Non-wrapping of hyperbolic interval bundles

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Abstract

We demonstrate a condition on the boundary at infinity of a hyperbolic interval bundle N that guarantees that for any associated geometric limit there is a compact core for N which embeds under the covering map. The proof involves an analysis of the geometry of torus cusps in a hyperbolic manifold, and techniques of Anderson, Canary and McCullough [7]. Together with results of Holt-Souto [20] this shows that the locus of non-local-connectivity of the space of once-punctured torus groups is not dense, and describes a relatively open subset of the boundary of the space of once-punctured torus groups consisting of points of non-self-bumping.

1 Introduction

We consider the following question:

How does the internal geometry of a hyperbolic 3-manifold influence the local topology of the space of deformations of the hyperbolic structure?

Many classical results in the deformation theory of hyperbolic 3-manifolds can be interpreted as addressing this question. Calabi-Weil rigidity [35] states that if the structure is on a closed manifold then there are no local deformations, a precursor to Mostow-Prasad rigidity ([31], [34]). Marden ([24]) and Sullivan [37]) have shown that if the structure is geometrically finite without annulus cusps then the structure has a maximal dimensional quasi-conformal deformation space of deformations, so that any small enough neighbourhood of the structure is an open ball.

Suppose that N is an orientable hyperbolic 3-manifold with finitely generated fundamental group. By Scott [36] there is a compact submanifold M of N whose inclusion into N is a homotopy equivalence.

In fact, by Agol [2], and Calegari-Gabai [15] (also see Choi [18], the interior of M is homeomorphic to N, so that we can view N as a hyperbolic structure on the interior of M. It is natural to consider the space of all such metrics on the interior of M, a set that we will call $H_0(M)$. Formally, a point in $H_0(M)$ is represented by a hyperbolic 3-manifold N and a homeomorphism $h : int(M) \longrightarrow N$ (a marking), and two such pairs represent the same point if there is an orientation-preserving isometry between the hyperbolic 3-manifolds that respects the markings. We topologize $H_0(M)$ by embedding $H_0(M)$ into the representation variety $R(\pi_1(M))$ with the compact-open topology: $[h : int(M) \longrightarrow N] \mapsto h_*$. We call the resulting space $AH_0(M)$.

Immediately one runs into issues, in that $AH_0(M)$ is not closed in $R(\pi_1(M))$. If ρ is the limit of a sequence $\rho_n \in AH_0(M) \subset R(\pi_1(M))$, then there is a hyperbolic 3-manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$, and a homotopy equivalence $h : M \longrightarrow N_\rho$ with $h_* = \rho$, but N may not be homeomorphic to $int(M)$ and/or $h|_{int(M)}$ may not be homotopic to a homeomorphism. Thus we consider the larger space $AH(M)$ consisting of orientation-preserving isometry classes of marked hyperbolic 3-manifolds homotopy equivalent to M, topologized with via the embedding of $AH(M)$ into the representation variety, as above. (This topology is called the *algebraic topology.*)

By work of Marden [24], and Sullivan [37], the interior of $AH(M)$ is the space $MP(M)$ of *geometrically finite, minimally parabolic* structures (a formal definition will be given in a later section). In the case that ∂M contains no tori, to be minimally parabolic and geometrically finite is to be convex co-compact, or to be conformally compact (in the sense of conformally compact Einstein manifolds), or to be structurally stable. Work of Ahlfors, Bers, Maskit, and Kra ([4], [9], [23], [25]) gives a parameterization of each component of $MP(M)$ as the quotient of a Teichmuller space $Teich(S)$ by the subgroup of the mapping class group of S generated by Dehn twists about compressible curves - see [28]. In particular, when M has no such compressible curves in its boundary, each component of $MP(M)$ is a topological ball.

By Bromberg, Brock-Bromberg, and Brock-Canary-Minsky ([13], [11], [12], together with results of Agol [2] and Calegari-Gabai [15], and Ohshika [33], Kleineidam-Souto [22], and Namazi-Souto [32], $AH(M)$ is the closure of its interior. Hence it is natural to study the topology of $AH(M)$ by considering the closure of a component of $MP(M)$ or the closure of a collection of such components.

