f of an annulus $(S^1 \times I, S^1 \times \partial I)$ or torus $(S^1 \times S^1, \emptyset)$ into $(M, \partial M)$ is admissibly homotopic to a map with image in Σ.

By combining corollaries 10.9 and 10.10 and theorem 12.5 of [15] one can show that every pared 3-manifold with incompressible boundary has a characteristic submanifold and it is unique up to admissible isotopy.

We have the following characterisation of the components of the characteristic submanifold in this pared manifold setting. See section 2.8 of [9], in particular theorem 2.37.

Theorem 2.21 *Let (M, P) be a pared 3-manifold with incompressible boundary. Let Σ denote the characteristic submanifold of (M, P). Then Σ can be fibred so that*

- *If $(R, \partial_0 R)$ is an I-bundle component of Σ then $\partial_0 R$ is contained in $\overline{M - P}$, and its base surface has negative Euler characteristic;*

- *If $(R, \partial_0 R)$ is a Seifert fibred component of Σ then R is homeomorphic to either $T^2 \times I$ or to a solid torus. If R is homeomorphic to $T^2 \times I$, then one of its boundary components lies in P and the other elements of $\partial_0 R$ lie in $\overline{M - P}$.*

The characteristic submanifold can be viewed as the obstruction to deforming a homotopy equivalence between two compact manifolds with incompressible boundary to a homeomorphism. The following theorem makes this more precise.

Theorem 2.22 *(Johannson's Deformation Theorem [15])*

Let (M, P) and (M', P') be pared 3-manifolds with incompressible boundary. Let $\Sigma \subset M$ and $\Sigma' \subset M'$ be the characteristic submanifolds of (M, P) and (M', P') respectively. Let $f : (M, P) \longrightarrow (M', P')$ be a homotopy equivalence of pairs. Then f can be homotoped through maps of pairs to a map $g : (M, P) \longrightarrow (M', P')$ so that

1. *$g|_{\Sigma} : \Sigma \longrightarrow \Sigma'$ is a homotopy equivalence; and*

2. *$g|_{\overline{M - \Sigma}} : \overline{M - \Sigma} \longrightarrow \overline{M' - \Sigma'}$ is a homeomorphism.*

Definition 2.23 Suppose $\mathcal{V}$ is a collection of disjoint, essential, admissibly embedded solid tori in a compact, orientable 3-manifold M such that each component of $\mathcal{V}$ is either a component of the characteristic submanifold of M or contained in an I-bundle component of the characteristic submanifold of M. Suppose further that if $V \in \mathcal{V}$ is contained in an I-bundle component of the characteristic submanifold of M then the frontier of V consists of exactly two annuli in M, and if V_1 and V_2 are two such solid tori in $\mathcal{V}$ then they are non-homotopic. In either case, also assume that for $V \in \mathcal{V}$ each component of $\partial M \cap V$ is an annulus whose inclusion into V is a homotopy equivalence (that is, each annulus in *primitive*).

Call such a collection *allowably contained* in M.

The following lemma is an easy exercise in 3-manifold topology. A full proof appears in [12].

Lemma 2.24 *Let (M, P) be a pared 3-manifold and suppose that $\mathcal{V}$ is collection of solid tori allowably contained in M, so that no component of $\mathcal{V}$ is homotopic into P. Let X be a component of $\overline{M - \mathcal{V}}$ and set $T = FrX \cup (X \cap P)$. Then (X, T) is a pared 3-manifold.*

Definition 2.25 A *primitive shuffle* $s : M_1 \longrightarrow M_2$ between two compact, oriented 3-manifolds is a homotopy equivalence for which there is a collection $\mathcal{V}_i$ of solid tori allowably contained in M_i, $i = 1, 2$, so that $s^{-1}(\mathcal{V}_2) = \mathcal{V}_1$ and s restricts to an orientation preserving homeomorphism from $\overline{M_1 - \mathcal{V}_1}$ to $\overline{M_2 - \mathcal{V}_2}$.

We say that s is *supported* on $\mathcal{V}_1$.

Remark Note that this definition differs from that given in [3]. The definition in [3] requires the support of a primitive shuffle to be a set of solid torus components of the characteristic submanifold. The reason for the change in definition is that in the absence of solid torus components of the characteristic submanifold, bumping phenomena still occurs when there is an I-bundle component of the characteristic submanifold. The reader will note that the primitive shuffle constructed by Anderson, Canary and McCullough in the proof of Theorem A of [3] satisfies the above definition.

Definition 2.26 We say that two elements $[(M_1, h_1)]$ and $[(M_2, h_2)]$ of $\mathcal{A}(M)$ are *primitive shuffle equivalent* if there is a primitive shuffle $s : M_1 \longrightarrow M_2$ so that $[(M_2, h_2)] = [(M_2, s \circ h_1)]$.

The notion of primitive shuffle equivalence as used in [3] is an equivalence relation on $\mathcal{A}(M)$. The set of equivalence classes of $\mathcal{A}(M)$ under this equivalence relation is denoted $\hat{\mathcal{A}}(M)$. We claim that two elements of $\mathcal{A}(M)$ are primitive shuffle equivalent under the definition in [3] if and only if they are primitive shuffle equivalent under the definition above. From this it will follow that the notion of primitive shuffle equivalence used here is an equivalence relation on $\mathcal{A}(M)$, and the set of equivalence classes is $\hat{\mathcal{A}}(M)$. Clearly if $[(M_1, h_1)]$ is equivalent to $[(M_2, h_2)]$ under the definition of primitive shuffle used by Anderson, Canary and McCullough then the two classes are primitive shuffle equivalent under the definition above. On the other hand, if $s : M_1 \longrightarrow M_2$ is a primitive shuffle as defined here, with $h_2 \circ s$ homotopic to h_1, then we need to demonstrate a primitive shuffle s' homotopic to s and supported on a collection of solid tori, all of which are components of the characteristic sub-manifold of M_1. Suppose that s is supported on $\mathcal{V}$ and let $\mathcal{V}_0$ consist of those solid tori which are themselves components of Σ_1 the characteristic submanifold of M_1. Let X be a component of $\overline{\Sigma_1 - \mathcal{V}_0}$ containing at least one solid torus in $\mathcal{V} - \mathcal{V}_0$. X is an I-bundle over a surface with boundary, and $s|X$ can be homotoped keeping the frontier constant to an orientation preserving homeomorphism. Repeating this for each component X, we obtain $s' : M_1 \longrightarrow M_2$, a primitive shuffle homotopic to s and supported on $\mathcal{V}_0$.

There is an induced map $\hat{\Theta}$ from $AH(\pi_1(M))$ onto $\hat{\mathcal{A}}(M)$. We have the following fundamental results on the global topology of $AH(\pi_1(M))$ due to Anderson, Canary and McCullough.

Theorem 2.27 *(Theorems A and B in [3]) Let M be a compact, orientable, atoroidal, irreducible 3-manifold with incompressible boundary. Then*

1. *$\hat{\Theta}$ is continuous on $AH(\pi_1(M))$;*

2. *Two components B_1 and B_2 of $MP(\pi_1(M))$ have intersecting closures in $AH(\pi_1(M))$ if and only if $\hat{\Theta}(B_1) = \hat{\Theta}(B_2)$.*

Hence the components of $\overline{MP(\pi_1(M))}$ are indexed by the elements of $\hat{\mathcal{A}}(M)$.

3 Bumping and Shuffling

In this chapter we will prove the main theorem. The main theorem states that when M has incompressible boundary, if $[(M_1, h_1)], \dots, [(M_m, h_m)]$ is any set of pair-wise primitive-shuffle-equivalent elements of $\mathcal{A}(M)$ then there is a representation ρ with $\Theta(\rho) = [(M_1, h_1)]$ which is in the closure of any component of $MP(\pi_1(M))$ indexed by one of $[(M_1, h_1)], \dots, [(M_m, h_m)]$. That is to say, for every j there is a sequence of representations $\{\rho_i\}$ converging to ρ with $\Theta(\rho_i) = [(M_j, h_j)]$.

We will attempt to present a cogent outline of the proof.

For each of the elements $[(M_j, h_j)]$ there is a primitive shuffle $s_j : M_1 \longrightarrow M_j$ with $s_j \circ h_1$ homotopic to h_j. The primitive shuffle s_j will be supported on a collection of solid tori $\mathcal{V}_j$, say. Let $\mathcal{V}$ be the "union" of the $\mathcal{V}_j$ (see the remark preceeding proposition 3.10 to make sense of this statement) and consider each s_j to be supported on $\mathcal{V}$. The proof will be based on induction on the number of solid tori in $\mathcal{V}$.

For simplicity we will consider the base case, when each primitive shuffle s_j is supported on the same solid torus V. By an abuse of notation we also denote by V the solid torus in M_j containing $s_j(V)$. The closure of the complement of V in M_1 is orientation preserving homeomorphic to the closure of the complement of V in M_j, for each j. Hence we can consider M_j as being obtained from the components of the closure of the complement of V in M_1 by attaching the annuli in the frontier of these components to the annuli in the frontier of V in a prescribed manner. We encode these gluing instructions with the notion of a *sewing kit* (see definition 3.1). For each j we obtain a different set of gluing instructions. The gluing instructions give us a recipe for using the Klein-Maskit combination theorems to construct a uniformization for the manifold in question.

We use the Klein-Maskit combination theorems to construct a uniformization of M_1, with uniformizing group Γ, so that for each j, $\mathbb{H}^3/\Gamma$ covers a geometrically finite hyperbolic 3-manifold, $\mathbb{H}^3/\widehat{\Gamma}_j$, which is homeomorphic to the interior of the complement in M_j of the core curve of V. The group $\widehat{\Gamma}_j$ will be generated by Γ and an additional parabolic element g_j, which we call a "shuffling parabolic". By construction g_j will have the effect of changing the gluing instructions to be those necessary to affect the required change in homeomorphism type. This is deliberately vague and the reader should see the statements of 3.7 and 3.10 for more precise statement. Performing $(1,n)$ hyperbolic Dehn surgery on the torus boundary component that resulted from the removal of the core curve of V produces a manifold homeomorphic to the interior of M_j. Composing the representations obtained from theorem 3.12 with the map of fundamental groups induced from the map from the interior of M_1 to $\mathbb{H}^3/\widehat{\Gamma}_j$ produces a sequence of representations which converges to the map on fundamental groups induced from the uniformization of M_1. This is the representation that we seek.

When the primitive shuffles are supported on a larger collection of solid tori the construction proceeds by induction, but the heart of the proof lies in the base step.

3.1 Sewing kits and uniformization

Definition 3.1 Let $\mathcal{C}$ be a finite collection of oriented, pared 3-manifolds with incompressible boundary and let V denote an oriented solid torus. Let $\mathcal{A}$ be the union of a fixed, finite collection of disjoint, parallel, longitudinal annuli in ∂V and p an orientation reversing embedding of $\mathcal{A}$ into

$$\mathcal{P} = \bigcup_{\substack{B \in P \\ (C,P) \in \mathcal{C}}} B$$

with the property that for a component A of $\mathcal{A}$, $p|_A$ is a homeomorphism onto a component of $\mathcal{P}$.

The triple $(\mathcal{C}, V, p)$ is a *sewing kit.*

Definition 3.2 Given a sewing kit $\kappa = (\mathcal{C}, V, p)$ the *manifold determined by* κ, denoted M_κ, is obtained from the disjoint union $V \cup \bigcup_{(C,P)\in\mathcal{C}} C$ by identifying an annulus A in the domain of p with $p(A)$ via p.

Fix (α, β) a meridian-longitude pair for ∂V oriented so that the orientation on V obtained by the right-hand rule is inward-pointing and agrees with the given orientation of V, and so that α bounds a disk in V. Choosing an annulus component A_1 in the domain of p fixes an ordering of the components of $\mathcal{A}$, given as the adjacency ordering induced by travelling around α starting in A_1. Let the annuli be $A_1, \ldots, A_k$, in this ordering (so that $A_{k+1} = A_1$). Then p induces a map from $\{1, \ldots, k\}$ into the union of the paring loci of the elements of $\mathcal{C}$, also called p. For the most part we will consider p in this manner, and when we consider p as a homeomorphism we will make note of it.

There is an annulus A_0 in ∂V separating A_1 and A_k. Let P_κ be the submanifold of ∂M_κ consisting of A_0 and those components of the paring locus of any element of $\mathcal{C}$ not meeting any $p(A_i)$.

With the annuli $\mathcal{A}$ in ∂V enumerated as above, we label the boundary components of each A_i, $i = 1, \ldots, k$, as follows: $\partial A_i = \partial_- A_i \cup \partial_+ A_i$, where the one of the two annuli in ∂V bounded by $\partial_+ A_i$ and $\partial_- A_{i+1}$ is disjoint from any $int(A_j)$, $j = 0, \ldots, k$. This unambiguously determines the boundary components of the A_i.

Definition 3.3 The pared manifold (M_κ, P_κ) is the *pared manifold determined by* κ.

It will be an immediate corollary of lemma 3.6 that (M_κ, P_κ) is indeed a pared 3-manifold.

Definition 3.4 For a pared uniformization $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ with $J =< z \mapsto z + 1 >\subset \Gamma$ a maximal parabolic subgroup, let $\chi(\phi) \subset PSL_2(\mathbb{C})$ be a minimal set such that

- for every component $P_0 \subset P$ there is a unique element $\gamma \in \chi(\phi)$ so that γ conjugates J to F a choice of representative of the conjugacy class of groups defined by $\phi_*(\pi_1(P_0))$;
- when $F = J$, γ, as above, is the identity.

Remark There is much choice in selecting the elements in $\chi(\phi)$: if γ is a choice of isometry conjugating J to F, then equally suitable is $\delta \circ \gamma \circ \delta^{-1}$ for $\delta \in F$. We do restrict the choice of $\chi(\phi)$ by requiring that the intersection of a Marden tube for J with $\widehat{\mathbb{C}}$ be mapped into the intersection of $\widehat{\mathbb{C}}$ with a Marden tube for F, as we now describe below.

Let $H^c = \{z \in \widehat{\mathbb{C}} \mid \Im z > c\}$ and let $H_c = \{z \in \widehat{\mathbb{C}} \mid \Im z < -c\}$.

Suppose that $c > 0$ is a constant so that, if $\gamma \in \chi(\phi)$ conjugates J to F, then $\gamma(H^c) \cup \gamma(H_c) \subset \tilde{T}_F \cap \widehat{\mathbb{C}}$, where T_F is the Marden tube in $\mathcal{T}(X_\Gamma)$ for F.

We say that ϕ is *c-normalized*, the dependence on $\chi(\phi)$ being implicit.

Suppose that $\kappa = (\mathcal{C}, V, p)$ is a sewing kit. For each $(C, P) \in \mathcal{C}$ let $\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C})$ be a pared uniformization of (C, P). For an element $(C, P) \in \mathcal{C}$ define $q(C)$ to be $\min\{j : p(A_j) \subset P\}$. The value $q(C)$ tells us where in the "queue" formed by $A_1, \ldots, A_k$ the component C is first attached under the sewing instructions.

