On the topology of the space of punctured torus groups

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1 Introduction

One of the central goals in the deformation theory of hyperbolic 3-manifolds is to find a continuous parameterization of the space of orientation-preserving isometry classes of marked hyperbolic 3-manifolds homotopy equivalent to a compact manifold M. Let us denote this space by $AH(M)$; a formal definition will be forthcoming. The analogous study in dimension two has been enormously fruitful; Thurston's compactification of Teichmuller space has been used to great effect in the study of the action of the modular group, for example. One reasonably expects that a continuous parameterization of $AH(M)$ would be extremely valuable.

Of course, there are topological restrictions on M that need to be satisfied if $AH(M)$ is to be non-empty. If M satisfies these requirements and has empty boundary or boundary consisting only of tori, then by Mostow-Prasad Rigidity (see [33], [35]) the space $AH(M)$ consists of exactly two points; if the boundary of M has higher genus components then there is a rich deformation theory and, in particular, the interior of $AH(M)$ consists of a collection of topological balls, or quotients thereof.

The solution of the Bers-Sullivan-Thurston Density conjecture (see [12], [9], [10], [1], [15]) tells us that $AH(M)$ is the closure of its interior. In a sense, $AH(M)$ is obtained from a collection of balls (or again, quotients of balls) by identifying points on the boundary. There are two phenomena that one would like to completely understand, namely that of *bumping* and *self-bumping.*

Bumping is when two distinct components of the interior of $AH(M)$ have closures that intersect. It was long suspected that bumping did not happen and Anderson and Canary were the first to demonstrate that bumping could occur (see [4]); together with McCullough they went on to give a complete classification of which components of $AH(M)$ bump, in the case that the boundary of M is incompressible (see [5]). In [21] it was shown that many distinct components can bump in a relatively open set of small codimension. A full understanding of the bumping locus is still far away.

The other phenomenon, self-bumping, is to do with "identifications" on the boundary of a single component of the interior, and is formulated as follows. Let B be a component of the interior of $AH(M)$ and suppose that ρ is in the closure of B. Then B self-bumps at ρ provided that for all sufficiently small neighbourhoods V of ρ in $AH(M)$, the intersection $V \cap B$ is disconnected - and ρ is called a *self-bumping point.* (Note that self-bumping at ρ does not imply nor rule out the possibility that $\overline{B}$ is non-locally connected at ρ). Self-bumping was first discovered by McMullen ([30]) using ideas of Anderson and Canary - McMullen showed that self-bumping occurs when M is a trivial interval bundle over a surface. Using different techniques, Bromberg and the first author ([14]) showed that self-bumping is fairly ubiquitous, where "ubiquitous" means that there are fairly non-stringent topological requirements that M must satisfy: M must contain a primitive, essential, properly embedded annulus not homotopic into the boundary.

A manifold is acylindrical provided that any properly embedded essential annulus is properly homotopic into the boundary. In [37] Thurston showed that $AH(M)$ is compact when M is acylindrical. In that paper he presented a heuristic picture of a typical component of $AH(M)$, again for M acylindrical, in order to illustrate the difference in various topologies that one could consider on $AH(M)$. In this heuristic each component is likened to a hard-boiled egg (with shell), in other words a topological closed ball. One wonders how far this analogy can be stretched: is every component of $AH(M)$ a closed ball when M is acylindrical?

In this paper we address the phenomenon of self-bumping. The paper was motivated by a desire to show that self-bumping never occurs when M is acylindrical. More precisely, we wished to address the question of when can we say that a point in $AH(M)$ is not a self-bumping point, for a general M. The criterion we present here for the once-punctured torus we call *non-wrapping* (see below). For topological reasons, every hyperbolic structure on an acylindrical manifold is non-wrapping, so this criterion seems like an apt one to apply to the acylindrical case. (However, as observed by R. Evans, any hyperbolic structure on a handlebody is non-wrapping, yet there is self-bumping in the deformation space associated to a handlebody. Perhaps there is a single criterion for self-bumping that reduces to non-wrapping in the case that M has incompressible boundary). While ultimately our theorem is only stated in the case that M is a trivial interval bundle over the once-punctured torus, there is but a single step of proof that currently doesn't generalize to general M with incompressible boundary - this step may in fact not generalize, but it is definitely worthy of further investigation.

Before continuing our discussion we need to introduce some terminology.

A *hyperbolic 3-manifold* is the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete, torsion free group of isometries. A compact 3-manifold M is said to be hyperbolizable provided that its interior can be identified with a hyperbolic 3-manifold. Such an identification $h : int(M) \longrightarrow \mathbb{H}^3/\Gamma$ is a *marked hyperbolic structure* on M. Using the upper-half space model of $\mathbb{H}^3$, a hyperbolic structure gives a faithful representation $h_* : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ with discrete image, and we say that two hyperbolic structures are isometrically equivalent provided that the induced representations are conjugate in $PSL_2(\mathbb{C})$. In this way we consider the set of isometry classes of marked hyperbolic structures on M to be a subset of the character variety $\chi(\pi_1(M))$ associated to representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$. This is not a closed subset of $PSL_2(\mathbb{C})$, since the limit of a sequence of hyperbolic structures on M may not be a hyperbolic 3-manifold homeomorphic to $int(M)$. Such a limit will be homotopy equivalent to M, however, so we consider the larger space $AH(M)$ consisting of orientation-preserving isometry classes of marked hyperbolic 3-manifolds homotopy equivalent to M. Again, $AH(M)$ embeds in the character variety, and by Jørgensen [22] it is a closed subset. The topology on $AH(M)$ that it inherits from this embedding is called the *algebraic topology.*

For an element ρ of $AH(M)$, let N_ρ denote the associated hyperbolic manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$.

Putting aside the issue of acylindricity for the moment, let M be arbitrary and let B be a component of $int(AH(M))$ - for the sake of argument, suppose that every element of B is homeomorphic to the interior of M. To study self-bumping of B one is led to consider representations $\rho \in \overline{B}$ and sequences $\rho_n \in B$ converging to ρ; if every such sequence is eventually contained in the same neighbourhood of ρ in B then we could conclude that ρ is not a self-bumping point.