When two components of $MP(M)$ have intersecting closures, then those components are said to *bump*. A component B of $MP(M)$ is said to *self-bump* at a point $\rho \in \overline{B}$ provided that for all sufficiently small neighbourhoods V of ρ, the intersection $V \cap B$ is disconnected. Bumping was first demonstrated by Anderson and Canary [6], and Anderson, Canary and McCullough [7] later characterized which components of $MP(M)$ bump, in the case that M has incompressible boundary. Self-bumping was first demonstrated by McMullen [30], in the case that M is a trivial interval bundle over a hyperbolic surface, and using different techniques Bromberg and Holt [14] showed that if M contains a properly embedded, primitive annulus not homotopic into a torus cusp, then every component of $MP(M)$ bumps.

The phenomena of bumping and self-bumping are related to how a structure ρ in the boundary of $AH(M)$ is related to the set of hyperbolic 3-manifolds which arise as *geometric limits* of sequences converging to ρ. Briefly, a sequence $\rho \in AH(M)$ has an associated sequence of groups $\Gamma_n = \rho_n(\pi_1(M))$ and the geometric limit of ρ is the Gromov-Hausdorff limit of Γ_n (as subsets of $PSL_2(\mathbb{C})$). Often we will refer to the associated quotient manifold as the geometric limit of ρ_n. If ρ_n converges to ρ, then up to subsequence ρ_n converges geometrically to $\hat{N}$ [21], and there is a locally isometric covering map $N_\rho \longrightarrow \hat{N}$.

We say that a representation $\rho \in AH(M)$ *does not wrap* provided that for any geometric limit $\hat{N}$ for a sequence converging algebraically to ρ, there is a compact core K for ρ so that the restriction of the covering map $\pi|_K : K \subset N_\rho \longrightarrow \hat{N}$ is an embedding. Wrapping is related to bumping and self-bumping.

Eventually the image $\pi(K)$ can be mapped into the approximating manifolds by a mapping in the correct homotopy type [8], [17], so that the marked, oriented homeomorphism type of the approximates is then the same as that of N_ρ, which implies that the sequence eventually lies in a component B of $MP(M)$, independent of the sequence, so that ρ is not a bumping point.

The relationship between non-wrapping and self-bumping is more delicate.

Let F_2 be the rank two free group, with generators α and β. A *punctured torus group* is a discrete and faithful representation $\rho : F_2 \longrightarrow PSL_2(\mathbb{C})$ so that $\rho([\alpha, \beta])$ is parabolic. We denote by $AH(S_{1,1})$ the space of punctured torus groups with the algebraic topology.

Theorem 1 *(Holt-Souto [20])*
Let ρ be a punctured torus group. Then if ρ does not wrap then ρ is not a self-bumping point for $AH(S_{1,1})$.

Hence the property of not wrapping is intimately connected to the small scale topology of $AH(M)$ - though we remark that every structure on a handlebody does not wrap, but by [14] there is self-bumping for the deformation space of the handlebody. Thus perhaps more restriction needs to be placed on the choice of compact core in the definition of non-wrapping.

Anderson and Canary [5] showed that if M is acylindrical, or an I-bundle and ρ has no accidental parabolics, then ρ does not wrap. The analysis of Anderson, Canary and McCullough in [7] shows that if M has incompressible boundary and ρ has no accidental parabolics, then ρ does not wrap.

In this paper we will show that if suitable constraints are placed on the conformal boundary ∂N_ρ of N_ρ and M is an I-bundle, then ρ does not wrap. In fact we will be able to bound the *amount* of wrapping, in the case that ρ does wrap. We now make this more precise.

Suppose that $\rho \in AH(M)$ has an associated geometric limit $\hat{N}$. By Abikoff and Maskit [1] $\Gamma = \rho(\pi_1(M))$ admits a decomposition into subgroups $\Gamma_1, \ldots, \Gamma_k$, where $\mathbb{H}^3/\Gamma_i$ has no accidental parabolics, and Γ is obtained by Klein-Maskit combinations of $\Gamma_1, \ldots, \Gamma_k$. Thus for any accidental parabolic γ in N_ρ there is a subgroup Γ_γ of Γ equal to $\Gamma_i *_{<\rho(\gamma)>} \Gamma_j$ for some $i \neq j$, or $\Gamma_i *_{<\delta>}$, for some i and some $\delta \in \Gamma$.

There is a covering map $\pi_\gamma : \mathbb{H}^3/\Gamma_\gamma \longrightarrow \hat{N}$. By Anderson, Canary and McCullough [7] if the annulus cusp in $\mathbb{H}^3/\Gamma_\gamma$ corresponding to γ does not cover a torus cusp in $\hat{N}$ then there is a compact core for $\mathbb{H}^3/\Gamma_\gamma$ which embeds in $\hat{N}$ under the covering map. Hence we will assume that there is a torus cusp T in $\hat{N}$ which is covered by the annulus cusp for γ. It follows that $\pi_1(\hat{N})$ can be written as $\hat{\Gamma}_0 *_{\delta_0}$ for some parabolic δ_0 which, together with $\rho(\gamma)$, generates $\pi_1(T)$.