An annulus A_i is mapped by p onto a component $B = p(i)$ of the paring locus of some $(C, P) \in \mathcal{C}$. Choose a lift of $\phi_C(B) = \phi_C \circ p(A_i)$ to $\mathbb{H}^3$, and let F_i be its stabilizer in G_C. The curve $\phi_C \circ p(\partial_- A_i)$ lifts to a F_i-invariant circle, α_- in $\widehat{\mathbb{C}}$, and $\phi_C \circ p(\partial_+ A_i)$ lifts to a F_i-invariant circle, α_+. Let S be a knuckle for F_i and X_{G_C} (some choice is involved here). There are two components of

$\widehat{\mathbb{C}} - S \cap \widehat{\mathbb{C}}$ which are open, precisely F_i-invariant disks. One of these disks has the property that its boundary together with α_+ bounds a component of $S \cap \widehat{\mathbb{C}}$. Call this disk H_i^+. The other disk has the property that its boundary together with α_- bounds the other component of $S \cap \widehat{\mathbb{C}}$. Call this other disk H_i^-.

Definition 3.5 We say that a collection $\{\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C}) \mid (C, P) \in \mathcal{C}\}$ is a *c-normalized family of pared uniformizations* for $\kappa = (\mathcal{C}, V, p)$ if following holds:

- each ϕ_C is c-normalized;
- if $\gamma_i \in \chi(\phi_C)$ conjugates J to F_i then $\gamma_i(H^c) \subset H_i^+$ and $\gamma_i(H_c) \subset H_i^-$;
- $F_{q(C)} = J$.

Note that when $F_i = J$ this says that $H^c \subset H_i^+$ and $H_c \subset H_i^-$.

Clearly if $d > c$ then a c-normalized family is also a d-normalized family.

The following lemma is nothing more than the Klein-Maskit combination theorems applied inductively to a c-normalized family of pared uniformizations. A c-normalized family has a limit set which is "uniformly c-thick" in that whenever you conjugate one of the groups G_C by an element γ which conjugates a subgroup of G_C to J, the width of the Euclidean strip containing the limit set of the resulting group is less than or equal to c. This normalization gives the degree of control that we need to effect the various shuffles. This is the reason for part 2 of the next lemma.

Lemma 3.6 *Let $\kappa = (\mathcal{C}, V, p)$ be a sewing kit. Let*

$$\{\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C}) \mid (C, P) \in \mathcal{C}\}$$

be a c-normalized family of pared uniformizations for κ.

Let $0 < a_1 < \ldots < a_k$ be a collection of real numbers so that $a_{i+1} - a_i \geq 2c+1$. Set $\xi_j(z) = z + a_j i$, $j = 1, \ldots, k$, $k = |\mathcal{A}|$.

Set $\delta_i = \xi_{qp(i)} \gamma_i \xi_i^{-1}$. (Note that $\delta_{qp(l)} = id$). Let Γ be generated by $\{\xi_{q(C)} G_C \xi_{q(C)}^{-1} \mid C \in \mathcal{C}\}$ and $\{\delta_i\}$.

Then

1. *Γ is discrete and geometrically finite;*
2. *There is a Marden tube T for J and Γ so that $\tilde{T} \supset \xi_1 H_c \cup \xi_k H^c$. In particular, $\xi_k H^c \cup \xi_1 H_c$ is precisely J-invariant in Γ;*

3. *Let π_C be the cover of $\mathbb{H}^3/\Gamma$ associated to $\xi_{q(C)} G_C \xi_{q(C)}^{-1}$. Then $\pi_C \circ \bar{\xi}_{q(C)}$ extends to an embedding when restricted to the union of those Marden tubes in $\mathcal{T}(G_C)$ that are not for J or for any F_i with $p(A_i) \subset C$. Let $\mathcal{T}_C$ be the image these Marden tubes under this embedding. Then the union $\mathcal{T} = (\bigcup_C \mathcal{T}_C) \cup \{T\}$ is a maximal, disjoint collection of of Marden tubes for Γ. Hence there is a Marden core X_Γ with $\mathcal{T}(X_\Gamma) = \mathcal{T}$, and for each C there is an inclusion $\mathbf{j}_C : X_{G_C} \longrightarrow X_\Gamma$;*

4. *There is a pared uniformization $\phi : (M_\kappa, P_\kappa) \longrightarrow (X_\Gamma, Q_\Gamma)$, so that $\phi \circ i_C = \boldsymbol{j}_C \circ \phi_C$, where $i_C : C \longrightarrow M_\kappa$ and $\boldsymbol{j}_C : X_{G_C} \longrightarrow X_\Gamma$ are inclusions.*

3.2 Primitive shuffles and shuffling parabolics

For a pared manifold (M, P) containing a collection of solid tori $\mathcal{V}$, let $\widehat{M}(\mathcal{V})$ denote the manifold obtained by removing disjoint, open tubular neighbourhoods of a collection of representatives of isotopy classes of the core curves of the solid tori in $\mathcal{V}$. Let $\widehat{P}(\mathcal{V})$ denote P together with the new torus boundary components of $\widehat{M}(\mathcal{V})$. When there can be no confusion we will refer to $\widehat{M}(\mathcal{V})$ and $\widehat{P}(\mathcal{V})$ as $\widehat{M}$ and $\widehat{P}$, respectively. There is a natural choice of meridian-longitude system on the resulting set of torus boundary components $\widehat{P}(\mathcal{V}) - P$. This was discussed earlier, but we reiterate for clarity. For a boundary torus T choose the meridian to be a simple closed curve on T which bounds a disk in M. Choose the longitude to be homotopic in M to the core curve of the component of $\mathcal{V}$ defining T, so that it intersects the meridinal curve exactly once. From now on we will consider such a choice of meridian-longitude system to be implicit.

Suppose now that $\mathcal{V}$ is allowably contained in M. If V is a solid torus in $\mathcal{V}$, then there is at least one embedded annulus in $\partial M \cap \partial V$. Choose one such and attach an annulus to M by identifying the boundary curves of the respective annuli by appropriate homeomorphisms. Thicken the result. Repeating this operation of attaching an annulus and thickening the result, one annulus for each solid torus in $\mathcal{V}$, we obtain a manifold homeomorphic to $\widehat{M}$. Hence there is an embedding $\iota : M \longrightarrow \widehat{M}$ which induces a monomorphism on the level of fundamental groups.

Suppose that $s : M_0 \longrightarrow M_1$ is a primitive shuffle supported on a single solid torus V_0 allowably embedded in M_0. Then $s|_{V_0}$ is a homotopy equivalence between V_0 and an allowably embedded torus V_1 in M_1. There is a sewing kit $\kappa_0 = (\mathcal{C}_0, V_0, p_0)$: the collection $\mathcal{C}$ consists of pairs (C, P) where C is a component of $\overline{M_0 - V_0}$ and $P = \partial C \cap \partial V$; the sewing instruction p is defined so that M_κ is orientation preserving homeomorphic to M_0. There is another sewing kit $\kappa_1 = (\mathcal{C}_1, V_1, p_1)$. With this, M_{κ_1} is orientation preserving homeomorphic to M_1. The map p_1 is an embedding of a set of frontier annuli in V_1 into the union of the paring loci in $\mathcal{C}_1$. The collection $\mathcal{C}_0$ is orientation preserving homeomorphic to $\mathcal{C}_1$ via s, and since our concern is only up to orientation preserving homeomorphism we identify these sets. Under this identification p_1 becomes $s \circ p_0$. Hence the sewing kit $\kappa = (\mathcal{C}_0, V_0, s \circ p_0)$ also produces a pared manifold orientation preserving homeomorphic to M_1. We will refer to κ as the sewing kit associated to s.

In the next lemma we begin our construction. We begin with a collection of primitive shuffles $s_j : M_0 \longrightarrow M_j$ $j = 1, \ldots, n$, of an oriented manifold M_0 and construct a hyperbolic structure on $int(M_0)$ and for each j, a hyperbolic structure on $int(\widehat{M_j})$ and an isometric immersion $int(M_0) \longrightarrow int(\widehat{M_j})$ which agrees up to homotopy with $\iota_j \circ s_j$ on each component of $int(M_0 - \mathcal{V})$. This will become important in the proof of the main theorem.

Lemma 3.7 *Let $\{\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C})\}$ be a c-normalized family of pared uniformizations for a sewing kit $\kappa = (\mathcal{C}, V, p)$. Let $s_j : M_\kappa \longrightarrow M_j$, $j = 1, \ldots, n$, be primitive shuffle equivalences supported on V. Let $\kappa_j = (\mathcal{C}, V, p_j)$ be the sewing kit associated to s_j.*

Set $c_1 = (2c+1)k + 2c$, where $\mathcal{A}$ contains k annuli. Then there is a geometrically finite c_1-normalized pared uniformization $\phi : (M_\kappa, P_\kappa) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that the following holds for each j:

1. *There is a parabolic element g_j so that the group $\widehat{\Gamma}_j = <\Gamma, g_j>$ is discrete and geometrically finite;*
2. *There is a geometrically finite group Γ_j so that $\widehat{\Gamma}_j = <\Gamma_j, g_j>$, and a c_1-normalized uniformization $\phi_j : (M_j, P_j) \longrightarrow (X_{\Gamma_j}, Q_{\Gamma_j})$;*
3. *There is a uniformization $\widehat{\phi}_j : (\widehat{M_j}, \widehat{P_j}) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$, so that for every non-trivial $\gamma \in \chi(\phi_j)$ the composition $\bar{\gamma}^{-1} \circ \widehat{\phi}_j$ is c-normalized. Moreover,*
   - *(a) $\widehat{\phi}_j \circ \iota_j = \boldsymbol{j} \circ \phi_j$, where $\boldsymbol{j} : X_{\Gamma_j} \longrightarrow X_{\widehat{\Gamma}_j}$ and $\iota_j : M_j \longrightarrow \widehat{M}_j$ are inclusions;*
   - *(b) $(\widehat{\phi}_j)^{-1} \circ \pi \circ \phi|_C$ is homotopic in $\widehat{M}_j$ to $\iota_j \circ s_j|C$, for every component C of $\mathcal{C}$. (Here $\pi : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ is an immersion whose restriction to $int(X_\Gamma)$ is homotopic to the restriction of the covering map $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_j$, and ι_j is the inclusion of M_j into $\widehat{M}_j$.)*

Proof

If we consider p and p_j to be maps from $\{1, \ldots, k\}$ into $\mathcal{P}$, then the map $t_j = p_j^{-1} \circ p = p^{-1} \circ s_j \circ p$ is a permutation of $1, \ldots, k$. If $B \subset P \subset C$ is sewn to A_i under the sewing instructions p, then B is sewn to $A_{t_j(i)}$ under the sewing instructions p_j.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct primes with the property that $\mathfrak{p}_j > (2c+1)k + 2c$, and so that the ratios $\frac{\mathfrak{p}_j}{\mathfrak{p}_l}$ are all greater than 2 or less than $\frac{1}{2}$. Choose real numbers $d_1, \ldots, d_n$ with the property that

- $d_m \equiv 1 \mod \mathfrak{p}_m$; and
- $d_m \equiv 0 \mod \mathfrak{p}_{m'}, m \neq m'$.

Define a_m by

$$a_m = \sum_{l=1}^{n} d_l(m\mathfrak{p}_l + (2c+1)t_l(m))$$

The particular form of the a_m is not important for the proof. The properties that the a_m have that we need are:

1. $a_m - a_{m-1} \geq 2c+1$;
2. $a_m \equiv (2c+1)t_l(m) \mod \mathfrak{p}_l$.

Any set of numbers satisfying 1 and 2 would do for our purposes. Consider the n open sets $R_i = (a_i - c, a_i + c)$ in the real line, $i = 1, \dots, n$. With the natural ordering $R_1 < R_2 < \cdots < R_n$. There is an action of $\mathbb{Z} \cong < \mathfrak{p}_l >$ on $\mathbb{R}$ by addition. The orbit of the sets R_i under this action is an infinite collection of sets, but cyclically ordered as $R'_{t_l^{-1}(1)} < R'_{t_l^{-1}(2)} < \cdots < R'_{t_l^{-1}(n)}$, where $R'_i = ((2c+1)i - c, (2c+1)i + c)$.

Set $\xi_j(z) = z + a_j i$ and let $\phi : (M_\kappa, P_\kappa) \longrightarrow (X_\Gamma, Q_\Gamma)$ be the pared uniformization given by lemma 3.6 applied to the c-normalized family $\{\phi_C\}$ for the sewing kit κ.

We can consider Γ as being (redundantly) generated by $\{\xi_i \gamma_i^{-1} G_{p(i)} \gamma_i \xi_i^{-1}\}_{i=1}^{k}$ and the set $\{\xi_{qp(i)} \gamma_i \xi_i^{-1}\}$.

The idea of the rest of the proof is as follows. There is a spine S for J and the set $\{\xi_i \gamma_i^{-1} G_{p(i)} \gamma_i \xi_i^{-1}\}$ in X_Γ. The complement of $\tilde{S}$ in the universal cover $\tilde{X}_\Gamma$ of X_Γ consists of k components. For each of these components there is a unique subgroup H of Γ so that $\tilde{X}_H$ is contained in that component, where H is of the form $\xi_i \gamma_i^{-1} G_{p(i)} \gamma_i \xi_i^{-1}$, for some i. Call such a subgroup H a *vertebral* subgroup if $\gamma_i = id$ (equivalently $qp(i) = i$). Think of $\tilde{X}_\Gamma$ as being formed by sewing each $\tilde{X}_H$ to $\tilde{S}$ in its proper position. We then introduce a "shuffling parabolic" to Γ. Such a parabolic, in this case, is of the form $g(z) = z + ai$, for some real number $a > 0$. When a is less than the "length" of $\tilde{S}$, g produces a cascading effect on the $\tilde{X}_H$, rearranging the (cyclic) order in which they are sewn onto the g-translates of $\tilde{S}$. If there are n components to $\mathcal{C}$, then the first n distinct subgroups H in the new ordering give rise to the vertebral subgroups of the shuffled group. The remaining $k-n$ groups define the gluing maps; that is, the elements inducing the HNN-extensions used to construct the shuffled group. Now choose suitable markings for the components corresponding to vertebral subgroups and sew everything back together to obtain the shuffled group. The trick is in constructing everything so that the same group Γ can be shuffled to uniformize each manifold that is primitive shuffle equivalent to M_κ.

Fix j for the remainder of the proof. The parabolic g_j that will effect the shuffle of Γ to the group Γ_j uniformizing M_j is defined by $g_j(z) = z + \mathfrak{p}_j i$, where i here is not an index, but instead is the square root of -1.

Let $\widehat{\Gamma}_j$ be the group generated by Γ and g_j. We now find a group $\Gamma_j \subset \widehat{\Gamma}_j$ so that Γ_j uniformizes M_j and $\widehat{\Gamma} =< \Gamma_j, g_j >$.

Γ can be considered as being (redundantly) generated by

$$\{\xi_i \gamma_i^{-1} G_{p(i)} \gamma_i \xi_i^{-1}\}_{i=1}^k$$

and

$$\{\xi_{qp(i)} \gamma_i \xi_i^{-1}\}_{i=1}^k,$$

though when $qp(i) = i$ (or equivalently, when $\gamma_i = \gamma_{qp(i)} = id$), the element $\xi_{qp(i)} \gamma_i \xi_i^{-1}$ is trivial.

Let β_m be defined by $\beta_m(z) = z + (2c+1)mi$, where again i is the square root of -1.