There is an associated sequence of discrete subgroups of $PSL_2(\mathbb{C})$, $\{\Gamma_n = \rho_n(\pi_1(M))\}$, and it is a theorem of Jørgensen and Marden [23] that suitable conjugates of the Γ_n can be found so that up to subsequence there is a Gromov-Hausdorff limit $\hat{\Gamma}$ of the sequence Γ_n. The discrete, torsion free group $\hat{\Gamma}$ is a *geometric limit* of the sequence. Sometimes the quotient manifold $\mathbb{H}^3/\hat{\Gamma}$ is called the geometric limit of the approximating manifolds $\mathbb{H}^3/\Gamma_n$.

Geometric limits $\hat{\Gamma}$ naturally contain the image of ρ, and so there is a covering map $N_\rho \longrightarrow \mathbb{H}^3/\hat{\Gamma}$, and the geometry and topology of N_ρ greatly constrains how a compact core for N_ρ immerses under the covering map.

Definition 1 A representation $\rho \in AH_0(M)$ *does not wrap* (or is *non-wrapping*) provided that for any sequence of representations ρ_n converging to ρ and any associated geometric limit $\hat{N}$ there is a compact core for N_ρ which embeds in $\hat{N}$ under the covering map.

The following theorems give an indication of how the geometry and topology of ρ dictate whether or not it can wrap. Both are stated in terms of the *conformal boundary* of N_ρ and the *accidental parabolics* (if any) in the conformal boundary.

The hyperbolic 3-manifold N_ρ has a smallest convex submanifold whose inclusion into N_ρ is a homotopy equivalence - such a manifold is called a *convex core*. The frontier of the convex core is a collection C_0 of embedded convex surfaces. For every $t > 0$ we let C_t be the surface external to the convex core, a distance t from C_0. Via the nearest-point retraction map the surfaces C_t are all homeomorphic to C_0. The induced metrics on C_t diverge as t tends to infinity, but the conformal classes converge to a conformal structure on C_0. It is a celebrated theorem of Sullivan [18] that this conformal structure "at infinity" is K-biLipshitz to the metric on C_0, and K is independent of M or ρ. This conformal structure is the conformal boundary ∂N_ρ of N_ρ.

A simple closed geodesic γ in ∂N_ρ is an accidental parabolic for N_ρ provided that γ is not homotopic into a puncture in ∂N_ρ, but γ can be homotoped in N_ρ to a curve of arbitrarily short length. When we speak of the length of a accidental parabolic, we mean its length in the metric on ∂N_ρ induced by the conformal structure - the Poincare metric.

A compact, orientable manifold M has incompressible boundary provided that the inclusion of ∂M into M induces an injection on the level of fundamental groups.

Note that if M is acylindrical then M has incompressible boundary and no $\rho \in AH(M)$ has accidental parabolics.

Theorem 1 *(Anderson-Canary [3]) Let M have incompressible boundary. A representation $\rho \in AH(M)$ does not wrap provided that $\partial N_\rho \neq \emptyset$ and N_ρ has no accidental parabolics.*

Note that if $\partial N_\rho = \emptyset$ then not only does ρ not wrap, but by the Covering Theorem [17] every geometric limit for N_ρ is equal to N_ρ.

Hence if M is acylindrical no structure $\rho \in AH(M)$ wraps. However, the existence of cylinders is not necessarily an obstruction to wrapping, provided that the geometry of the conformal boundary is suitably constrained.

Theorem 2 *(Evans-Holt [19]) Let S be a surface of negative Euler characteristic. There exists a constant ϵ_0 (depending only on S) with the following property.*

A structure $\rho \in AH(S)$ does not wrap provided that for any accidental parabolic $\gamma \in \partial N_\rho$ there is a simple closed curve in ∂N_ρ transverse to γ and with length at most ϵ_0.

We now arrive at the question that this paper begins to address:

If $\rho \in \overline{B}$ does not wrap then must it follow that B does not self-bump at ρ?

Though we do not have a complete answer, we can answer in the affirmative in the case that M is an oriented interval bundle over the once-punctured torus $S = S_{1,1}$. Here $AH(M)$ is actually a relative deformation space $AH(S) := AH(S \times [-1,1]), \partial S \times [-1,1]$, wherein for all $\rho \in AH(S)$, $\rho(\partial S)$ is parabolic.

Theorem 3 *Let S be the once-punctured torus. Suppose that $\rho \in AH(S)$ does not wrap. Then $QF(S) = int(AH(S))$ does not self-bump at ρ.*

Combining our main theorem with the main theorem of [19] we obtain the following corollaries. The first follows from noting that a small neighbourhood of a representation that fulfills the requirement for not wrapping given in theorem 2 consists of non-wrapping representations. The second follows from this and from noting that a representation that is a non-bumping point is also not a point where $AH(S)$ is non-locally connected.

Corollary 1 *Self-bumping points are not dense in $AH(S)$, for S the once-punctured torus.*

Corollary 2 *The locus of non-local connectivity in $AH(S)$ is not dense, for S the once-punctured torus.*

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2 Preliminaries and outline of proof

Let S denote an orientable surface with negative Euler characteristic, possibly with boundary. Later we will require that S be the once-punctured torus, but for now we will work in this generality.

As we remarked above, the deformation space of S, $AH(S)$, is actually a relative deformation space. That is, $AH(S) := AH(S \times [-1,1], \partial S \times [-1,1])$, the space of conjugacy classes of discrete and faithful representations $\rho : \pi_1(S) \longrightarrow PSL_2(\mathbb{C})$, with the requirement that $\rho(\gamma)$ is parabolic whenever γ is conjugate into the fundamental group of a component of $\partial S \times [-1,1]$.

A representation ρ in $AH(S)$ determines a hyperbolic 3-manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ - more, it determines a homotopy equivalence (a *marking*) $h : S \longrightarrow N_\rho$, with h_* conjugate to ρ.

An element ρ of $AH(S)$ is *geometrically finite* if the convex core of N_ρ has finite volume. If the only elements of $\pi_1(S)$ mapped by ρ to a parabolic are homotopic into a puncture of S, then we say that ρ is *minimally parabolic.* By Marden [27] and Sullivan [36] the interior of $AH(S)$ consists of the geometrically finite and minimally parabolic representations. The interior of $AH(S)$ is the space of quasi-Fuchsian representations, denoted $QF(S)$. Bers' celebrated simultaneous uniformization [8] gives a homeomorphism

$$Q : Teich(S) \times Teich(S) \longrightarrow QF(S).$$

Thus the interior of $AH(S)$ is homeomorphic to an open ball.