In $\widehat{\Gamma}_j$ the subgroup $\xi_i \gamma_i^{-1} G_{p(i)} \gamma_i \xi_i^{-1}$ can be written as

$$g_j^m \beta_{t_j(i)} \gamma_i^{-1} G_{p(i)} \gamma_i \beta_{t_j(i)}^{-1} g_j^{-m}$$

since $a_i \equiv t_j(i)(2c+1) \mod \mathfrak{p}_j$, where

$$m = \frac{a_i - (2c+1)t_j(i)}{\mathfrak{p}_j}.$$

The integer m is clearly dependent upon i, but we suppress that for ease of notation.

and $\xi_{qp(i)} \gamma_i \xi_i^{-1}$ is $g_j^{m'} \beta_{t_j qp(i)} \gamma_i \beta_{t_j(i)}^{-1} g_j^{-m}$, where

$$m' = \frac{a_{qp(i)} - (2c+1)t_j(qp(i))}{\mathfrak{p}_j}$$

We conclude that $\widehat{\Gamma}_j$ is generated by g_j and

$$\{\beta_{t_j(i)} \gamma_i^{-1} G_{p(i)} \gamma_i \beta_{t_j(i)}^{-1}\}$$

and

$$\{\beta_{t_j qp(i)} \gamma_i \beta_{t_j(i)}^{-1}\}.$$

Let Γ_j be the group generated by $\{\beta_{t_j(i)} \gamma_i^{-1} G_{p(i)} \gamma_i \beta_{t_j(i)}^{-1}\}$ and $\{\beta_{t_j qp(i)} \gamma_i \beta_{t_j(i)}^{-1}\}$.

This generating set is redundant, and is not yet in the form required for lemma 3.6. The element $\xi_i \gamma_i^{-1} \xi_{qp(i)}^{-1}$ of Γ conjugates $J \subset \xi_{qp(i)} G_C \xi_{qp(i)}^{-1}$ to $\xi_i F_i \xi_i^{-1} \subset \xi_i \gamma_i^{-1} G_C \gamma_i \xi_i^{-1}$, where $p(i) \subset C$. In Γ_j this gives rise to $\beta_{t_j(i)} \gamma_i^{-1} \beta_{t_j qp(i)}^{-1}$, conjugating $J \subset \beta_{t_j qp(i)} G_C \beta_{t_j qp(i)}^{-1}$ to $\beta_{t_j(i)} F_i \beta_{t_j(i)}^{-1} \subset \beta_{t_j(i)} \gamma_i^{-1} G_C \gamma_i \beta_{t_j(i)}^{-1}$. If

$t_jq(C) = t_jqp(i)$ equaled $q_j(C)$ for every $C \in \mathcal{C}$ then we would be in a position to apply lemma 3.6. As it is, we may need to modify the set $\{\beta_{t_j(i)}\gamma_i^{-1}\beta_{t_jqp(i)}^{-1}\}$, producing a set $\{\beta_i\delta_i^{-1}\beta_{q_jp_j(i)}\}$ (for some elements δ_i), so that δ_i conjugates J to some infinite cyclic maximal parabolic subgroup of a vertebral subgroup. As an example we have figure 3.2.

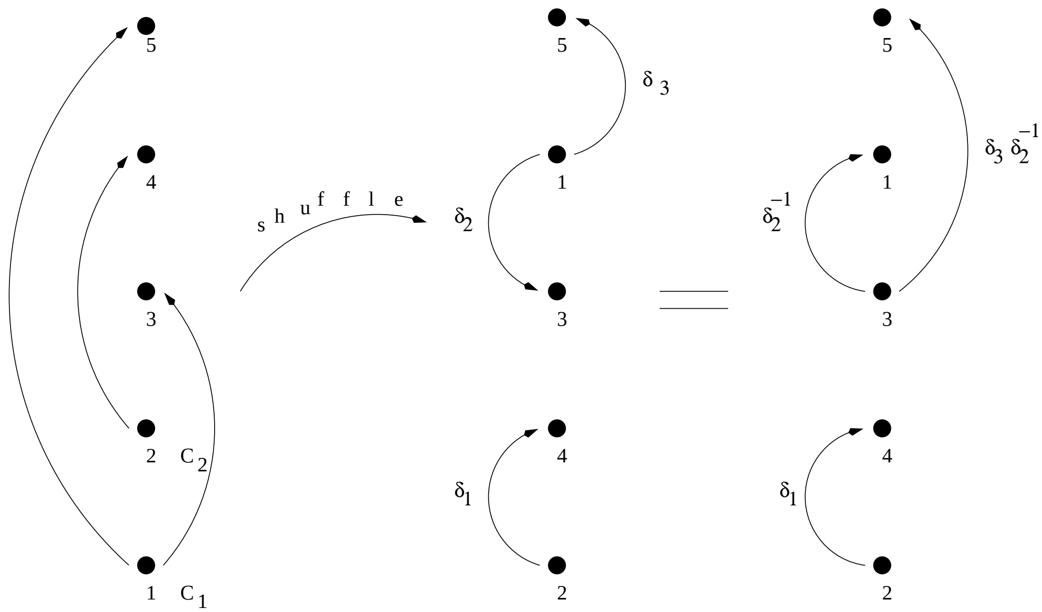


Figure 5: Shuffling

An example where there are two components to $\mathcal{C}$ and five annuli in $\mathcal{A}$, with t_j corresponding to the permutation (214). Here $q_j(C_2) = t_jq(C_2)$, but $q_j(C_1) \neq t_jq(C_1)$.

Set $n = t_j(i)$ to rewrite $\beta_{t_j(i)}\gamma_i^{-1}G_{p(i)}\gamma_i\beta_{t_j(i)}^{-1}$ as

$$\beta_n\gamma_{t_j^{-1}(n)}^{-1}G_{t_j^{-1}(n)}\gamma_{t_j^{-1}(n)}\beta_n^{-1}.$$

But $t_j = p_j^{-1} \circ p$, so that $t_j^{-1}(n) = p^{-1} \circ p_j(n)$. Hence the above group is

$$\beta_n\gamma_{p^{-1}p_j(n)}^{-1}G_{p_j(n)}\gamma_{p^{-1}p_j(n)}\beta_n^{-1}.$$

Similarly, $\beta_{t_jqp(i)}\gamma_i\beta_{t_j(i)}^{-1}$ can be written as

$$\beta_{t_jqp(t_j^{-1}(n))}\gamma_{t_j^{-1}(n)}\beta_{t_jt_j^{-1}(n)}^{-1} = \beta_{t_jqp_j(n)}\gamma_{p^{-1}p_j(n)}\beta_n^{-1}.$$

Hence Γ_j is generated by

$$\{\beta_n \gamma^{-1}_{p^{-1}p_j(n)} G_{p_j(n)} \gamma_{p^{-1}p_j(n)} \beta_n^{-1}\} \text{ and } \{\beta_{t_j q p_j(n)} \gamma_{p^{-1}p_j(n)} \beta_n^{-1}\}.$$

Suppose that $n = q_j(C)$ for some component C. Then the group element $\beta_{t_j q p_j(n)} \gamma_{p^{-1}p_j(n)} \beta_n^{-1}$ can be written as

$$\beta_{t_j q p_j q_j(C)} \gamma_{p^{-1}p_j q_j(C)} \beta^{-1}_{q_j(C)} = \beta_{t_j q(C)} \gamma_{p^{-1}p_j q_j(C)} \beta^{-1}_{q_j(C)}.$$

This element is hence the identity only when $pq(C) = p_j q_j(C)$. We multiply every element of the form $\beta_{t_j q p_j(n)} \gamma_{p^{-1}p_j(n)} \beta_n^{-1}$ by

$$\beta_{q_j p_j(n)} \gamma^{-1}_{p^{-1}p_j q_j p_j(n)} \beta^{-1}_{t_j q p_j(n)} = \beta_{q_j(C)} \gamma^{-1}_{p^{-1}p_j q_j(C)} \beta^{-1}_{t_j q(C)},$$

when $p_j(n) \subset C$. Take note of this element, since it will appear again in the final step of the argument. The element $\beta_{q_j(C)} \gamma^{-1}_{p^{-1}p_j q_j(C)} \beta^{-1}_{t_j q(C)}$ is the element conjugating

$$\beta_{t_j q(C)} F_{p^{-1}p_j q_j(C)} \beta^{-1}_{t_j q(C)} \subset \beta_{t_j q(C)} G_C \beta^{-1}_{t_j q(C)}$$

to J, which we view as a subgroup of

$$\beta_{q_j(C)} \gamma^{-1}_{p^{-1}p_j q_j(C)} G_C \gamma_{p^{-1}p_j q_j(C)} \beta^{-1}_{q_j(C)},$$

which is a vertebral subgroup.

Thus consider the element

$$\sigma_i = \beta_{q_j p_j(i)} \gamma^{-1}_{p^{-1}p_j q_j p_j(i)} \gamma_{p^{-1}p_j(i)} \beta_i^{-1}.$$

It is the product of $\beta_{q_j p_j(i)} \gamma^{-1}_{p^{-1}p_j q_j p_j(i)} \beta^{-1}_{t_j q p_j(i)}$ and $\beta_{t_j q p_j(i)} \gamma_{p^{-1}p_j q_j p_j(i)} \beta_i^{-1}$. For a component C of $\mathcal{C}$, $\sigma_{q_j(C)}$ is the identity. The group generated by $\{\sigma_i\}$ is the same as the group generated by $\{\beta_{t_j q p_j(n)} \gamma_{p^{-1}p_j(n)} \beta_n^{-1}\}$, since $\beta_{t_j q(C)} \gamma_i \beta^{-1}_{p_j^{-1}p(i)}$ is the identity when $i = q(C)$.

For an element γ of $PSL_2(\mathbb{C})$ conjugating a group G to a group G' recall that we denote by $\bar{\gamma}$ the homeomorphism between X_G and $X_{G'}$, whose induced map on fundamental groups is conjugation by γ. Let

$$\delta_i = \gamma^{-1}_{p^{-1}p_j q_j p_j(i)} \gamma_{p^{-1}p_j(i)},$$

and set

$$K_C = \gamma^{-1}_{p^{-1}p_j q_j(C)} G_C \gamma_{p^{-1}p_j q_j(C)}.$$

Then δ_i conjugates J to

$$O_i = \gamma^{-1}_{p^{-1}p_j q_j p_j(i)} F_{p^{-1}p_j(i)} \gamma_{p^{-1}p_j q_j p_j(i)} \subset K_C.$$

Noting that $t_j = p_j^{-1} \circ p$ we can rewrite the above elements and groups as:

$$\delta_i = \gamma^{-1}_{t_j^{-1} q_j p_j(i)} \gamma_{t_j^{-1}(i)}$$

$$O_i = \gamma^{-1}_{t_j^{-1} q_j p_j(i)} F_{t_j^{-1}(i)} \gamma_{t_j^{-1} q_j p_j(i)}$$

$$K_C = \gamma^{-1}_{t_j^{-1} q_j(C)} G_C \gamma_{t_j^{-1} q_j(C)}.$$

Define cusp neighbourhoods for O_i by $I_i^{\pm} = \gamma^{-1}_{t_j^{-1} q_j p_j(i)}(H^{\pm}_{t_j^{-1}(i)})$.

Then δ_i maps the outside of $I_1^+ \subset H_1^+$ onto the inside of I_i^-.

Set $\psi_C = \bar{\gamma}^{-1}_{t_j^{-1} q_j(C)} \circ \phi_C \circ (s_j|_C)^{-1}$. Then when $p_j(A_i) \subset C$ we have that $\psi_C(p_j(\partial_1 A_i)) = \bar{\gamma}^{-1}_{t_j^{-1} q_j(C)} \circ \phi_C \circ s_j^{-1} \circ p_j(\partial_1 A_i) = \bar{\gamma}^{-1}_{t_j^{-1} q_j(C)} \circ \phi_C \circ p(\partial_+ A_i)$. Hence $\psi_C \circ p_j(\partial_+ A_i)$ lifts to a $O_{t_j(i)}$-invariant curve in $\widehat{\mathbb{C}}$ which bounds a disk which is disjoint from X_{K_C} and containing $I^+_{t_j(i)}$. Hence we conclude that $\{\psi_C : (C,P) \longrightarrow (X_{K_C}, Q_{K_C})\}$ is a c-normalized family of pared uniformizations for κ_j.

Now apply lemma 3.6 to the normalized family of pared uniformizations $\{\psi_C : (C,P) \longrightarrow (X_{K_C}, Q_{K_C})\}$, with δ_i playing the role of γ_i, and β_i playing the role of ξ_i. We find that the group Γ_j generated by $\{\beta_{q_j(C)} K_C \beta^{-1}_{q_j(C)}\}$ and $\{\sigma_i\}$ is discrete and geometrically finite and that the set $\beta_k H^c \cup \beta_1 H_c$ is precisely J-invariant in Γ_j.

Moreover, there is a orientation preserving homeomorphism $\phi_j : (M_j, P_j) \longrightarrow (X_{\Gamma_j}, Q_{\Gamma_j})$ such that, for each $C \in \mathcal{C}$, $\phi_j \circ i_C = \boldsymbol{j}'_C \circ \psi_C$, where i_C denotes the inclusion of C into M_j, and $\boldsymbol{j}'_C : X_{K_C} \longrightarrow X_{\Gamma_j}$ is inclusion (note that the homomorphism $(\boldsymbol{j}'_C)_*$ is conjugation by $\beta_{q_j(C)}$). It follows that ϕ_j is $((2c+1)k + 2c)$-normalized.

The number $\mathfrak{p}_j > (2c+1)k+2c$, so that g_j maps the outside of $\beta_1 H_c$ into the inside of $\beta_k H^c$. Hence by an application of the second Klein-Maskit combination theorem we find that $\widehat{\Gamma}_j$ is discrete and geometrically finite. Moreover, there is an orientation preserving homeomorphism $\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ so that $\widehat{\phi}_j \circ \iota_j = \boldsymbol{j} \circ \phi_j$, where $\boldsymbol{j} : X_{\Gamma_j} \longrightarrow X_{\widehat{\Gamma}_j}$ is inclusion (really the extension of the restriction of a covering map - see our earlier remarks).

Let $\phi : (M_\kappa, P_\kappa) \longrightarrow (X_\Gamma, Q_\Gamma)$ be the pared uniformization given for by lemma 3.6.

Let M_Γ be a relative compact core for $(int(X_\Gamma), Q_\Gamma)$ and let $j : X_\Gamma \longrightarrow M_\Gamma$ be a homotopy inverse to the inclusion map. The covering map $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_j$ immerses M_Γ in $X_{\widehat{\Gamma}_j}$ and there is a compact core $M_{\widehat{\Gamma}_j}$ for $int(X_{\widehat{\Gamma}_j})$ containing the image of M_Γ. Let $\pi : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ be the composition of j, the covering map and the inclusion of $M_{\widehat{\Gamma}_j}$ into $X_{\widehat{\Gamma}_j}$. It remains to show that $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C$ is homotopic to $\iota_j \circ s_j|_C$.

We know that $\phi \circ i_C = \boldsymbol{j}_C \circ \phi_C$, where $\boldsymbol{j}_C$ is inclusion of X_{G_C} into X_Γ. (Note that $(\boldsymbol{j}_C)_*$ is conjugation by $\xi_{q(C)}$.) The group $\pi_* \circ (\boldsymbol{j}_C)_* \circ (\phi_C)_*(\pi_1(C))$ is the group

$$g_j^m \beta_{t_j q(C)} G_C \beta^{-1}_{t_j q(C)} g_j^{-m}.$$

If we insert the identity element $\gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)} \beta_{q_j(C)} \gamma^{-1}_{t_j^{-1} q_j(C)}$ into this expression we obtain:

$$g_j^m \beta_{t_j q(C)} (\gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)} \beta_{t_j(C)} \gamma^{-1}_{t_j^{-1} q_j(C)}) G_C (\gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)} \beta_{q_j(C)} \gamma^{-1}_{t_j^{-1} q_j(C)}) \beta_{t_j q(C)} g_j^{-m}.$$

Conjugation by $\beta_{q_j(C)} \gamma^{-1}_{t_j^{-1} q_j(C)}$ is $(\boldsymbol{j}'_C)_* \circ (\bar{\gamma}_{t_j^{-1} q_j(C)})^{-1}_*$. Hence $\pi_* \circ (\boldsymbol{j}_C)_* \circ (\phi_C)_*$ is

$$g_j^m \beta_{t_j q(C)} \gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)} ((\boldsymbol{j}'_C)_* \circ (\bar{\gamma}_{t_j^{-1} q_j(C)})^{-1}_*) \circ (\phi_C)_* \beta_{q_j(C)} \gamma^{-1}_{t_j^{-1} q_j(C)} \beta^{-1}_{t_j q(C)} g_j^{-m}.$$

As we noted previously, the element $\beta_{t_j q(C)} \gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)}$ is in $\widehat{\Gamma}_j$, and hence so is $g_j^m \beta_{t_j q(C)} \gamma_{t_j^{-1} q_j(C)} \beta^{-1}_{q_j(C)}$. Hence the map $\pi \circ \boldsymbol{j}_C \circ \phi_C$ is homotopic in $X_{\widehat{\Gamma}_j}$ to $\boldsymbol{j}'_C \circ \bar{\gamma}^{-1}_{t_j^{-1} q_j(C)} \circ \phi_C$. Since $\bar{\gamma}^{-1}_{t_j^{-1} q_j(C)} \circ \phi_C$ is $\psi_C \circ s_j|_C$ we find that

$$\pi \circ \boldsymbol{j}_C \circ \phi_C \simeq \boldsymbol{j}'_C \circ \psi_C \circ s_j|_C.$$

But $\boldsymbol{j}'_C \circ \psi_C$ is $\phi_j|_C$, which in turn is $\widehat{\phi}_j \circ \iota_j|_C$. Thus we conclude that

$$\pi \circ \boldsymbol{j}_C \circ \phi_C \simeq \widehat{\phi}_j \circ \iota_j \circ s_j|_C,$$

which implies that $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C$ is homotopic to $\iota_j \circ s_j|_C$, as required.

lemma 3.7

Definition 3.8 Suppose V is a solid torus and $\hat{V}$ the result of removing a regular neighbourhood of the core curve of V from V.

Let $f : V \longrightarrow \hat{V}$ be a π_1-injective immersion, so that $f_*(\pi_1(V))$ is a maximal cyclic subgroup of $\pi_1(\hat{V})$. Suppose further that there is a collection $A_1, \dots, A_k$ of parallel, essential annuli in ∂V so that $f(A_1), \dots, f(A_k)$ is a collection of distinct, embedded, parallel, essential annuli in $\partial \hat{V}$. We will define a k-tuple of integers which encodes how f "wraps" V around $\hat{V}$, called the *wrapping* of f.

The map f lifts to $\tilde{f} : V \longrightarrow \tilde{V}$, where $\tilde{V}$ is the cover of $\hat{V}$ corresponding to $f_*(\pi_1(V))$. Let x be a point in $int(V)$ and for each i let c_i be a path in V from x to A_i. The path $f(c_i)$ lifts to a path $\tilde{c}_i$ joining $\tilde{f}(x)$ to a component of $\bigcup_i \pi^{-1}(A_i)$, where $\pi : \tilde{V} \longrightarrow \hat{V}$ is the covering map. Let B_0 be the component containing the other endpoint of $\tilde{c}_1$. Let β be a generator for the deck transformations of $\tilde{V}$. Label the components of $\bigcup_i \pi^{-1}(A_i)$ as $\dots, B_{-n}, \dots, B_0, \dots, B_n, \dots$, where

$B_n = \beta(B_{n-k})$, and the components $B_1, \ldots, B_{k-1}$ appear, in order, between B_0 and B_k.

If $\tilde{c}_i$ connects $\tilde{f}(x)$ to B_{n_i}, then the wrapping of f is

$$w(f) = (0, n_2, \ldots, n_k).$$

Note that $w(f)$ is only well-defined up to choice of cyclic ordering of $A_1, \ldots, A_k$ and deck transformation β. Hence we consider that $(0, -n_2, -n_3, \ldots, -n_k)$ to be equivalent to $(0, n_2, \ldots, n_k)$. Moreover, if $\tau \in S_k$ is the long cycle $(12 \cdots k)$ then we consider $(0, n_2, \ldots, n_k)$ to be equivalent to $(0, m_2, \ldots, m_k)$ provided that for some $t \neq 0$

$$n_{\tau^t(i)} - n_{\tau^t(1)} = m_i,$$

$i = 1, \ldots, k$. (We have adopted the convention that n_1 and m_1 are both 0.) This last equivalence reflects a change in cyclic ordering of the annuli $A_1, \ldots, A_k$. Hence, more formally, we define a wrapping to be an equivalance class of such k-tuples. Since f is an embedding restricted to the union of the annuli $A_1, \ldots, A_k$, the k-tuple $(0, n_2 \mod k, n_3 \mod k, \ldots, n_k \mod k)$ has no repeated entries.

It is clear that if f_1 and f_2 are maps of V into $\hat{V}$ as above, and that f_1 and f_2 are homotopic $\mathrm{rel}(A_1 \cup \cdots \cup A_k)$ then $w(f_1) = w(f_2)$.

For the converse, suppose that f_1 and f_2 are two maps of V into $\hat{V}$ with the same wrapping. There are lifts $\tilde{f}_i : V \longrightarrow \tilde{V}$ which are homotopy equivalences taking $A_1 \cup \ldots \cup A_k$ into $\partial \tilde{V}$. Since the wrapping is the same, there is a (possibly orientation reversing) homeomorphism $\tilde{g} : \tilde{V} \longrightarrow \tilde{V}$ so that $\tilde{g}(\tilde{f}_1(A_i)) = \tilde{f}_2(A_i)$, for each i. The composition

$$(\tilde{f}_2)^{-1} \circ \tilde{g} \circ \tilde{f}_1 : V \longrightarrow V$$

is a homotopy equivalence which is a homeomorphism on each A_i and hence homotopic $\mathrm{rel}(A_1, \ldots, A_k)$ to a self-homeomorphism $h : V \longrightarrow V$.

The homeomorphism $\tilde{g}$ descends to a homeomorphism $g : \hat{V} \longrightarrow \hat{V}$, allowing us to conclude that $g \circ f_2$ is homotopic rel $(A_1, A_2, \ldots, A_k)$ to $f_1 \circ h$.

We have the following corollary to the proof of 3.7.

Corollary 3.9 *With the notation from lemma 3.7, for each $j = 1, \ldots, n$ let f_j be the restriction of $(\hat{\phi}_j)^{-1} \circ \pi_j \circ \phi$ to V. Then f_j is an immersion of V into $\hat{V}$ as above and*

$$w(f_j) \textit{ is not equivalent to } w(f_i)$$

for $j \neq i$.

Proof

With the fixed ordering of $A_1, \ldots, A_k$ the wrapping number $w(f_j) = (0, n_2, \ldots, n_k)$ is seen to be defined by

$$n_i = \frac{a_i - a_1 - ((a_i - a_1) \mod \mathfrak{p}_j)}{\mathfrak{p}_j}$$

Suppose there is some l so that for each i,

$$\frac{a_i - a_1 - ((a_i - a_1) \mod \mathfrak{p}_j)}{\mathfrak{p}_j} = \frac{a_i - a_1 - ((a_i - a_1) \mod \mathfrak{p}_l)}{\mathfrak{p}_l}.$$

Then the fraction $\frac{\mathfrak{p}_j}{\mathfrak{p}_l}$ is the same as

$$\frac{a_i - a_1 - (2c+1)(t_l(i) - t_l(1))}{a_i - a_1 - (2c+1)(t_j(i) - t_j(1))}.$$

Using the expressions for a_i and a_1 we see that the above fraction is always greater than $\frac{1}{2}$ or less than 2, since $k \geq 2$, $c \geq 1$, and $\mathfrak{p}_i \geq (2c+1)k + 2c$. The details are left to the reader.

Since the primes $\mathfrak{p}_j$ and $\mathfrak{p}_l$ were chosen so that their quotients were always greater than 2 or less than $\frac{1}{2}$ we obtain a contradiction.

Hence for $w(f_j)$ to be equivalent to $w(f_i)$ there must be a shift in cyclic ordering of $A_1, \ldots, A_k$) and/or a change of β to β^{-1} (as discussed above) which transforms the representative for $w(f_i)$ to $(0, n_2, \ldots, n_k)$. Since the representative for $w(f_i)$, given by considering the above ordering of $A_1, \ldots, A_k$ and choice of β over β^{-1}, is a k-tuple of non-negative integers with increasing value, changing the ordering and choice of generator for the deck group will not transform the reprentative for $w(f_i)$ to the representative for $w(f_j)$.

corollary 3.9

The next proposition is the inductive step in our construction. It essentially produces the representation that is to be our bumping representation, and a set of geometrically finite hyperbolic 3-manifolds that are the geometric limits of sequences of minimally parabolic representations converging to the bumping representation. Each such sequence will lie in a component of the interior of $AH(\pi_1(M))$ indexed by an element of $\mathcal{A}(M)$ which is primitive shuffle equivalent to $[(M_0, h_0)]$.

Remark: Suppose $s_i : M_0 \longrightarrow M_i$, $i = 1, \ldots, m$, is a collection of primitive shuffles. Suppose s_i is supported on $\mathcal{V}_i$. If there is an i and a j so that $\mathcal{V}_i$ and $\mathcal{V}_j$ intersect, then we may admissibly isotope s_j, say, so that any element of $\mathcal{V}_j$ intersecting an element of $\mathcal{V}_i$ is contained in that element, or visa versa. We will always assume that when considering such collections of primitive shuffles we have made this modification. In particular this means that the union of the $\mathcal{V}_i$ makes sense as the support of a primitive shuffle.

Proposition 3.10 *Suppose that M_0 is compact, oriented, atoroidal, irreducible and has incompressible boundary and let $s_i : M_0 \longrightarrow M_i$, $i = 1, \ldots, n$, be a* *primitive shuffle. Suppose s_i is supported on a collection of solid tori $\mathcal{V}_i$ and set $\mathcal{V} = \bigcup_{i=1}^{n} \mathcal{V}_i$. Let $P_i \subset \partial M_i$ be a collection of annuli, one chosen from the components of $\partial V \cap \partial M_i$ for each $V \in \mathcal{V}$.*

Then there is a pared uniformization $\phi : (M_0, P_0) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that the following holds for each $j = 1, \ldots, n$:

1. *there are parabolic elements $g_1, \ldots, g_{|\mathcal{V}|}$, so that the group $\widehat{\Gamma}_j =< \Gamma, g_1, \ldots, g_{|\mathcal{V}|} >$ is discrete and geometrically finite;*
2. *there is a pared uniformization $\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ so that $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_{\overline{M_0 - \mathcal{V}}}$ is homotopic to $\iota_j \circ s_j|_{\overline{M_0 - \mathcal{V}}}$. Here $\pi : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ and is an immersion whose restriction to $int(X_\Gamma)$ is homotopic to the restriction of the covering map $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_j$ and $\iota_j : M_j \longrightarrow \widehat{M}_j$ is the inclusion defined previously.*

Proof

We prove this inductively, inducting on the number of solid tori in $\mathcal{V}$. In the base case of the induction $\mathcal{V}$ consists of one solid torus V and all the primitive shuffles s_i, $i = 1, \ldots, n$, are supported on V. There is a sewing kit $\kappa = (\mathcal{C}, V, p)$ defined so that $M_\kappa = M_0$: the collection $\mathcal{C}$ is the set of components of $\overline{M_0 - V}$, with paring locus equal to $FrV \cap FrC$; p is defined so that $M_\kappa = M_0$. Let $\kappa_i = (\mathcal{C}, V, p_i)$ be the sewing kit determined by s_i. For each $C \in \mathcal{C}$ choose a pared uniformization $\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C})$. We may assume, by conjugating G_C if necessary, that $(\phi_C)_* \circ p_*(\pi_1(A_{q(C)})$ is J, the cyclic parabolic group generated by $z \mapsto z + 1$. Choose a set $\chi(G_C)$ of elements in $PSL_2(\mathbb{C})$ conjugating J to the maximal cyclic parabolic subgroups of G_C, and choose a constant c_0 so that the set $\{\phi_C\}$ is a c_0-normalized family of pared uniformizations for κ.

Applying lemma 3.7 we obtain a constant c_1, depending only on c_0 and M_0, and a c_1-normalized uniformization $\phi : (M_0, P_0) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that for each j:

1. There is a parabolic g_j with $\widehat{\Gamma}_j =< \Gamma, g_j >$ discrete and geometrically finite;
2. There is a geometrically finite group Γ_j so that $\widehat{\Gamma}_j =< \Gamma_j, g_j >$, a c_1-normalized uniformization $\phi_j : (M_j, P_j) \longrightarrow (X_{\Gamma_j}, Q_{\Gamma_j})$;
3. There is a pared uniformization $\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ so that for each non-trivial $\gamma \in \chi(\phi_j)$ the composition $\bar{\gamma}^{-1} \circ \widehat{\phi}_j$ is a c_0-normalized pared uniformization. Moreover,
   (a) $\widehat{\phi}_j \circ \iota_j = \boldsymbol{j}_j \circ \phi_j$, where $\iota_j : M_j \longrightarrow \widehat{M}_j$ and $\boldsymbol{j}_j : X_{\Gamma_j} \longrightarrow X_{\widehat{\Gamma}_j}$ are inclusions;

(b) $\phi_j|_C = \boldsymbol{j}_{C,j} \circ \phi_C \circ (s_j|_C)^{-1}$, for every $C \in \mathcal{C}$, where $\boldsymbol{j}_{C,j} : X_{G_C} \longrightarrow X_{\Gamma_j}$ is inclusion;

(c) $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C \simeq \iota_j \circ s_j|_C$, for all $C \in \mathcal{C}$, where $\pi : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ is an immersion as describe earlier.

Note that parts 3a, 3b and 3c together imply that

(d) $\pi \circ \phi|_C \simeq \boldsymbol{j}_j \circ \phi_j \circ s_j|_C$.