Let γ be a non-peripheral, simple closed curve in S and let $\hat{M} = S \times [-1,1] \setminus \mathcal{N}(\gamma \times \{0\})$, where $\mathcal{N}(\gamma \times \{0\})$ is a regular neighbourhood of $\gamma \times \{0\}$. Now a structure ρ in $AH(\hat{M})$ is minimally parabolic provided that the only parabolics in N_ρ correspond to the fundamental group of components of $\partial S \times [-1,1]$ and the fundamental group of $\partial \mathcal{N}(\gamma \times \{0\})$. It is worth noting that by Ahlfors-Bers theory ([2], [7], [28], [26]) the geometrically finite, minimally parabolic representations in $AH(\hat{M})$ are also prescribed by the conformal boundary, so that any component of $MP(\hat{M}) := int(AH(\hat{M}))$ is homeomorphic to $Teich(S) \times Teich(S)$. There is a component $\hat{B}$ of $int(AH(\hat{M}))$ characterized by the property that $\rho \in \hat{B}$ if and only if there is a homeomorphism $h : int(\hat{M}) \longrightarrow N_\rho$ with $h_* = \rho$. With this notation, let

$$\hat{Q} : Teich(S) \times Teich(S) \longrightarrow \hat{B}$$

be the homeomorphism of simultaneous uniformization.

The work of Brock-Bromberg [9] or the solution to Thurston's Ending Lamination Conjecture due to Brock, Canary and Minsky [32][10] provides a solution to the Density Conjecture of Bers-Thurston-Sullivan: $\overline{QF(S)} = AH(S)$. Thus we will always refer to the closure of $QF(S)$ as $AH(S)$.

In our investigation we use the following straightforward lemma to detect whether $\rho \in AH(S)$ is a self-bumping point for $QF(S)$.

Lemma 2 *A structure $\rho \in AH(S)$ is not a self-bumping point for $QF(S)$ provided that for any two sequences $\rho_n, \rho_n' \in QF(S)$ converging to ρ there is a sequence of paths $\sigma_n : [0,1] \longrightarrow QF(S)$ so that*

1. *$\sigma_n(0) = \rho_n$ and $\sigma_n(1) = \rho_n'$;*
2. *For any sequence $\{t_n\}$, $t_n \in [0,1]$, the sequence $\{\sigma_n(t_n)\}$ converges to ρ.*

(A proof is left to the reader).

We say that a sequence ρ_n *converges strongly* if it converges algebraically to a representation ρ and geometrically to $\rho(\pi_1(M))$.

In the aim of applying lemma 2 we will suppose that ρ_n and ρ_n' are two sequences in $QF(S)$ converging to ρ and we will find a sequence of paths σ_n satisfying the hypotheses of lemma 2. The proof divides into the consideration of two cases. In the first, most straightforward case, we assume that both ρ_n and ρ_n' converge strongly to ρ - an extremely strong form of non-wrapping. In the second case we assume that some sequence, ρ_n say, does not converge strongly.

We will end this section with an outline of the approach in this second case, as it is here that it will ultimately be necessary for us to assume that M is a trivial interval bundle over the once-punctured torus.

The approach in the non-strongly convergent case is to find a third sequence of representations ρ''_n converging to ρ strongly and a sequence of paths σ''_n joining ρ_n to ρ''_n so that any sequence of representations obtained by sampling points in σ''_n converges to ρ, a la lemma 2. Being able to find such a third sequence reduces us to the strongly-convergent case.

We deform the non-strongly convergent sequence ρ_n to a strongly convergent one using the technology of hyperbolic cone-manifolds developed by Hodgson and Kerckhoff, with further developments by Bromberg and Brock-Bromberg. The hypothesis that ρ does not wrap means that a fixed compact core for N_ρ can be embedded in each N_{ρ_n} via a bi-Lipshitz diffeomorphism with bi-Lipshitz constant tending to one; call this embedded manifold M_n. Because the sequence ρ_n does not converge strongly we can use a result of Kleineidam's [25] to find short geodesics in N_{ρ_n} in the complement of M_n. If this geodesic is sufficiently short we can apply the Drilling Theorem of Brock and Bromberg [9] to drill out the curve to produce a geometrically finite, minimally parabolic structure $\hat{N}_n$ on the complement of these geodesics; the manifold M_n embeds in $\hat{N}_n$, again via a bi-Lipshitz diffeomorphism with bi-Lipshitz constant tending to one as the length of the geodesics to be drilled tends to zero; call the image of M_n in $\hat{N}_n$, M'_n.

For simplicity, suppose that a single curve γ on S is becoming short in N_n, though its length on ∂N_n is bounded away from zero; thus $\hat{N} \simeq S \times (-1,1) \setminus \gamma \times \{0\}$. The fundamental group of $\hat{N}_n$ can be written as an HNN-extension $\pi_1(M'_n)*_\delta$, where δ is a parabolic isometry commuting with $\rho(\gamma)$. The torus cusp in $\hat{N}_n$ separates M'_n from a component X_n of ∂N_n and the element δ determines the length of γ on X_n.

There is a one-complex dimensional set U of parabolic isometries such that, for each $\tau \in U$, $\pi_1(M'_n)*_\tau$ is a geometrically finite, minimally parabolic uniformization of $S \times (-1,1) \setminus \{\gamma\} \times \{0\}$. For $\tau \in U$ let $\hat{N}^\tau_n$ be this geometrically finite, minimally parabolic structure. Let X^τ_n be the component of $\partial\hat{N}^\tau_n$ analogous to X_n. If there is a path in U joining δ to a parabolic δ_n so that the length of γ on $X^{\delta_n}_n$ is less than $1/n$, say, then we can "push X_n down" to deform $\hat{N}_n$ to $\hat{N}^{\delta_n}_n$. M'_n isometrically embeds in each manifold along this path. Done appropriately, each manifold along this path can be Dehn filled to produce a path in $QF(S)$ joining ρ_n to a new representation ρ''_n. The resulting sequence ρ''_n converges strongly to ρ. Moreover, because the image of M_n has been controlled throughout the deformation of ρ_n to ρ'_n, any sequence obtained by sampling structures from these deformations converges to ρ.