Our inductive assumption is that for collections $\{d_j : N \longrightarrow N_j\}$ of primitive shuffles of a compact, irreducible, atoroidal, oriented 3-manifold N with boundary, supported on at most l solid tori $\mathcal{V}_l$ the following holds:

There is a constant c_1 and a c_1-normalized uniformization $\phi : (N, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that for each j

1. There are parabolic elements $g_1^j, \dots, g_l^j$ with $\widehat{\Gamma}_j =< \Gamma, g_1^j, \dots, g_l^j >$ is discrete and geometrically finite;

2. There is a geometrically finite group Γ_j so that $\widehat{\Gamma}_j =< \Gamma_j, g_1^j, \dots, g_l^j >$, and a c_1-normalized uniformization $\phi_j : (N_j, P_j) \longrightarrow (X_{\Gamma_j}, Q_{\Gamma_j})$;

3. There is a pared uniformization $\widehat{\phi}_j : (\widehat{N}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ so that for each non-trivial $\gamma \in \chi(\phi_j)$ the composition $\bar{\gamma}^{-1} \circ \widehat{\phi}_j$ is a c_1-normalized family of pared uniformizations. Moreover,

   (a) $\widehat{\phi}_j \circ \iota_j = \boldsymbol{j}_j \circ \phi_j$, where $\iota_j : N_j \longrightarrow \widehat{N}_j$ and $\boldsymbol{j}_j : X_{\Gamma_j} \longrightarrow X_{\widehat{\Gamma}_j}$ are inclusions;

   (b) $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C \simeq \iota_j \circ d_j|_C$, for all $C \in \mathcal{C}$, where $\pi : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ is the immersion describe earlier;

   (c) $\pi \circ \phi|_C \simeq \boldsymbol{j}_j \circ \phi_j \circ s_j$.

Suppose now that the primitive shuffles s_j are supported on $l+1$ solid tori $\mathcal{V}$. Choose V_1 from $\mathcal{V}$ arbitrarily. Inductively, choose V_i from $\mathcal{V} - V_1 \cup \cdots V_{i-1}$ by requiring that there is a component C of $\overline{M_0 - \mathcal{V}}$ and an integer $j < i$ so that $FrV_i \cap C \neq \emptyset$ and $FrV_j \cap C \neq \emptyset$. Let $\mathcal{V}_l$ denote $V_1 \cup \cdots \cup V_l$. By construction there is a unique component of $\overline{M_0 - V_{l+1}}$ containing $\mathcal{V}_l$. Call this component Y_0. Let R_0 be the frontier of Y_0 together with $P_0 \cap Y_0$. By lemma 2.24 we obtain a pared manifold (Y_0, R_0). Let $\widehat{Y_0}$ denote $\widehat{Y}_0(\mathcal{V}_l)$.

We claim that $s_j : Y_0 \longrightarrow Y_j = s_j(Y_0)$ is a primitive shuffle supported on $\mathcal{V}_l$. To see this we need to show that $\mathcal{V}_l$ is allowably contained in Y_0. Let Σ denote a characteristic submanifold for (Y_0, R_0). The disjoint union $V_1 \sqcup V_2 \sqcup \ldots \sqcup V_l$ is an admissibly embedded Seifert fibred space in $(Y_0, \partial Y_0)$. By the Engulfing property $V_1 \sqcup V_2 \sqcup \ldots \sqcup V_l$ is admissibly isotopic into Σ. Reversing this isotopy produces a characteristic submanifold containing $V_1 \sqcup V_2 \sqcup \ldots \sqcup V_l$.

Hence the components of $\mathcal{V}_l$ are contained in Σ a characteristic submanifold of (Y_0, R_0). Suppose that $V \subset \mathcal{V}_l$ is not a component of Σ. It follows that V is contained in an I-bundle component of Σ. There is an I-bundle component of the characteristic submanifold of M_0 containing the I-bundle in Σ which contains V. Since V is allowably contained in this I-bundle, it is allowably contained in the intersection of the I-bundle with Y_0. It follows that $\mathcal{V}_l$ is allowably contained in Y_0.

Hence, by our inductive assumption, there is a constant c_1 and c_1-normalized pared uniformization $\theta_0 : (Y_0, R_0) \longrightarrow (X_{\Gamma_{Y_0}}, Q_{\Gamma_{Y_0}})$ with the following holding for each j:

1. There are parabolic elements $g_1^j, \ldots, g_l^j$ so that $\widehat{\Gamma}_{Y_j} =< \Gamma_{Y_0}, g_1^j, \ldots, g_l^j >$ is discrete and geometrically finite;

2. There is a geometrically finite group Γ_{Y_j}, and a c_1-normalized uniformization $\theta_j : (Y_j, R_j) \longrightarrow (X_{\Gamma_{Y_j}}, Q_{\Gamma_{Y_j}})$;

3. There is a pared uniformization $\widehat{\theta}_j : (\widehat{Y}_j, \widehat{R}_j) \longrightarrow (X_{\widehat{\Gamma}_{Y_j}}, Q_{\widehat{\Gamma}_{Y_j}})$ so that for any non-trivial $\gamma \in \chi(\theta_j)$ the composition $\bar{\gamma}^{-1} \circ \widehat{\theta}_j$ is a c_1-normalized uniformization. Moreover,

$$\widehat{\theta}_j \circ \iota_{Y_j} = \boldsymbol{j}_{Y_j} \circ \theta_j, \tag{1}$$

where $\iota_{Y_j} : Y_j \longrightarrow \widehat{Y}_j$ and $\boldsymbol{j}_{Y_j} : X_{\Gamma_{Y_j}} \longrightarrow X_{\widehat{\Gamma}_{Y_j}}$ are inclusions;

$$\widehat{\theta}_j^{-1} \circ \pi_{Y_0,j} \circ \theta_0|_C \simeq \iota_{Y_j} \circ s_j|_C, \tag{2}$$

for all components C of $\overline{Y_0 - \mathcal{V}_l}$, where $\pi_{Y_0,j} : X_{\Gamma_{Y_0}} \longrightarrow X_{\widehat{\Gamma}_{Y_j}}$ is an immersion whose restriction to $int(X_{\Gamma_{Y_0}})$ is homotopic to the restriction of the covering map $\mathbb{H}^3/\Gamma_{Y_0} \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_{Y_j}$;

$$\pi_{Y_0,j} \circ \theta_0|_C \simeq \boldsymbol{j}_{Y_j} \circ \theta_j \circ s_j|_C. \tag{3}$$

Let $\mathcal{C}$ consist of all the components of $\overline{M_0 - V_{l+1}}$. For $C \in \mathcal{C}$ let $P = P(C)$ be the frontier of C together with $C \cap P_0$. By lemma 2.24 (C, P) is a pared 3-manifold. Define a sewing kit $\kappa = (\mathcal{C}, V_{l+1}, p)$ by requiring that p be defined in such as way as to ensure that $M_\kappa = M_0$. For each $C \neq Y_0$ choose a pared uniformization $\psi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C})$, chosen so that

$(\psi_C)_* \circ p_*(\pi_1(A_{q(C)}))$ is the maximal parabolic subgroup J. Choose sets $\chi(\psi_C)$ so that for each maximal cyclic parabolic subgroup of G_C there is a unique element in $\chi(\psi_C)$ conjugating J to it. Let $c_2 \geq c_1$ be chosen so that each ψ_C is c_2-normalized.

Let $\gamma_0 \in \chi(\theta_0)$ conjugate J to $(\theta_0)_* \circ p_*(\pi_1(A_{q(Y_0)}))$. Set ψ_{Y_0} to be $\bar{\gamma}_0^{-1} \circ \theta_0$ and set $G_{Y_0} = \gamma_0^{-1}\Gamma_{Y_0}\gamma_0$. We define $\chi(\psi_{Y_0})$ as $\chi(\psi_{Y_0}) = \{\gamma_0^{-1} \circ \gamma \mid \gamma \in \chi(\theta_0)\}$. Let γ_i be the element of $\chi(\psi_{Y_0})$ which conjugates J to $(\psi_{Y_0})_* \circ p_*(\pi_1(A_i))$. As it stands, we may need to alter the set $\chi(\psi_{Y_0})$ still further; there is the possibility that an element $\gamma_i \in \chi(\psi_{Y_0})$ maps H^{c_1} into H_i^-, instead of H_i^+. If this is the case, make the alteration $\gamma_i \mapsto \gamma_i \circ r$, where r is a reflection which maps H^{c_1} to H_{c_1} and visa versa. When all this is done we find that ψ_{Y_0} is a c_1-normalized pared uniformization, and hence a c_2-normalized pared uniformization.

The set $\{\psi_C\}$, as C ranges through $\mathcal{C}$, is a c_2-normalized family of pared uniformizations for κ. Define, for each $j = 1, \ldots, n$, a sewing kit $\kappa_j = (\mathcal{C}, V_{l+1}, p_j)$, where $p_j = s_j \circ p$. Hence the manifold determined by κ_j is obtained by attaching the components of $\overline{M_0 - V_{l+1}}$ to V_{l+1} in the order in which they appear in M_j; it is an intermediate stage in the full shuffling process. We will define a primitive shuffle d_j between M_κ and M_{κ_j} supported on V_{l+1}. We will then be in a position to invoke lemma 3.7. Since we wish, at this stage, to only shuffle the position of the components C of $\mathcal{C}$ with respect to V_{l+1}, we want to leave the marking on Y_0 unchanged, whilst changing the marking on the other elements of $\mathcal{C}$. Hence define $d_j : M_\kappa \longrightarrow M_{\kappa_j}$ by setting $d_j = s_j$ off of Y_0, and homotopic to the identity on Y_0. Then d_j is homotopic to a primitive shuffle supported on V_{l+1}. Homotope d_j to a primitive shuffle, for each j, and again call the resulting map d_j.

Lemma 3.7 supplies us with the following:

1. Prime numbers $\mathfrak{p}_j$, $j = 1, \ldots, n$, and constants a_i, $i = 1, \ldots, k$, $k = |\mathcal{A}|$, so that

   (a) $a_i \equiv (2c_2 + 1)t_j(i) \mod \mathfrak{p}_j$, $j = 1, \ldots, n$;

   (b) $a_{i+1} - a_i \geq 2c + 1$;

   (c) $\mathfrak{p}_j \geq (2c_2 + 1)k + 2c_2$.

2. With $\xi_m(z) = z + a_m i$, $m = 1, \ldots, k$, there is a group Γ generated by

$$\{\xi_{q(C)} G_C \xi_{q(C)}^{-1}\}_{C \in \mathcal{C}} \quad \text{and} \quad \{\xi_{qp(i)} \gamma_i \xi_i^{-1}\}_{i=1}^k$$

   so that there is a $(a_k - a_1 + 2c_1)$-normalized pared uniformization $\phi : (M_0, P_0) \longrightarrow (X_\Gamma, Q_\Gamma)$, with $\phi|_C = \boldsymbol{j}_C \circ \psi_C$, where $\boldsymbol{j}_C : X_{G_C} \longrightarrow X_\Gamma$ is inclusion;

3. With $g_j(z) = z + \mathfrak{p}_j i$, $j = 1, \ldots, n$, the group $\widehat{\Gamma}_{\kappa_j} =< \Gamma, g_j >$ is discrete and geometrically finite. Setting $\beta_m(z) = z + (2c_1+1)mi$, $m = 1, \ldots, k$, and recalling the permutation $t_j = p_j^{-1} \circ p$, we have the subgroup Γ_{κ_j} of $\widehat{\Gamma}_{\kappa_j}$ generated by

$$\{\beta_{q_j(C)}\gamma^{-1}_{t_j^{-1}q_j(C)} G_C \gamma_{t_j^{-1}q_j(C)}\beta^{-1}_{q_j(C)}\}_{C\in\mathcal{C}} \quad \text{and} \quad \{\beta_{q_jp_j(i)}\gamma^{-1}_{t_jq_jp_j(i)}\gamma_{t_j^{-1}(i)}\beta_i^{-1}\}_{i=1}^k$$

is discrete and geometrically finite, and $\widehat{\Gamma}_{\kappa_j} =< \Gamma_{\kappa_j}, g_j >$;

4. There is a $((2c_1+1)k+2c_1)$-normalized pared uniformization $\phi_{\kappa_j} : (M_{\kappa_j}, P_{\kappa_j}) \longrightarrow (X_{\Gamma_{\kappa_j}}, Q_{\Gamma_{\kappa_j}})$, so that

$$\phi_{\kappa_j}|_C = \boldsymbol{j}_{C,\kappa_j} \circ \bar{\gamma}^{-1}_{t_j^{-1}q_j(C)} \circ \psi_C \circ (d_j|_C)^{-1}, \tag{4}$$

where $\boldsymbol{j}_{C,\kappa_j} : X_{G_C} \longrightarrow X_{\Gamma_{\kappa_j}}$ is an embedding;

5. There is a pared uniformization $\widehat{\phi}_{\kappa_j} : (\widehat{M}_{\kappa_j}, \widehat{P}_{\kappa_j}) \longrightarrow (X_{\widehat{\Gamma}_{\kappa_j}}, Q_{\widehat{\Gamma}_{\kappa_j}})$ so that for each $\gamma \in \chi(\phi_{\kappa_j})$ the composition $\bar{\gamma}^{-1} \circ \widehat{\phi}_{\kappa_j}$ is a c_2-normalized uniformization. Moreover,

$$\widehat{\phi}_{\kappa_j} \circ \iota_{\kappa_j} = \boldsymbol{j}_{\kappa_j} \circ \phi_{\kappa_j}, \tag{5}$$

where $\iota_{\kappa_j} : M_{\kappa_j} \longrightarrow \widehat{M}_{\kappa_j}$ and $\boldsymbol{j}_{\kappa_j} : X_{\Gamma_{\kappa_j}} \longrightarrow X_{\widehat{\Gamma}_{\kappa_j}}$ are inclusions;

$$\widehat{\phi}^{-1}_{\kappa_j} \circ \pi_{\kappa_j} \circ \phi|_C \simeq \iota_{\kappa_j} \circ d_j|_C, \tag{6}$$

for all $C \in \mathcal{C}$, where $\pi_{\kappa_j} : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_{\kappa_j}}$ is an immersion whose restriction to $int(X_\Gamma)$ is homotopic to the restriction of the covering map $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_{\kappa_j}$.

The pared uniformization ϕ is the uniformization we seek. To justify this we need to show that, for any given j, we can find a pared uniformization $\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ that satisfies the inductive hypothesese and hence the statement of the proposition.

From now on we consider a fixed j.

Let $\widehat{\mathcal{C}}$ be the set $(\mathcal{C} - \{Y_0\}) \cup \{\widehat{Y}_j\}$. That is, $\widehat{\mathcal{C}}$ consists of the components of $\overline{\widehat{M}_j(\mathcal{V}_l) - V_{l+1}}$. The paring locus $\widehat{R}_j$ of $\widehat{Y}_j$ is the union of the paring locus R_j of Y_j and the torus components of $\partial\widehat{Y}_j$. Define a sewing kit $\widehat{\kappa} = (\widehat{\mathcal{C}}, V_{l+1}, p_j)$. Then $M_{\widehat{\kappa}}$ is $\widehat{M}_j(\mathcal{V}_l)$.

We want a c_2-normalized family of pared uniformizations for $\widehat{\kappa}_j$, so that we may apply lemma 3.6 to obtain a pared uniformization for $\widehat{\kappa}$.

Recall the pared uniformization $\widehat{\theta}_j : (\widehat{Y}_j, \widehat{R}_j) \longrightarrow (X_{\widehat{\Gamma}_{Y_j}}, Q_{\widehat{\Gamma}_{Y_j}})$ given by the inductive hypothesis. We know that for any non-trivial $\gamma \in \chi(\theta_j)$ the composition $\bar{\gamma}^{-1} \circ \widehat{\theta}_j$ is c_1-normalized. Since $\pi_{Y_0,j} \circ \theta_0|_C \simeq \boldsymbol{j}_{Y_j} \circ \theta_j \circ s_j|_C$, for every component C of $\overline{Y_0 - \mathcal{V}_l}$, it follows that for each $\gamma_1 \in \chi(\theta_0)$ there is a $\gamma_2 \in \chi(\theta_j)$ and an element $\sigma \in \widehat{\Gamma}_{Y_j}$ so that $\gamma_1 = \sigma\gamma_2$. This implies that $\bar{\gamma}_1^{-1} \circ \widehat{\theta}_j$ (with $\chi(\bar{\gamma}_1^{-1} \circ \widehat{\theta}_j) = \{\gamma_1^{-1} \circ \gamma \mid \gamma \in \chi(\widehat{\theta})\}$) is c_1-normalized, since the image group is $\gamma_2^{-1}\widehat{\Gamma}_{Y_j}\gamma_2$, which is c_1-normalized by assumption.

Thus, in particular, recalling the element $\gamma_0 \in \chi(\theta_0)$ which conjugates J to $(\theta_0)_* \circ p_*(\pi_1(A_{q(Y_0)}))$, the uniformization $\bar{\gamma}_0^{-1} \circ \widehat{\theta}_j$ is c_1-normalized. Considering the element $\gamma_{t_j^{-1}q_j(Y_0)} \in \chi(\psi_{Y_0})$ it follows that $\bar{\gamma}^{-1}_{t_j^{-1}q_j(Y_0)}\bar{\gamma}_0^{-1}\widehat{\theta}_j$ is c_1-normalized, and hence c_2-normalized.

If $p_j(A_i) \subset Y_j$ for some i, then let $\eta_i \in \chi(\bar{\gamma}^{-1}_{t_j^{-1}q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \widehat{\theta}_j)$ conjugate J to $(\bar{\gamma}_{t_j^{-1}q_j(Y_0)})_* \circ (\bar{\gamma}_0^{-1})_* \circ (\widehat{\theta}_j)_* \circ (p_j)_*(\pi_1(A_i))$. We may need to alter η_i so that it maps H^{c_2} into the appropriate cusp neighbourhood. Since s_j is orientation preserving off of $\mathcal{V}$, the modification made here agrees with the modification made when $\chi(\psi_{Y_C})$ was being altered.

Suppose $p_j(A_i) \subset C \neq Y_0$ for some $i = 1, \dots, k$. Let $\eta_i \in \chi(\bar{\gamma}^{-1}_{t_j^{-1}q_j(C)} \circ \psi_C \circ s_j^{-1})$ conjugate J to $(\bar{\gamma}^{-1}_{t_j^{-1}q_j(C)})_* \circ (\psi_C)_* \circ (p_j)_*(\pi_1(A_i))$. Consider $\bar{\gamma}^{-1}_{t_j^{-1}q_j(C)} \circ \psi_C \circ s_j^{-1}$; it is a c_2-normalized uniformization.

The set $\{\bar{\gamma}^{-1}_{t_j^{-1}q_j(C)} \circ \psi_C \circ (s_j|_C)^{-1}\}_{C \neq \widehat{Y}_j} \cup \{\bar{\gamma}^{-1}_{t_j^{-1}q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \widehat{\theta}_j\}$ is a c_2-normalized family of pared uniformizations for $\widehat{\kappa}$. We apply lemma 3.6 with $(2c_2+1)m$ playing the role of a_m, to obtain a pared uniformization $\phi_{\widehat{\kappa}} : (M_{\widehat{\kappa}}, P_{\widehat{\kappa}}) \longrightarrow (X_{\Gamma_{\widehat{\kappa}}}, Q_{\Gamma_{\widehat{\kappa}}})$. This uniformization has the property that for $C \in \widehat{\mathcal{C}}$ either

$$C = \widehat{Y}_j \text{ and } \phi_{\widehat{\kappa}}|_{\widehat{Y}_j} = \boldsymbol{j}_{\widehat{\kappa},\widehat{Y}_j} \circ \bar{\gamma}^{-1}_{t_j^{-1}q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \widehat{\theta}_j, \tag{7}$$

where $\boldsymbol{j}_{\widehat{\kappa},\widehat{Y}_j}$ is inclusion; or

$$C \neq \widehat{Y}_j \text{ and } \phi_{\widehat{\kappa}}|_C = \boldsymbol{j}_{\widehat{\kappa},C} \circ \bar{\gamma}^{-1}_{t_j^{-1}q_j(C)} \circ \psi_C \circ (s_j|_C)^{-1}, \tag{8}$$

where $\boldsymbol{j}_{\widehat{\kappa},C}$ is inclusion.

The group $\Gamma_{\widehat{\kappa}}$ is generated by

$$\{\beta_{q_j(C)}\gamma^{-1}_{t_j^{-1}q_j(C)}G_C\gamma_{t_j^{-1}q_j(C)}\beta^{-1}_{q_j(C)}\}_{C \in \widehat{\mathcal{C}} - \{\widehat{Y}_j\}} \cup \{\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_0)}\gamma_0^{-1}\widehat{\Gamma}_{Y_j}\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_j)}\}$$

and

$$\{\beta_{q_j p_j(i)}\eta_i\beta_i^{-1}\}.$$

Also, by lemma 3.6, part 2, we know that the set $\beta_k H^{c_2} \cup \beta_1 H_{c_2}$ is contained in the Marden tube for J in $\mathcal{T}(X_{\Gamma_{\widehat{\kappa}}})$. Hence the set $\beta_k H^{c_2} \cup \beta_1 H_{c_2}$ is precisely J-invariant in $\Gamma_{\widehat{\kappa}}$.

The parabolic element g_j is $z \mapsto z + \mathfrak{p}_j i$. The prime $\mathfrak{p}_j$ was chosen so that $\mathfrak{p}_j \geq (2c_2+1)k + 2c_2$. Hence we find that g_j maps the outside of the cusp neighbourhood $\beta_1 H_{c_2}$ into the inside of the cusp neighbourhood $\beta_k H^{c_2}$. We apply the second Klein-Maskit combination theorem to conclude that the group $\widehat{\Gamma}_j$ generated by $\Gamma_{\widehat{\kappa}}$ and g_j is discrete and geometrically finite. Moreover, there is a pared uniformization

$$\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j}) \text{ where } \widehat{\phi}_j \circ \iota_{\widehat{\kappa}} = \boldsymbol{j}_{\widehat{\kappa}} \circ \phi_{\widehat{\kappa}}. \tag{9}$$

This is the uniformization of $\widehat{M}_j$ that we seek. We need to show that it has the required properties.

To begin with, we show that $\widehat{\Gamma}_j$ contains Γ, and that Γ together with a set of $l+1$ parabolic elements generates $\widehat{\Gamma}_j$. Recall that Γ is generated by

$$\{\xi_{q(C)} G_C \xi_{q(C)}^{-1}\}_{C\in\mathcal{C}} \quad \text{and} \quad \{\xi_{qp(i)}\gamma_i\xi_i^{-1}\}_{i=1}^k$$

where G_{Y_0} is the group $\gamma_0^{-1}\Gamma_{Y_0}\gamma_0$, and for each i there is a unique C so that $\gamma_i \in \chi(\psi_C)$.

Consider the group G generated by Γ, g_j and $\{\beta_{q_j(Y_j)}\gamma_0^{-1}g_i^j\gamma_0\beta_{q_j(Y_j)}^{-1}\}_{i=1}^l$.

Then rewriting G_{Y_0} as $\gamma_0^{-1}\Gamma_{Y_0}\gamma_0$ and recalling the relationship between the ξ_i and $\beta_{t_j(i)}$ we find that G is generated by

$$\{\xi_{q(C)}^{-1} G_C \xi_{q(C)}\}_{C\neq Y_0},$$

$$g_j,$$

$$\beta_{t_j q(Y_0)}\gamma_0^{-1}\Gamma_{Y_0}\gamma_0\beta_{t_j q(Y_0)}^{-1},$$

the set of parabolics

$$\{\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_j)}^{-1}\gamma_0^{-1}g_i^j\gamma_0\gamma_{t_j^{-1}q_j(Y_j)}\beta_{q_j(Y_j)}^{-1}\}_{i=1}^l,$$

and the set of elements

$$\{\beta_{t_j qp(i)}\gamma_i\beta_{t_j(i)}^{-1}\}_{i=1}^k.$$

If we set $m = t_j(i)$ this last set can be rewritten as $\{\beta_{t_j qp_j(m)}\gamma_{t_j^{-1}(m)}\beta_m^{-1}\}_{m=1}^k$. This set contains $\beta_{t_j^{-1}q(Y_0)}\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_0)}^{-1}$.

Thus G contains the group generated by

$$\beta_{q_j(Y_0)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}\Gamma_{Y_0}\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_0)}^{-1}$$

and the parabolics

$$\{\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}g_i^j\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_j)}^{-1}\}_{i=1}^l.$$

The map q_j is defined with respect to the sewing kit κ_j so that $q_j(Y_j) = q_j(Y_0)$. It follows that this last group is

$$\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}\widehat{\Gamma}_{Y_j}\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_j)}^{-1}.$$

Hence G is generated by g_j, $\{\beta_{t_j q(C)}G_C\beta_{t_j q(C)}^{-1}\}_{C\neq Y_0}$, the group

$$\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}\widehat{\Gamma}_{Y_j}\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_j)}^{-1},$$

and the elements $\{\beta_{t_j qp_j(m)}\gamma_{t_j^{-1}(m)}\beta_m^{-1}\}_{m=1}^k$.

As in lemma 3.7 the group generated by $\{\beta_{t_j qp_j(m)}\gamma_{t_j^{-1}(m)}\beta_m^{-1}\}$ is the same as the group generated by $\{\beta_{q_j p_j(m)}\eta_m\beta_m^{-1}\}$ (recalling that $\eta_m = \gamma_{t_j^{-1}q_j p_j(m)}^{-1}\gamma_0^{-1}$). It follows that $G = \widehat{\Gamma}_j$.

Moreover, letting Γ_j be the group generated by the set of groups

$$\{\beta_{q_j(C)}\gamma_{t_j^{-1}q_j(C)}G_C\gamma_{t_j^{-1}q_j(C)}\beta_{q_j(C)}^{-1}\}\cup\{\beta_{q_j(Y_j)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\Gamma_{Y_j}\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_j)}^{-1}\}$$

and the set of elements $\{\beta_{q_j p_j(i)}\eta_i\beta_i^{-1}\}$ by lemma 3.6 there is a pared uniformization $\phi_j : (M_j, P_j) \longrightarrow (X_{\Gamma_j}, Q_{\Gamma_j})$ with

$$\widehat{\phi}_j \circ \iota_j = \boldsymbol{j}_j \circ \phi_j, \text{ where } \boldsymbol{j}_j : X_{\Gamma_j} \longrightarrow X_{\widehat{\Gamma}_j}. \tag{10}$$

Finally we show that $\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C \simeq \iota_j \circ s_j|_C$, where π is the immersion described earlier whose restriction to $int(X_\Gamma)$ is the homotopic to the restriction of the covering map.

We have an immersion $\widehat{\pi}_{\kappa_j} : X_{\widehat{\Gamma}_{\kappa_j}} \longrightarrow X_{\widehat{\Gamma}_j}$, since $\widehat{\Gamma}_{\kappa_j}$ is a subgroup of $\widehat{\Gamma}_j$. Moreover, the immersion π factors (up to homotopy) as $\pi = \widehat{\pi}_{\kappa_j} \circ \pi_{\kappa_j}$.

Associated to the uniformization of M_{κ_j} we have the inclusions $\boldsymbol{j}_{\widehat{\kappa}} : X_{\Gamma_{\widehat{\kappa}}} \longrightarrow X_{\widehat{\Gamma}_j}$, and $\iota_{\widehat{\kappa}} : M_{\widehat{\kappa}} \longrightarrow \widehat{M}_j$.

The group Γ_{κ_j} contains the group $\beta_{q_j(Y_0)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}\Gamma_0\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_0)}^{-1}$. The group $\widehat{\Gamma}_j$ contains $\beta_{q_j(Y_0)}\gamma_{t_j^{-1}q_j(Y_0)}^{-1}\gamma_0^{-1}\widehat{\Gamma}_j\gamma_0\gamma_{t_j^{-1}q_j(Y_0)}\beta_{q_j(Y_0)}^{-1}$. Hence we have the identity

$$\widehat{\pi}_{\kappa_j} \circ \widehat{\phi}_{\kappa_j} \circ \iota_{\kappa_j}|_C = \boldsymbol{j}_{\widehat{\kappa}} \circ \boldsymbol{j}_{\widehat{Y}_j,\widehat{\kappa}} \circ \bar{\gamma}^{-1}_{t_j^{-1} q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \pi_{Y_0,j} \circ \theta_0|_C \tag{11}$$

(Note that $(\boldsymbol{j}_{\widehat{Y}_j,\widehat{\kappa}})_* = (\boldsymbol{j}_{Y_0,\kappa_j})_*$ is conjugation by $\beta_{q_j(Y_0)}$.)

Let C be a component of $\overline{M_0 - \mathcal{V}}$. There are two cases: either C is contained in Y_0 or it is not. To begin with, suppose that C is contained in Y_0.

Then

$$\begin{aligned}
\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C &= \widehat{\phi}_j^{-1} \circ \widehat{\pi}_{\kappa_j} \circ \pi_{\kappa_j} \circ \phi|_C && \text{using the factorization of } \pi \\
&\simeq \widehat{\phi}_j^{-1} \circ \widehat{\pi}_{\kappa} \circ \widehat{\phi}_{\kappa_j} \circ \iota_{\kappa_j} \circ d_j|_C && \text{by 6} \\
&\simeq \widehat{\phi}_j^{-1} \circ \widehat{\pi}_{\kappa_j} \circ \widehat{\phi}_{\kappa_j} \circ \iota_{\kappa_j}|_C && \text{since } d_j|_{Y_0} \simeq id \\
&= \widehat{\phi}_j^{-1} \circ \boldsymbol{j}_{\widehat{\kappa}} \circ \boldsymbol{j}_{\widehat{Y}_j,\widehat{\kappa}} \circ \bar{\gamma}^{-1}_{t_j^{-1} q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \pi_{Y_0,j} \circ \theta_0|_C && \text{substituting 11} \\
&\simeq \widehat{\phi}_j^{-1} \circ \boldsymbol{j}_{\widehat{\kappa}} \circ \boldsymbol{j}_{\widehat{Y}_j,\widehat{\kappa}} \circ \bar{\gamma}^{-1}_{t_j^{-1} q_j(Y_0)} \circ \bar{\gamma}_0^{-1} \circ \widehat{\theta}_j \circ \iota_{Y_j} \circ s_j|_C && \text{using 2} \\
&= \widehat{\phi}_j^{-1} \circ \boldsymbol{j}_{\widehat{\kappa}} \circ \phi_{\widehat{\kappa}} \circ \iota_{Y_j} \circ s_j|_C && \text{using 7} \\
&= \widehat{\phi}_j^{-1} \circ \widehat{\phi}_j \circ \iota_{\widehat{\kappa}} \circ \iota_{Y_j} \circ s_j|_C && \text{using 9} \\
&= \iota_{\widehat{\kappa}} \circ \iota_{Y_j} \circ s_j|_C \\
&= \iota_j \circ s_j|_C
\end{aligned}$$

as required. Thus we are done in the case that C is contained in Y_0.