Thus the problem of deforming a non-strongly convergent sequence to a strongly convergent sequence is reduced to finding appropriate paths in U. It is not known if this can be done in general; it is certainly possible for the once-punctured torus, and in general whenever the non-wrapping representation has conformal boundary a collection of once-punctured tori and four-times-punctured spheres.

3 The strongly convergent case

From now on we let S denote the once-punctured torus. The results in this section hold in greater generality, but for the sake of a clearer exposition we would like to minimize and standardize our notation for the rest of the paper.

If ρ is a strong limit of ρ_n and ρ'_n, both sequences in B, then to show that ρ is not a self-bumping point we must evidence paths σ_n connecting ρ_n and ρ'_n, as in lemma 2. At this point we make first use of a beautiful result of Kleineidam which characterizes strong convergence of sequences in $QF(S)$.

To state Kleineidam's results we need to introduce some notation. Let $\mathcal{D}_c \subset \pi_1(S)$ be the set of (isotopy classes of) simple closed curves γ in S, not homotopic into a puncture, so that $\rho(\gamma)$ can be represented on the boundary of the convex core of N_ρ by a curve which is homotopic into a puncture.

Let $\rho = Q(X_1, X_2)$, $X_i \in Teich(S)$. Let γ be a closed curve on S and define $l_\rho(\gamma)$ to be $\min_i\{length_{X_i}(\gamma)\}$, where $length_{X_i}(\gamma)$ is the length in X_i of the geodesic representative of γ.

We are now in a position to state

Theorem 4 *(Kleineidam [25])*
Let $\rho_n \in QF(S) = int(AH(S)$ converge algebraically to ρ.
Then ρ_n converges strongly to ρ if and only if for all $\gamma \in \mathcal{D}_c$, $l_{\rho_n}(\gamma) \to 0$, $n \to \infty$.

A path σ_n between ρ_n and ρ'_n where the curves in $\mathcal{D}_c$ are short is easy to find: simply extend the curves in $\mathcal{D}_c$ to a pair of pants decomposition for S and use Fenchel-Nielson coordinates on Teichmuller space. There is no need for a sequence of such paths to satisfy the hypotheses of lemma 2 - sequences formed by sampling these paths may not converge, and if they do, they need not converge to ρ. Thus we need to choose the paths σ_n so that the paths in $Teich(S)$ corresponding to σ_n converge to the end invariants for ρ. Thurston's double-limit theorem allows us to conclude that a sequence sampled from a suitably constructed sequence of paths converges. If this sequence converges strongly then the following proposition, combined with Minsky's solution to the Ending Lamination Conjecture for punctured-torus groups, tells us that the limit is ρ.

The Ending Lamination Theorem [31], [32], [10], states that the isometry type of a hyperbolic 3-manifold is determined uniquely by a collection of topological, analytic and combinatorial data. The topological data is its homeomorphism type. The analytic and combinatorial data form what are referred to as the *ending invariants*.

Let N be a hyperbolic 3-manifold (with finitely generated fundamental group). Let $\mathcal{H}$ be a $\pi_1(N)$-invariant family of horoballs in $\mathbb{H}^3$, so that the stabilizer of a horoball in $\mathcal{H}$ is a maximal parabolic subgroup of $\pi_1(N)$ and the horoballs in $\mathcal{H}$ are pair-wise disjoint. Let $N^0 = (\mathbb{H}^3 \setminus \mathcal{H})/\pi_1(N)$; the frontier FrN^0 of N^0 in N is a collection of open annuli and tori. By [29] there is a relative compact core M for (N^0, FrN^0). The components of $\partial M \setminus \partial M \cap FrN^0$ are in one-to-one correspondence with the ends of N^0. Let S_0 denote the union of the components of $\partial M \setminus \partial M \cap FrN^0$ which do not correspond to geometrically finite ends. The end invariants for N consist of ∂N together with a geodesic lamination λ on S_0, called the *ending lamination*.

The lamination λ is *maximally arational*: every component of the complement of λ in S_0 is either simply connected or a once-punctured disk. Moreover, if μ is any measured lamination with support equal to λ, then μ is *non-realizable* in N: there is no pleated surface in N with pleating lamination μ. Finally, λ is minimal and every leaf is dense.

It is worthwhile to discuss how a sequence of structures in $Teich(S)$ converges to a projective measured lamination- the Thurston compactification of Teichmuller space [38]. There are cleaner approaches (see for example [24], [34] where $\mathbb{R}$-trees are used), but what follows is to at least one of the authors' liking.

Fix an arbitrary maximal lamination λ on S, so that $S \setminus \lambda$ is a collection of ideal triangles - stronger than we need: it would suffice to have every component of $S \setminus \lambda$ be simply connected. Fix a hyperbolic structure X on S and foliate each component of $S \setminus \lambda$ with segments of horocycles centered at the ideal vertices - there is a three-pronged singularity to the foliation. In this way we obtain a singular measured foliation on S, where the measure of a curve transverse to the foliation is its length in X. The singular locus of the measured foliation can be blown up, producing a measured lamination $\mu(X)$ on S. The intersection with $\mu(X)$ of a curve transverse to $\mu(X)$ is within a bounded error of the length of the curve in X.

In Thurston's compactification of Teichmuller space, a divergent sequence $X_n \in Teich(S)$ converges to $[\mu] \in \mathcal{PML}(S)$ iff the sequence $\mu(X_n)$ converges projectively to $[\mu]$. Note that in particular a simple closed curve γ on S has $length_{X_n}(\gamma)$ bounded if and only if $i(\gamma, \mu) = 0$ for any μ in the projective class $[\mu]$.

The following lemma is true in more general circumstances. Mimicking the proof below for the analogous statement for manifolds with incompressible boundary requires a strengthening of Kleineidam's result.

Lemma 3 *Suppose that $\rho_n \in QF(S)$ converges to ρ strongly. Then the end invariants of ρ_n converge to the end invariants of ρ.*

Proof

Let $\rho_n = Q(X_n, Y_n)$ for conformal structures $X_n, Y_n \in Teich(S)$. Let $\mathcal{D} \subset \pi_1(S)$ be the set of curves in S, not homotopic into a puncture of S, so that every curve in $\mathcal{D}$ is mapped to a parabolic isometry by ρ. $\mathcal{D}$ contains $\mathcal{D}_c$, the set of curves which are parabolic under ρ, and which have representatives in ∂N_ρ that are homotopic in ∂N_ρ into a puncture that is not a component of ∂S.