Suppose that C is not contained in Y_0. Thus C is shuffled only with respect to V_{l+1}, so that

$$\boldsymbol{j}_{\widehat{\kappa}} \circ \phi_{\widehat{\kappa}}|_C = \widehat{\pi}_{\kappa_j} \circ \boldsymbol{j}_{\kappa_j} \circ \phi_{\kappa_j}|_C \tag{12}$$

Hence in this case we find that

$$\begin{aligned}
\pi \circ \phi|_C &= \widehat{\pi}_{\kappa_j} \circ \pi_{\kappa_j} \circ \phi|_C && \text{using the factorization of } \pi \\
&\simeq \widehat{\pi}_{\kappa_j} \circ \widehat{\phi}_{\kappa_j} \circ \iota_{\kappa_j} \circ s_j|_C && \text{by 6} \\
&= \widehat{\pi}_{\kappa_j} \circ \boldsymbol{j}_{\kappa_j} \circ \phi_{\kappa_j} \circ s_j|_C && \text{by 5.} \\
&= \boldsymbol{j}_{\widehat{\kappa}} \circ \phi_{\widehat{\kappa}}|_C && \text{using 12} \\
&= \widehat{\phi}_j \circ \iota_{\widehat{\kappa}} && \text{using 9}
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\widehat{\phi}_j^{-1} \circ \pi \circ \phi|_C &\simeq \widehat{\phi}_j^{-1} \circ \widehat{\phi}_j \circ \iota_{\widehat{\kappa}} \circ s_j|_C \\
&= \iota_{\widehat{\kappa}} \circ s_j|_C \\
&= \iota_j \circ s_j|_C && \text{since } \iota_{\widehat{\kappa}} \text{ is } \iota_j \text{ when restricted to those } C \text{ such that } C \cap \mathcal{V}_l = \emptyset
\end{aligned}$$

And hence we are done.

proposition 3.10

Again we have the following corollary regarding wrapping numbers.

Corollary 3.11 *With the notation of proposition 3.10, let* $\pi_j : X_\Gamma \longrightarrow X_{\hat{\Gamma}_j}$ *be an immersion whose restriction to* $int(X_\Gamma)$ *is the restriction of the covering map* $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\hat{\Gamma}_j$. *For each* $j = 1, \ldots, n$ *set* $f_j = (\hat{\phi}_j)^{-1} \circ \pi_j \circ \phi$. *Then for each* j *and each* $V \in \mathcal{V}$, *the map* $f_j|_V$ *is a* π_1*-injective immersion into* $\hat{V}$ *which is an embedding on the frontier of* V. *Moreover,*

$$w(f_j|_V) \text{ is not equivalent to } w(f_i|_V)$$

for each $i \neq j$, *where* $w(f)$ *is the wrapping of* f *defined previously.*

corollary 3.11

3.3 Bumping

To obtain our point of bumping we want to Dehn fill the boundary tori in $\hat{M}_i$ so that the resulting manifold is homeomorphic to M_i, and then we want to find a hyperbolic structure on the result. In fact, we want to do this for a sequence of Dehn filling coefficients to obtain a sequence converging geometrically to the geometric structure on $\hat{M}_i$ covered by our bumping representation.

More precisely, we want to make use of the Hyperoblic Dehn surgery theorem, as discussed below.

Let $\widehat{M}$ be a compact, irreducible, oriented 3-manifold whose boundary contains a collection of tori $T_1, \ldots, T_n$. (There may be other surfaces in the boundary). Choose a meridian m_i and a longitude l_i for the torus T_i and consider them as a basis for $\pi_1(T_i)$. For an n-tuple of pairs of relatively prime integers $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; p_2, q_2; \ldots; p_n, q_n)$ denote by $\widehat{M}(\mathbf{p}, \mathbf{q})$ the manifold obtained by performing, in turn, (p_i, q_i) Dehn filling on $\widehat{M}$ along T_i. That is, $\widehat{M}(\mathbf{p}, \mathbf{q})$ is obtained by sewing n solid tori $V_1, \ldots, V_n$ to $\widehat{M}$ along $T_1, \ldots, T_n$, respectively, via an orientation reversing homeomorphism which maps the meridian of ∂V_i to a simple closed curve in the homotopy class of $m_i^{p_i} l_i^{q_i}$ on T_i. The following generalization of Thurston's Hyperbolic Dehn Surgery Theorem is due to T. Comar. See also Bonahon and Otal [4] and Bromberg [6].

Theorem 3.12 *(The Hyperbolic Dehn Surgery Theorem [10])*

Let $\widehat{M}$ *be a compact, oriented 3-manifold and let* $T = \{T_1, \ldots, T_k\}$ *be a non-empty collection of tori in the boundary of* $\widehat{M}$. *Let* $\widehat{N} = \mathbb{H}^3/\widehat{\Gamma}$ *be a minimally parabolic hyperbolic 3-manifold and* $\phi : int(\widehat{M}) \longrightarrow \widehat{N}$ *an orientation preserving homeomorphism between the interior of* $\widehat{M}$ *and* $\widehat{N}$. *Let* $\{(\mathbf{p}_n, \mathbf{q}_n) = (p_n^1, q_n^1; \ldots; p_n^k, q_n^k)\}$ *be a sequence of k-tuples of pairs of relatively prime integers such that, for each* j, $|p_n^j| + |q_n^j|$ *converges to* ∞ *as* $n \to \infty$.

Then, for all sufficiently large n, *there exists a representation* $\beta_n : \widehat{\Gamma} \longrightarrow PSL_2(\mathbb{C})$ *with discrete image such that:*

1. *$\beta_n(\widehat{\Gamma})$ is minimally parabolic and uniformizes $\widehat{M}(\mathbf{p}_n, \mathbf{q}_n)$;*

2. *the kernel of β_n is normally generated by $\{m^{p_n^1} l^{q_n^1}, \ldots, m^{p_n^k} l^{q_n^k}\}$; and*

3. *$\{\beta_n\}$ converges to the identity representation of $\widehat{\Gamma}$.*

Moreover, if we let i_n denote the inclusion of $\widehat{M}$ into $\widehat{M}(\mathbf{p}_n, \mathbf{q}_n)$, then there exists an orientation preserving homeomorphism $\phi_n : int(\widehat{M}(\mathbf{p}_n, \mathbf{q}_n)) \longrightarrow \mathbb{H}^3/\beta_n(\widehat{\Gamma})$ such that $\beta_n \circ \phi_*$ is conjugate to $(\phi_n)_* \circ (i_n)_*$.

theorem 3.12

Theorem 3.13 *Suppose $[(M_0, h_0)]$ is primitive shuffle equivalent to $[(M_j, h_j)]$, $j = 1, \ldots, n$, via primitive shuffles $s_j : M_0 \longrightarrow M_j$. Then there is a geometrically finite representation ρ with $\Theta(\rho) = [(M_0, h_0)]$ so that for each $[(M_j, h_j)]$ there are sequences ρ_n^j with $\Theta(\rho_n^j) = [(M_j, h_j)]$ and $\rho_n^j \longrightarrow \rho$.*

Proof

By Proposition 3.10 there is a pared uniformization $\phi : (M_0, P_0) \longrightarrow (X_\Gamma, Q_\Gamma)$ so that for each j there are elements $g_1^j, \ldots, g_k^j$ so that $\widehat{\Gamma}_j = < g_1^j, \ldots, g_k^j, \Gamma >$ is geometrically finite and uniformizes $\widehat{M}_j$. The uniformizing map $\widehat{\phi}_j$ satisfies $(\widehat{\phi}_j)^{-1} \circ \pi \circ \phi|_C$ is homotopic to $\iota_j \circ s_j|_C$ for each component C of $\overline{M_0 - \mathcal{V}}$, where $\mathcal{V}$ is the set of solid tori that support the primitive shuffles.

Set

$$\rho = (\phi)_* \circ (h_0)_*.$$

Then

$$\Theta(\rho) = [(X_\Gamma, \phi \circ h_0)] = [(M_0, h_0)].$$

Fix j now.

Perform $(\mathbf{1}, \mathbf{n})$ hyperbolic Dehn surgery on $\widehat{\phi}_j : (\widehat{M}_j, \widehat{P}_j) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$ to obtain representations $\beta_n : \widehat{\Gamma}_j \longrightarrow PSL_2(\mathbb{C})$, such that for sufficiently large n there are uniformizations $\phi_n : int(M_j) \longrightarrow \mathbb{H}^3/\beta_n(\widehat{\Gamma}_j)$. These homeomorphisms ϕ_n are such that $\beta_n \circ (\widehat{\phi}_j)_*$ is conjugate to $(\phi_n)_* \circ (\boldsymbol{i}_n)_*$, where $\boldsymbol{i}_n : \widehat{M}_j \longrightarrow \widehat{M}_j(1, n)$.

For sufficiently large n set

$$\rho_n^j = \beta_n \circ \rho.$$

Then by theorem 3.12 ρ_n^j is conjugate to $(\phi_n)_* \circ (\boldsymbol{i}_n)_* \circ (\widehat{\phi}_j)_*^{-1} \circ \pi_* \circ \phi_* \circ (h_0)_*$.

Consider the maps $\boldsymbol{i}_n \circ (\widehat{\phi}_j)^{-1} \circ \pi \circ \phi$ and $\boldsymbol{i}_n \circ \iota_j \circ s_j$. Both are primitive shuffles supported on $\mathcal{V}$ and up to homotopy are equal on $\overline{M_0 - \mathcal{V}}$. Hence by lemma 7.1 of [3] there is an orientation preserving homeomorphism $r_n : \widehat{M}_j(\mathbf{1}, \mathbf{n}) \longrightarrow \widehat{M}_j(\mathbf{1}, \mathbf{n})$ so that $r_n \circ \boldsymbol{i}_n \circ \iota \circ s_j$ is homotopic to $\boldsymbol{i}_n \circ (\widehat{\phi}_j)^{-1} \circ \pi \circ \phi$.

Thus ρ_n^j is conjugate to $(\phi_n)_* \circ (r_n)_* \circ (\boldsymbol{i}_n)_* \circ (\iota_j)_* \circ (s_j)_* \circ (h_0)_*$.

We conclude that ρ_n^j is faithful and has image $\beta_n(\widehat{\Gamma}_j)$, since $\boldsymbol{i}_n \circ \iota_j$ is homotopic to an orientation preserving homeomorphism, and each of the other homomorphisms in the above composition are isomorphisms induced by homeomorphisms or homotopy equivalences.

Moreover,

$$\begin{aligned} \Theta(\rho_n^j) &= [(\widehat{M}_j(\mathbf{1}, \mathbf{n}), r_n \circ \boldsymbol{i}_n \circ \iota_j \circ s_j \circ h_0)] && \text{since } \phi_n \text{ is an orientation preserving homeomorphism} \\ &= [(\widehat{M}_j(\mathbf{1}, \mathbf{n}), \boldsymbol{i}_n \circ \iota_j \circ s_j \circ h_0)] && \text{since } r_n \text{ is an orientation preserving homeomorphism} \\ &= [(M_j, s_j \circ h_0)] && \text{since } \boldsymbol{i}_n \circ \iota_j \text{ is homotopic to an orientation preserving homeomorphism (see lemma 10.3, [3])} \end{aligned}$$

Thus

$$\Theta(\rho_n^j) = [(M_j, h_j)]$$

as required. Clearly $\{\rho_n^j\}$ converges to ρ.

theorem 3.13

3.4 Quasiconformal deformations

In this section we wish to explore how open a condition it is to be a bumping point. In particular, we will construct bumping points for which large quasiconformal deformations are also bumping points. For basic definitions on quasiconformal mappings and quasiconformal deformations of Kleinian groups see [19].

If f is a quasiconformal deformation of a group Γ then f continuously extends to a group equivariant bi-Lipshitz homeomorphism of $\mathbb{H}^3$ to itself, denoted $\tilde{f}$. See [28], [11] and [24].

For our purposes, a *normalized* quasiconformal map is one that fixes $0, 1$ and ∞.

For a fixed $K \geq 1$, normalized K-quasiconformal maps form a normal family (see Lehto [19], theorem I.2.2).

The proof of the following lemma is the same as one in [13] with some simple changes made to cover groups other than Fuchsian groups.

Lemma 3.14 *Let $K \geq 1$. Let Γ be a geometrically finite Kleinian group containing ξ_1 as a primitive element, and so that $< \xi_1 >= J$ is a maximal parabolic subgroup of Γ.*

Then there is a constant $c = c(K,\Gamma)$ so that if f is any normalized K-quasiconformal deformation of Γ then the sets $H^c = \{z \in \widehat{\mathbb{C}} \mid \Im z > c\}$ and $H_c = \{z \in \widehat{\mathbb{C}} \mid \Im z < -c\}$ are precisely J-invariant in $f \circ \Gamma \circ f$.

The above lemma is instrumental in the proof of the following theorem.

Theorem 3.15 *Suppose $B_0, \dots, B_k$ is a collection of components of $MP(\pi_1(M))$ so that $\bar{B}_i \cap \bar{B}_j \neq \emptyset$, for all i, j. Then for every $K \geq 1$ there is a geometrically finite representation $\rho_K \in \overline{MP(\pi_1(M))}$ so that every K-quasiconformal deformation of ρ_K is contained in $\bigcap \bar{B}_i$.*

Proof

Let $s_i : M_0 \longrightarrow M_i$ be a primitive shuffle so that $\Theta(B_i) = [(M_i, s_i \circ h_0)]$, where $\Theta(B_0) = [(M_0, h_0)]$. Let $\mathcal{V}$ be the smallest allowably embedded collection of solid tori containing the support of each s_i (up to isotopy). Homotope each s_i, if necessary, so that the support of s_i is contained in $\mathcal{V}$. As before we induct on $n = |\mathcal{V}|$.

Suppose that $\mathcal{V}$ consists of a single solid torus V. Let $\mathcal{C}$ be the components of $\overline{M_0 - V}$. This defines a sewing kit $\kappa = (\mathcal{C}, V, p)$, where p is defined as in the proofs of previous theorems. Let $\{\phi_C : (C, P) \longrightarrow (X_{G_C}, Q_{G_C})\}$ be a normalized family of pared uniformizations for κ. Set

$$c = \max_{\substack{\gamma \in \chi(\phi_C) \\ C \in \mathcal{C}}} c(\gamma G_C \gamma^{-1}, K),$$

where $c(G, K)$ is the constant given by lemma 3.14 applied to G and K.

Let $\phi_K : (M_0, P_0) \longrightarrow (X_{\Gamma_K}, Q_{\Gamma_K})$ be the pared uniformization given by lemma 3.7, applied to the c-normalized family defined above, and set

$$\rho_K = \phi_* \circ (h_0)_*.$$

Note that for each C, $\phi_K|_C = j_C \circ \phi_C$, where $j_C : X_{G_C} \longrightarrow X_{\Gamma_K}$ is inclusion.

Let ρ be a representation arising as a K-quasiconformal deformation of ρ_K.