For $\gamma \in \mathcal{D}_c$ let $\gamma_n \subset X_n \sqcup Y_n$ be the shortest curve in $X_n \sqcup Y_n$ in the homotopy class of γ. Theorem 4 tells us that the length of γ_n tends to zero as n tends to infinity. Let $\mathcal{D}_c^n \subset X_n \sqcup Y_n$ be the set of such γ_n.

Given $\gamma \in \mathcal{D}_c$, there exists $n_0 > 0$ so that for all $n > n_0$, γ_n is contained in X_n, say. Thus for n sufficiently large, $X_n \sqcup Y_n \setminus \mathcal{D}_c$ is a sequence of conformal structures on a disconnected surface T. Let S' be a component of T and let $Z_n \subset X_n \sqcup Y_n \setminus \mathcal{D}_c$ be the associated sequence of conformal structures on S'.

Suppose that there is an element $\gamma \in \mathcal{D} \setminus \mathcal{D}_c$ which has a representative on S'. If the length of γ in Z_n tends to zero, then the induced conformal structures on some subsurface S' containing the representative of γ diverge in the appropriate Teichmuller space, by Kleineidam's theorem. It follows that the subsurface must be all of S'.

Thus we assume that any representative in S' of a curve in $\mathcal{D} \setminus$

$\mathcal{D}_c$ has length bounded away from zero in Z_n. Since the genus of S' is fixed, convergence of Z_n is independent of the choice of basepoint, provided that the basepoint is appropriately chosen in the thick part of Z_n. That is, if w_n and w'_n are both points in the same component of the thick part of Z_n and the sequences (Z_n, w_n) and (Z_n, w'_n) both converge geometrically to Riemann surfaces, then there are basepoint preserving conformal maps between the two limits. Thus we suppress the choice of basepoint.

If Z_n converges to a conformal structure Z, then by strong convergence, Z is a structure on S' and is a component of ∂N_ρ.

If Z_n does not converge to a conformal structure, then Z_n converges to a projective measured lamination $[\mu]$ on S'. We want to show that the ending lamination on $S_0 \cap S' \subset S' \subset M \setminus \partial N_\rho$ is the support of μ. Let μ be a measured lamination in the projective class of $[\mu]$. We want to show that μ is non-realizable and maximally arational.

Let μ be the limit $\mu = c_n \alpha_n$ of weighted simple closed curves α_n. By a theorem of Sullivan [18] there is a uniform constant $K \geq 1$ so that for any geodesic β in Z_n, $length_{Z_n}(\beta) \geq \frac{1}{K} length_{N_{\rho_n}}(\beta^*)$, where β^* is the geodesic representative of β in N_{ρ_n}. Hence the length of μ in N_ρ is at most a bounded constant times the length of μ in Z_n.

We now show that μ is non-realizable. Using continuity of the length function $length : AH(S) \times \mathcal{ML}(S) \longrightarrow \mathbb{R}$ (see Brock [11]) it is enough to show that the length of μ in N_{ρ_n} tends to zero as n tends to infinity. The length of μ in N_{ρ_n} is a bounded multiplicative factor of the length of μ in Z_n. By convergence of Z_n to $[\mu]$ we know that there is a sequence $d_n > 0$ tending to infinity so that $length_{Z_n}(d_n \mu)$ is bounded. Hence it follows that $length_{Z_n}(\mu)$ tends to zero, so that $length_{N_\rho}(\mu) = 0$.

We claim that μ is maximally arational in S'. Seeking a contradiction, suppose that there is an essential closed curve γ in S' which is not a component of μ, but which has zero intersection number with μ. From the definition of convergence of Z_n to $[\mu]$ it follows that the length of γ in Z_n is bounded above throughout the sequence. If the length of γ is bounded away from zero then it follows that if a basepoint w_n is chosen for Z_n on γ, that (Z_n, w_n) converges geometrically to a surface, which is absurd since S' has no subsurface contained in the conformal boundary of N_ρ. So the length of γ must tend to zero, or γ must not exist. Hence every component of $S' \setminus \mu$ is simply connected or a punctured disk.

Choosing a maximal ρ-invariant family of pair-wise disjoint horoballs $\mathcal{H}$, it follows that the punctured disk components of $S' \setminus \mu$ correspond to components of $\partial N^0 = Fr\mathcal{H}/\rho$, so that μ is a lamination on $S_0 \cap S'$.

It follows that the support of μ is a component of the ending lamination for N_ρ. Repeating the above analysis for the other components of $\partial M \setminus \partial N_\rho$ we obtain the result.

q.e.d.

The next lemma completes the strongly-convergent case.

Lemma 4 *Suppose ρ_n and ρ'_n both converge strongly to ρ. Then for each n there a path σ_n in $QF(S)$ joining ρ_n to ρ'_n. Moreover, any sequence of the form $\{\rho_n(t_n)\}$, $\{t_n\} \subset [0,1]$ converges, and converges to ρ.*

Proof

By lemma 3 the end invariants for ρ_n and ρ'_n converge to those of ρ. For each n let $\sigma_n : [0,1] \longrightarrow QF(S)$ be a path so that

1. $\sigma_n(0) = \rho_n; \sigma_n(1) = \rho'_n$;
2. $l_{\sigma_n(t)}(\gamma) \leq \max\{l_{\rho_n}(\gamma), l_{\rho'_n}(\gamma)\}$ for all $t \in [0,1]$ and $\gamma \in \mathcal{D}_c$;
3. For any sequence $\{t_n\}$, $t_n \in [0,1]$, the sequence of end invariants for $\sigma_n(t_n)$ converges to the end invariants for ρ.

Now apply Minsky's lemma 12.1 of [31] to conclude that each such sequence $\{\sigma_n(t_n)\}$ converges, and converges to ρ_n.

q.e.d.

3.1 The non-strong case

Suppose now that ρ_n converges to ρ, but not strongly.