The quasiconformal deformation between ρ and ρ_K extends to an equivariant map $\tilde{\psi} : \mathbb{H}^3 \cup \widehat{\mathbb{C}} \longrightarrow \mathbb{H}^3 \cup \widehat{\mathbb{C}}$, which descends to an orientation preserving homeomorphism $\psi : N(\Gamma_K) \longrightarrow N(\Gamma)$.

As in lemma 3.7 define constants

$$a_j = \sum_i d_i(j\mathfrak{p}_i + (2c+1)t_i(j))$$

where d_i is congruent to 1 modulo $\mathfrak{p}_i$ and 0 modulo $\mathfrak{p}_j$, $j \neq i$.

Recall from the proof of lemma 3.7 that Γ_K is generated by $\{\xi_{q(C)} G_C \xi_{q(C)}^{-1}\}$ and $\{\xi_{qp(i)} \gamma_i \xi_i^{-1}\}$, where

$$\xi_j(z) = z + a_j i$$

and $\gamma_i \in \chi(\phi_C)$ (for some C) conjugates $J \subset G_C$ to $F_i = (\phi_C)_*(\pi_1(p(i))$.

For each i, define δ_i by requiring that

$$\xi_{qp(i)}\delta_i\xi_i^{-1} = \rho \circ (\rho_K)^{-1}(\xi_{qp(i)}\gamma_i\xi_i^{-1}).$$

For each i there are subgroups $\delta_i^{-1}G'_{p(i)}\delta_i$ of $\Gamma = \rho(\pi_1(M))$ obtained by requiring that

$$\xi_i\delta_i^{-1}G'_{p(i)}\delta_i\xi_i^{-1} = \rho \circ \rho_K^{-1}(\xi_i\gamma_i^{-1}G_{p(i)}\gamma_i\xi_i^{-1}).$$

By construction lemma 3.14 implies that the set $\xi_i(H^c \cup H_c)$ is precisely J-invariant in $\xi_i\delta_i^{-1}G'_{p(i)}\delta_i\xi_i^{-1}$.

Let $\xi_{qp(i)}F'_i\xi_{qp(i)}^{-1} = \rho \circ (\rho_K)^{-1}(\xi_{qp(i)}F_i\xi_{qp(i)}^{-1})$, for each i. Let T be a Marden tube for F_i in $G_{p(i)}$. Then $\tilde{T} \cap \widehat{\mathbb{C}}$ contains $\gamma_i(H^c \cup H_c)$, since ϕ_C is c-normalized. The set $\tilde{\psi}(\xi_{qp(i)}\tilde{T})$ is precisely $\xi_{qp(i)}F'_i\xi_{qp(i)}^{-1}$-invariant in $\xi_{qp(i)}G'_{p(i)}\xi_{qp(i)}^{-1}$, and hence there is a lift of a Marden tube $\xi_{qp(i)}\tilde{T}'$ contained in $\tilde{\psi}(\xi_{qp(i)}\tilde{T})$ and the two are isotopic. There is a lift of a Marden tube $\xi_{qp(i)}\tilde{T}''$ which contains $\xi_{qp(i)}\delta_i(H^c \cup H_c)$. Up to isotopy $\xi_{qp(i)}\tilde{T}''$ is contained in $\tilde{\psi}(\xi_{qp(i)}\tilde{T})$, for each i, so we may isotope ψ to a homeomorphism $\psi : (X_{\Gamma_K}, Q_{\Gamma_K}) \longrightarrow (X_\Gamma, Q_\Gamma)$, where X_Γ is defined by these choices of Marden tubes for Γ.

Define ϕ'_C by setting $j'_C \circ \phi'_C = \psi \circ \phi_K|_C$. Where $j'_C : X_{G'_C} \longrightarrow X_\Gamma$ is inclusion.

By the above comments the set $\{\phi'_C\}$ is a c-normalized family of pared uniformizations for κ. By the proof of lemma 3.7 and theorem 3.13 we find that ρ is in the intersection of the closures of all the components B_i.

The inductive case is a repeat of the above argument combined with the inductive step of proposition 3.10

theorem 3.15

4 Shuffle Immersions

Suppose that (M, h) represents an element of $\mathcal{A}(M)$ and (M_0, h_0) represents an element of $\mathcal{A}(M)$ primitive shuffle equivalent to $[(M, h)]$. Using the techniques of this paper, there is a uniformization $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ and a uniformization $\widehat{\phi} : (\widehat{M}_0, \widehat{P}_0) \longrightarrow (X_{\widehat{\Gamma}}, Q_{\widehat{\Gamma}})$ so that the composition

$$\widehat{\phi}^{-1} \circ \pi \circ \phi$$

is an immersion of M into $\widehat{M}_0$ which is an embedding on each of the components of $\overline{M - \mathcal{V}}$, where $\mathcal{V}$ is the support of s, and π is an immersion whose restriction to $int(X_\Gamma)$ is homotopic to the restriction of the covering map $\mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_j$.

Equivalently, there is a topological immersion $f : M \longrightarrow \widehat{M}_0$ which is an embedding on $\overline{M - \mathcal{V}}$, and a uniformization $\widehat{\phi} : (\widehat{M}_0, \widehat{P}_0) \longrightarrow (X_{\widehat{\Gamma}}, Q_{\widehat{\Gamma}})$, so that the (incomplete) metric on M obtained by pulling back the metric on $int(\widehat{M}_0)$ with f extends to a complete hyperbolic metric on an open collar of M.

There is evidence (see [3] and [8]) that the set of (equivalence classes of) pairs $(f, \widehat{\phi})$ determines the topology of $\overline{MP(\pi_1(M))}$. In particular, if there are n distinct pairs $(f_1, \widehat{\phi}_1), \ldots, (f_n, \widehat{\phi}_n)$ so that

$$\rho = (\widehat{\phi}_1)_* \circ (f_1)_* = \cdots = (\widehat{\phi}_n)_* \circ (f_n)_*$$

then for any sufficiently small neighbourhood V of ρ in $AH(\pi_1(M))$ the intersection $V \cap B$ has at least n components, where B is the component of $MP(\pi_1(M))$ indexed by $[(M_0, h_0)]$.

Before our next definition we introduce some notation. For a collection $\mathcal{V}$ of allowably embedded solid tori in a given manifold let $\Delta(\mathcal{V})$ denote the collection of (homotopy classes) of core curves of solid tori in $\mathcal{V}$. For a compact manifold M let $Collar(M)$ be the union $M \cup (\partial M \times [0,1)$ where ∂M is identified with $\partial M \times \{0\}$. Note that M can be considered to be a compact core for $Collar(M)$.

Definition 4.1 Let M and M_0 be compact hyperbolizable 3-manifolds with non-empty boundary. Suppose that $\mathcal{V}$ is a collection of solid tori allowably embedded in M. A *shuffle immersion* between M and M_0 *supported on* $\mathcal{V}$ is an immersion

$$f : M \longrightarrow int(M_0) - \Delta(\mathcal{V}) = int(\widehat{M_0}) \subset int(M_0)$$

such that

- $f(M) \subset int(M_0) - \Delta(\mathcal{V}) = int(\widehat{M_0})$. That is, we can consider f as a map

$$f : M \longrightarrow int(\widehat{M_0});$$

- f factors through the restriction to M of a cover $\pi : Collar(M) \longrightarrow int(\widehat{M_0})$ which is an embedding on $\overline{M - \mathcal{V}}$. That is, $f = i \circ \pi|_M$ where $i : int(\widehat{M_0}) \longrightarrow int(M_0)$ is inclusion,and π is an embedding when restricted to $\overline{M - \mathcal{V}}$.

Note that the notion of the wrapping of a solid torus by a shuffle immersion makes sense.

It follows that a shuffle immersion $f : M \longrightarrow int(\widehat{M_0})$ lifts to a map

$$\tilde{f} : M \longrightarrow int(M')$$

so that $f|_{int(M)}$ is a homeomorphism onto its image, where $int(M')$ is the cover of $\widehat{M_0}$ corresponding to $f_*(\pi_1(M))$. Hence if $int(\widehat{M_0})$ admits a complete geometrically finite hyperbolic structure, so too does $int(M)$ via pull-back by f. That is to say, a pair $(\widehat{\rho}, f)$ determines a hyperbolic structure on $int(M)$, where f is a shuffle immersion between M and M_0 and $\widehat{\rho}$ is a complete geometrically finite representation in $MP(\widehat{M_0})$.

We say that two shuffle immersions

$$f_1 : M \longrightarrow int(\widehat{M}_1)$$

and

$$f_2 : M \longrightarrow int(\widehat{M}_2)$$

are equivalent provided f_1 and f_2 are homotopic in $int(\widehat{M}_1) = int(\widehat{M}_2)$.

In this section we use the techniques from earlier sections to produce many (inequivalent) pairs $(\widehat{\rho}, f)$ inducing the same hyperbolic structure on $int(M)$. In a later paper this might be used to show the existence of complicated bumping phenomena in the case that M has incompressible boundary.

Theorem 4.2 *Let M be compact, atoroidal, irreducible with non-empty and incompressible boundary of negative Euler characteristic. Moreover, assume that there exists a solid torus component or an I-bundle component to the characteristic submanifold of M.*

For a positive integer k, let $B_1, \ldots, B_k$ be a collection of distinct components of $MP(\pi_1(M))$ so that for each i and j, $B_i \cap B_j \neq \emptyset$. Moreover, let $n_1, \ldots, n_k$ be any collection of positive integers.

Then there exists a geometrically finite representation $\rho \in \bigcap \overline{B}_i$ so that the following holds.

For each i set $\Theta(B_i) = [(M_i, s_i \circ h)]$, for a shuffle immersion s_i supported on $\mathcal{V}$. Then for each i there are n_i distinct pairs $(\widehat{\rho}_j, f_j)$, $j = 1, \ldots, n_i$, inducing ρ. That is $\rho = \widehat{\rho}_j \circ (f_j)_*$, $j = 1, \ldots, n_i$, and each f_j is a shuffle immersion between $M = M_0$ and M_i, so that no f_j is equivalent to any f_k, $j \neq k$. Moreover, each f_j is homotopic in $int(\widehat{M}_i)$ to the composition of s_i with the inclusion of M_i into $int(M_i)$.

Proof

In the case that $k = 1$, if necessary choose $s = s_1$ to be supported on a non-empty collection of solid tori allowably contained in M.

We apply theorem 3.13 to the redundant collection

$$B_1, \ldots, B_1, B_2, \ldots, B_2, \ldots, B_k \ldots, B_k,$$

where each B_i appears in the list exactly n_i times.

This produces $\rho \in \bigcap \overline{B}_i$. Moreover, for each i there are n_i uniformizations $\widehat{\phi}_j : (\widehat{M}_i, \widehat{P}_i) \longrightarrow (X_{\widehat{\Gamma}_j}, Q_{\widehat{\Gamma}_j})$, $j = 1, \ldots, n_i$. Here $\widehat{\Gamma}_j$ is a minimally parabolic uniformization of $int(\widehat{M}_i)$.

Moreover, there exists $\phi : (M, P) \longrightarrow (X_\Gamma, Q_\Gamma)$ uniformizing M, so that $\rho = \phi_*$. There are immersions $\pi_j|_{X_\Gamma} : X_\Gamma \longrightarrow X_{\widehat{\Gamma}_j}$ which are restrictions of covering maps $\pi_j : \mathbb{H}^3/\Gamma \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_j$. Define a shuffle immersion f_j by

$$f_j = (\widehat{\phi}_j)^{-1} \circ \pi_j \circ \phi.$$

We will be done once we have shown that for each j there is no $l \neq j$ so that f_j is equivalent to f_l. If f_j is equivalent to f_l then for every V in the support of s_j, the wrapping number $w(f_j|_V)$ would be equivalent to $w(f_l|_V)$. But corollary 3.11 rules out this possibility.

theorem 4.2

When M is an I-bundle we can construct our shuffle immersions with more control over the wrapping numbers. The ideas used here are similar to those used in Kerckhoff-Thurston [18], McMullen [25], and Ito [14].

Let $\lambda_1, \ldots, \lambda_k$ be k collections of positively weighted, homotopically distinct, pair-wise non-intersecting, simple closed curves on a surface S of negative Euler characteristic. (Sometimes such collections are refered to as integral measured laminations.) Suppose that for each i and j, the geometric intersection number $i(\lambda_i, \lambda_j) = 0$, that is to say, any curve in λ_i is either disjoint from or parallel to every curve in λ_j. Let Λ be the union of all the curves in the k collections.

Definition 4.3 A shuffle immersion $f : S \times I \longrightarrow int(S \times I - \Lambda \times \{\frac{1}{2}\})$ *carries* the integral measured lamination λ_i provided that for each curve α in Λ, if $w(f|_{\mathcal{N}(\alpha)}) = (0, n)$ is the wrapping of f with respect to a solid torus neighbourhood of $\alpha \times \{\frac{1}{2}\}$ with $n > 0$, then the weighting of α in λ_i (which we decree to be zero when λ_i does not contain α) is $n - 1$.

Theorem 4.4 *We assume the above notation.*

Given $K \geq 1$ there exists a geometrically finite representation $\rho_0 \in \overline{QF(S)}$ so that for every K-quasiconformal deformation ρ of ρ_0 the following holds:

For each $i = 1, \ldots, k$, there is a pair $(\widehat{\rho}_i, f_i)$ inducing ρ, where $\widehat{\rho}_i$ is a geometrically finite uniformization of $int(S \times I - \Lambda \times \{\frac{1}{2}\})$, and $f_i : S \times I \longrightarrow int(S \times I - \Lambda \times \{\frac{1}{2}\})$ is a shuffle immersion carrying λ_i,

Proof

Let $\mathcal{V}$ be a regular neighbourhood of $\Lambda \times I$. Choose a normalized family of pared uniformizations for the sewing kit associated to the splitting of $S \times I$ along $\mathcal{V}$, and sewing up along a solid torus $V \subset \mathcal{V}$. Such a normalized family contains at most two uniformizations. As usual we induct on the number of solid tori in $\mathcal{V}$.

Let $\alpha \times \{\frac{1}{2}\}$ be the core curve of V. For λ_i let m_i denote the weighting of α in λ_i, and reindex so that $m_1 \geq m_2 \geq \cdots m_k \geq 0$.

Let c_0 be a constant greater than the constant supplied by lemma 3.14 applied to any group appearing in the normalized family and the quasi-conformal constant K. Choose an integer I greater than c_0.

Define rational numbers $I = I_1 < I_2 < \cdots < I_{m_1}$ by requiring that

$$m_j I_j = m_1 I_1$$

for all j such that $m_j \neq 0$.

Now repeat the proof of 3.10 using the constants $a_1 = 0$ and

$$a_2 = m_1 I + c_0,$$

and the shuffling parabolics

$\beta_j(z) = z + I_j i$. When $m_j = 0$ set $\beta_j(z) = z + i(m_1 I + 3c_0)$.

For each j we obtain a shuffle immersion $f_j : M \longrightarrow \widehat{M}$, as in previous theorems.

By construction, for any j for which m_j is non-zero, a_2 is congruent mod I_j to c_0 and the wrapping of f_j is given by $w(f_j) = (0, n_j)$ where

$$n_j = \frac{a_2 - (a_2 mod I_j)}{I_j} = m_j$$

as required.

The inductive step follows from the proofs of previous theorems and a similar construction to the one above.

theorem 4.4

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