The manifolds N_{ρ_n} are defined only up to orientation preserving isometry. Equivalently, $\rho_n(\pi_1(S))$ is only determined up to conjugation. We fix a baseframe ω_n for N_{ρ_n} by requiring that it lifts to a fixed baseframe $\tilde{\omega}$ for $\mathbb{H}^3$. In this way, ρ_n gives rise to a sequence of manifolds with baseframe (N_n, ω_n). Up to a choice of subsequence, we may assume that these manifolds with baseframe converge geometrically to a hyperbolic manifold with baseframe $(\hat{N}, \hat{\omega})$.

Let $B(x, R)$ denote the ball of radius R in $\hat{N}$ centered at x. Suppose that $\hat{\omega}$ is a basis for $T_p(\hat{N})$. A more precise formulation of the convergence of (N_n, ω_n) to $(\hat{N}, \hat{\omega})$ is that there exists two sequences of real numbers $K_n \geq 1$, $R_n > 0$, and smooth K_n-biLipschitz embeddings

$$f_n : B(p, R_n) \longrightarrow N_n$$

such that

1. $Df_p(\tilde{\omega}) = \omega_n$;
2. $R_n \to \infty$, $K_n \to 1$ as $n \to \infty$

(See [16], [6] for more details).

There is a covering map $\pi : N_\rho \longrightarrow \hat{N}$. If $K \subset N_\rho$ is a compact subset of N_ρ then a lemma of Canary and Minsky allows us to relate the homotopy class of $f_n \circ (\pi|_K)$ to the homomorphism $\rho_n \circ \rho^{-1}$. We record it as a lemma for later reference.

Lemma 5 *With the notation used above, the composition* $\rho_n^{-1} \circ (f_n)_* \circ (\pi|_K)_*$ *converges algebraically to* $\rho^{-1}|_{\pi_1(K)}$.

The compact subset of N_ρ that we are going to be concerned with is a compact core M of N_ρ which embeds in $\hat{N}$ under the covering map π.

Notation Let $\mathcal{D}'_c \subset \mathcal{D}_c$ be the collection of curves γ whose length in the conformal boundary of N_n is bounded away from zero throughout the sequence, though the length in the manifold of the geodesic representative γ_n^* of γ is tending to zero. Since the convergence of ρ_n to ρ is not strong, there is at least one curve in $\mathcal{D}'_c$. For simplicity we will consider that $\mathcal{D}'_c$ is a single curve γ - the general case is not more difficult, it only involves more notation.

As we said in the proof outline, we will make use of the Drilling Theorem of Brock and Bromberg [9].

Theorem 5 *(Drilling Theorem) Given* $L > 1$ *and* $\epsilon_0 < \mathcal{M}_3$*, there is an* $\epsilon > 0$ *so that if* N *is a minimally parabolic, geometrically finite hyperbolic manifold and* γ *is a geodesic in* N *with length at most* ϵ*, then there is a L-biLipschitz diffeomorphism of pairs*

$$h : (N \setminus \mathbb{T}^{\epsilon_0}(\gamma), \partial\mathbb{T}^{\epsilon_0}(\gamma)) \longrightarrow (\hat{N} \setminus \mathbb{P}^{\epsilon_0}(\gamma), \partial\mathbb{P}^{\epsilon_0}(\gamma))$$

where $\hat{N}$ *is the complete hyperbolic structure on* $N \setminus \gamma$ *with the same conformal boundary, and* $\mathbb{P}^{\epsilon_0}$ *is the rank-2 cusp component of the thin part* $(\hat{N})^{\leq \epsilon_0}$ *corresponding to* γ.

The map h is actually the end point of a one-parameter family of maps obtained by reducing the hyperbolic cone angle of γ from 2π to zero. Thus, if $i : \hat{N} \setminus \mathbb{P}^{\epsilon_0} \longrightarrow N$ is "inclusion" (considering $\hat{N}$ as $N \setminus \gamma$), the composition $i \circ h$ extends to a self-map of N which is homotopic to the identity.

We will want to reverse the effect of drilling, and the reverse of drilling is Dehn-filling. Thus we will make use of the Dehn filling theory of Hodgson and Kerckhoff [20]. In particular we will use the following theorem. Note that in the original formulation of Hodgson and Kerckhoff's theorem, the hyperbolic manifold is required to be finite volume, but using work of Bromberg [13] one can extend the theorem to the following.

Theorem 6 *(Filling Theorem) Let N be a complete, minimally parabolic, geometrically finite hyperbolic 3-manifold with a torus cusp $\mathbb{P}$. Let δ be a simple closed curve on $\partial\mathbb{P}$ whose Euclidean length on $\partial\mathbb{P}$ is denoted by L. If the normalized length of δ, $\hat{L} = \frac{L}{\sqrt{area(\partial\mathbb{P})}}$ is at least 7.515 then the manifold $N(\delta)$ obtained by Dehn filling along δ is a minimally parabolic, geometrically finite hyperbolic 3-manifold.*

Recall the following notation introduced earlier. $Q : Teich(S) \times Teich(S) \longrightarrow QF(S)$ is the Bers homeomorphism of simultaneous uniformization. If $\hat{B}$ denotes the component of $MP(\hat{M})$ consisting of geometrically finite, minimally parabolic structures with markings homotopic to homeomorphisms, then we have another homeomorphism from simultaneous uniformization $\hat{Q} : Teich(S) \times Teich(S) \longrightarrow \hat{B}$. We observe that the Drilling theorem gives a path from $Q(X,Y)$ to $\hat{Q}(X,Y)$ through cone structures, and the Dehn-filling theorem is a path in the other direction.

To apply theorem 6 in our situation we will need the following estimate.

Lemma 6 *Choose $L > 1$ and $\epsilon_0 < \mathcal{M}_3$. There is a proper continuous decreasing function $G : (0,\infty) \longrightarrow (0,\infty)$, dependent only on ϵ_0, so that the following holds.*

Let $\rho = Q(X,Y)$, and let γ be a geodesic in N_ρ of length at most ϵ, where ϵ is given by the Drilling Theorem. Let $\hat{N} = \hat{Q}(X,Y)$ be a minimally parabolic structure on $N \setminus \gamma$.

Then the torus cusp in $\hat{N}$ the meridian curve is normalized flat length at least

$$\frac{1}{L^2} G(length(\gamma))$$

In particular, for γ sufficiently short the normalized flat length of the meridian curve in $\hat{N}$ is greater than 7.515.

Proof

A version of the Margulis lemma implies that γ has a large tube radius R. Thus any closed geodesic intersecting the R-neighbourhood of γ must pass through it. Let $r_0 \leq R$ be the radius of $\mathbb{T}^{\epsilon_0}(\gamma)$. If $R - r_0 > \epsilon_0/2$ then any closed geodesic that intersects the Margulis tube for γ has length greater than ϵ_0, a contradiction. Thus the radius r_0 of the Margulis tube is a proper, continuous, decreasing function of the length of γ.

The drilling theorem gives an L-biLipschitz diffeomorphism of pairs between $\partial\mathbb{T}^{\epsilon_0}(\gamma)$ and $\partial\mathbb{P}^{\epsilon_0}(\gamma)$. The length of the meridian curve in $\partial\mathbb{T}^{\epsilon_0}(\gamma)$ is $\pi\sinh(2r_0)$, and the area of $\partial\mathbb{T}^{\epsilon_0}(\gamma)$ is $\pi\sinh(2r_0)length(\gamma)$. Thus the length of the meridian of $\partial\mathbb{P}^{\epsilon_0}(\gamma)$ is at least $\frac{1}{L}\pi\sinh(2r_0)$, and the area of $\partial\mathbb{P}^{\epsilon_0}(\gamma)$ is at most $L^2\pi\sinh(2r_0)length(\gamma)$. Thus the normalized flat length of the meridian in $\partial\mathbb{P}^{\epsilon_0}(\gamma)$ is at least $\frac{1}{L^2}\sqrt{\frac{\pi\sinh(2r_0)}{length(\gamma)}}$ as required.

q.e.d.

We need to discuss the*Maskit* slice of $AH(S)$, S the once-punctured torus. Fix a generating set A, B for $\pi_1(S)$, so that $[A, B]$ is the puncture in S. For a complex number μ with positive imaginary part, let ρ_μ be the representation of $\pi_1(S)$ defined by

$$\rho_\mu(A) = \begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix}$$

and

$$\rho_\mu(B) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

The Maskit slice $\mathcal{M} \subset \hat{\mathbb{C}}$ consists of those μ for which ρ_μ is discrete and faithful. A point in the interior of $\mathcal{M}$ determines a geometrically finite hyperbolic 3-manifold with one conformal boundary component a thrice-punctured sphere and the other conformal boundary component a once-punctured torus. The map which assigns to a point in the interior of $\mathcal{M}$ the conformal structure of the punctured torus component is a homeomorphism onto $Teich(S)$, and we denote its inverse by $m : Teich(S) \longrightarrow int(\mathcal{M})$.

Consider $\hat{Q}(X, Y)$, a minimally parabolic structure on $\hat{N}$. The boundary of the convex core of $\hat{Q}(X, Y)$ has a component facing X

and a component facing Y. The cover of $\hat{Q}(X, Y)$ corresponding to the fundamental group of the former component has holonomy $\rho_{m(X)}$ and the cover corresponding to the latter has holonomy $\rho_{\overline{m(\overline{Y})}}$. Thus there is a map $\hat{B} \longrightarrow AH(S)$ by $\hat{Q}(X, Y) \mapsto \rho_{\overline{m(\overline{Y})}}$.

We record this and other information in the following lemma.

Lemma 7 *$\hat{B}$ is homeomorphic to $int(\mathcal{M}) \times int(\mathcal{M})$ by the composition $(\mu, \eta) \mapsto \hat{Q}(m^{-1}(\mu), m^{-1}(\eta))$.*

Moreover, if $\hat{N}$ corresponds to $(\mu, \eta) \in int(\mathcal{M}) \times int(\mathcal{M})$, then the normalized flat length of the meridian curve in $\hat{N}$ is $\mathcal{L}(\mu, \eta) = \frac{1}{\sqrt{2}} \frac{|\mu - \overline{\eta}|}{\sqrt{\Im(\mu - \overline{\eta})}}$.

Proof

$\hat{Q}(X, Y)$ is obtained by amalgamating $\rho_m u$ and $\rho_{\overline{\eta}}$ along the common thrice-punctured sphere group, where $\mu = m(X)$ and $\eta = m(\overline{Y})$.

Note that

$$\rho_\mu(A)\rho_{\overline{\eta}}(A)^{-1} = \begin{pmatrix} 1 & \mu - \overline{\eta} \\ 0 & 1 \end{pmatrix}$$

so that the normalized length of the meridian curve is

$$\frac{1}{\sqrt{2}} \frac{|\mu - \overline{\eta}|}{\Im(\mu - \overline{\eta})}$$

q.e.d.

Lemma 8 *Fix $\eta \in int(\mathcal{M})$ and let $c > 0$ be given. There is a constant $k = k(c)$ so that if $\mu \in int(\mathcal{M})$ is such that $\mathcal{L}(\mu, \eta) > k$ then there is a path $\sigma : [0, \infty) \longrightarrow int(\mathcal{M})$ so that $\sigma(0) = \mu$, $\mathcal{L}(\sigma(t), \eta) > c$, and $\Im(\sigma(t)) \to \infty$ as $t \to \infty$.*

Proof

The paths in $int(\mathcal{M})$ on which $\mathcal{L}(\cdot, \eta)$ is constant are the intersections of $int(\mathcal{M})$ with horocycles based at $\overline{\eta}$. In fact, if $\eta = (x, y)$, then the paths on which $\mathcal{L}(\cdot, \eta)$ are a constant r lie on the circle centered at $(x, r - y)$ of radius r. Seeking a contradiction, we suppose that any path σ joining μ_0 to a point μ_1 with imaginary part of $\mu_1 > K$ must always enter the region in $int(\mathcal{M})$ cut off by the circle C centered at $(x, c - y)$ of radius c.

By theorem B of [31] $\mathcal{M}$ is homeomorphic to a closed disk less a boundary point, said boundary point corresponding to infinity. Let

B be the closed disk bounded by C and consider the components of $\mathcal{M} \setminus \mathcal{M} \cap B$.

If the imaginary part of η is sufficiently large then $B \cap \mathcal{M} = \emptyset$, and there is nothing to prove, as in this case any path in $\mathcal{M}$ will do. Moreover, $\mathcal{M}$ is invariant under translation by $z \mapsto z + 2$, so we may assume that η lies in a compact subset of $\mathcal{M}$. Hence if we can show that a bound exists for any given η, by compactness we will have proven that the bound is independent of η.

If there are finitely many components to $B \cap \mathcal{M}$ and the lemma is false then necessarily at least two of these components must be unbounded, so that a neighbourhood of infinity in $\mathcal{M}$ is disconnected, contradicting the above theorem of Minsky. If there are infinitely many components then they must accumulate on the intersection of C and $\mathcal{M}$, which is compact and locally connected, providing the contradiction in this case. We conclude that $\mathcal{M} \setminus \mathcal{M} \cap B$ has finitely many components, only one of which is unbounded. Thus for $\mathcal{L}(\mu, \eta)$ sufficiently large, μ must lie in the unbounded component.

q.e.d.

Theorem 7 *Suppose that $\rho \in AH(S)$ does not wrap. Let ρ_n be a sequence in $QF(S)$ that converges algebraically to ρ but not strongly. Then there is a sequence of paths $\sigma_n : [0,1] \longrightarrow QF(S)$ so that*

1. *$\sigma_n(0) = \rho_n$;*
2. *For all sequences $\{t_n\}$, $t_n \in [0,1]$, the sequence $\{\sigma_n(t_n)\}$ converges to ρ;*
3. *The sequence $\{\sigma_n(1)\}$ converges strongly to ρ.*

Proof

As above, let M be a compact core for $N = \mathbb{H}^3/\rho(\pi_1(S))$. Let $N_n = \mathbb{H}^3/\rho_n(\pi_1(S))$ and, suppressing the choice of baseframe, suppose that N_n converges geometrically to $\hat{N}$. Let f_n be the biLipschitz maps given from geometric convergence, and let $\pi : N \longrightarrow \hat{N}$ be the covering map. By assumption we may assume that M embeds in $\hat{N}$ under π, and for large enough n, $\pi(M)$ lies in the domain of f_n, so that $M_n = f_n(\pi(M))$ is a compact submanifold of N_n.

Since the convergence of ρ_n to ρ is not strong, there is a curve $\gamma \in \mathcal{D}'_c$. We assume, for simplicity, that $\{\gamma\} = \mathcal{D}'_c$. The manifold M_n is disjoint from the geodesic representative γ_n^* of γ in N_n.

Choose ϵ_0 so that for all n, M_n is disjoint from the Margulis tube $\mathbb{T}^{\epsilon_0}(\gamma_n^*)$. Let ϵ_n be given by the Drilling Theorem applied to $L_n = 1+1/n$ and ϵ_0. Let m_n be a positive integer so that the length of γ_n^* is less than ϵ_{m_n}.

Let $\hat{N}_n$ be the minimally parabolic, geometrically finite structure on $N_n \setminus \gamma_n^*$ with the same conformal boundary as N_n. Let h_n be the biLipschitz maps given by the Drilling Theorem.

Suppose that $\hat{N}_n = \hat{Q}(X_n, Y_n)$. Let k be the constant given by lemma 8 applied to $c = 7.515...$

Let δ_n be the meridian curve for the solid torus $\mathbb{T}^{\epsilon_0}(\gamma_n^*)$. By lemma 6 the normalized flat length of δ_n in $\hat{N}_n$ is at least

$$\frac{m_n^2}{(1+m_n)^2} G(length(\gamma_n^*))$$

which for sufficiently large n is larger than k.

Thus by lemma 8 there is a path $\beta_n : [0,\infty) \longrightarrow int(\mathcal{M})$ starting at $m(X_n)$ with the following properties:

- In $\hat{Q}(m^{-1}(\beta_n, Y_n)$ the normalized flat length of the meridian curve for the torus cusp is at least 7.515...
- The length of γ in $m^{-1}(\beta_n(t))$ tends to 0 with t.

There is a constant T_n so that for $t > T_n$, the length of γ in $m^{-1}(\beta_n(t))$ is less than $1/n$. We now truncate the path, re-parameterize, and compose with m^{-1} to produce a path $\alpha_n : [0,1] \longrightarrow Teich(S)$ by $\alpha_n(t) = m^{-1}(\beta_n(tT_n))$.

In $QF(S)$ there is a path $\rho_{n,t}$ given by $\rho_{n,t} = Q(\alpha_n(t), Y_n)$, and in $\hat{B}$ there is the analogous path $\hat{\rho}_{n,t}$ given by $\hat{\rho}_{n,t} = \hat{Q}(\alpha_n(t), Y_n)$. Note that $\hat{\rho}_{n,t}$ restricts to the identity representation on $\pi_1(h_n(M_n))$, for all t.

By the Filling Theorem there is a diffeomorphism $k_{n,t} : \mathbb{H}^3/\hat{\rho}_{n,t}(\delta_{n,t}) \longrightarrow \mathbb{H}^3/\rho_{n,t}(\pi_1(S))$, which is continuous in t, and for each t is $(1+e_n)$-biLipschitz off of $\mathbb{P}^{\epsilon_0}(\gamma_n^*)$, where $e_n \to 0$ as $n \to \infty$.

Set $\sigma_n(t) = (k_{n,t})_* \circ \hat{\rho}_{n,t} \circ (h_n)_* \circ (f_n)_* \circ \pi_* \circ \rho$.

Observe that $\sigma_n(0)$ is the representation ρ_n. Moreover, for each t, the complex length in $\sigma_n(t)$ of any curve in $\pi_1(S)$ is within $(1+1/m_n)(1+e_n)$ of that of the curve in N_n. It follows that sequences obtained by sampling the paths σ_n converge to ρ, as required.

q.e.d.

Theorem 8 *If $\rho \in AH(S)$ does not wrap then ρ is not a self-bumping point.*

Proof

Suppose ρ_n and ρ'_n, both sequences in $QF(S)$ converge to ρ. If both converge strongly then we are done by lemma 4. If ρ_n does not converge strongly, then by theorem 7 there is a sequence of paths $\sigma_n : [0,1] \longrightarrow QF(S)$ such that sequences obtained by sampling the paths σ_n converge ρ, and so that the sequence $\rho''_n = \sigma_n(1)$ converges strongly. Thus we reduce the non-strongly convergent case to the strongly convergent case.

q.e.d.

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