We define the *wrapping of* γ in $\hat{N}$ to be the smallest n such that π_γ factors through $\pi : \mathbb{H}^3/\Gamma_\gamma \longrightarrow \bar{N}$ and $\bar{\pi} : \bar{N} \longrightarrow \hat{N}$ so that

1. There is a compact core K for $\mathbb{H}^3/\Gamma_\gamma$ so that $\pi|_K$ is an embedding
2. $\pi_1(\bar{N}) = \hat{\Gamma}_0 *_{\delta_0^{n+1}}$.

Then the *wrapping of* γ is the largest n for which there exists $\hat{N}$ such that the wrapping of γ in $\hat{N}$ is n.

If $\rho \in AH(M)$ and $\gamma \subset \partial N_\rho$ is an accidental parabolic, let $\mathrm{t}(\gamma)$ be the infimum of lengths of curves in ∂N_ρ transverse to γ. We are now in a position to state the results of this paper.

Theorem 10 *There is a constant $K = K(\pi_1(S)) > 0$ so that if $\mathrm{t}(\gamma) \leq K$ then the wrapping of γ is 0.*

In fact we will prove the following theorem, from which theorem 10 follows as an immediate corollary.

Theorem 11 *Let n_0 be a non-negative integer. Then there is a constant $K = K(n_0, S)$ so that if $\mathrm{t}(\gamma)) \leq K$ then the wrapping of γ is at most n_0.*
The constant K grows logarithmically with n_0.

Work of Anderson, Canary and McCullough [7] shows that if the wrapping of every accidental parabolic in ∂N_ρ is zero, then ρ does not wrap. Hence we have

Corollary 12 *There is a constant $L = L(S)$ so that if $\mathrm{t}(\gamma) \leq L$ for all accidental parabolics γ in ∂N_ρ, then ρ does not wrap.*

Applying Holt and Souto [20] we have the following corollary.

Corollary 14 *Let $S_{1,1}$ be the once-punctured torus. There is a constant C so that $\rho \in AH(S_{1,1})$ is not a self-bumping point provided that any accidental parabolic for ρ has length at least C in ∂N_ρ.*

Note that this is condition on the accidental parabolics is an open condition. Hence we obtain the following corollary.
Corollary 15 *The locus of non-local connectivity in $\partial AH(S_{1,1})$ is not dense.*

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2 Preliminaries

2.1 The deformation space

A hyperbolic structure on the interior of M defines and is defined by the holonomy representation, a faithful representation $\rho : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ with discrete image. Thus it is natural to consider the space $AH(M)$ consisting of conjugacy classes of faithful representations $\rho : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ with discrete image. Associated to a point $\rho \in AH(M)$ is a hyperbolic 3-manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$. If ρ lies in $AH(M)$, then there is a homotopy equivalence $h_\rho : M \longrightarrow N_\rho$ with $(h_\rho)_* = \rho$. When M is a trivial interval bundle over a surface S, such a representation is always homotopic to a homeomorphism, so $AH(S) := AH(M)$ is the set of marked hyperbolic manifolds homeomorphic to $int(M) = S \times (-1, 1)$.

The topology on $AH(S)$ is the *algebraic topology* wherein $[\rho], [\rho'] \in AH(S)$ are close provided that there are representatives $\rho \in [\rho]$ and $\rho' \in [\rho']$ so that for all $\gamma \in \pi_1(S)$, $\rho(\gamma)$ is near to $\rho'(\gamma)$. Thus a sequence $[\rho_n]$ converges to $[\rho]$ if and only if there are representations $\rho_n \in [\rho_n]$ and $\rho \in [\rho]$ so that ρ_n converges to ρ in the compact-open topology.

We remark that while an element of $AH(S)$ is only defined up to conjugation in $PSL_2(\mathbb{C})$, we frequently consider that we have made a choice of representative, so that an element of $AH(M)$ is a representation $\rho : \pi_1(S) \longrightarrow PSL_2(\mathbb{C})$. This assumption presents no bias, since we will be studying sequences in $AH(S)$ converging to $[\rho] \in AH(S)$, so a choice of $\rho \in [\rho]$ pins down representatives for the elements in the sequence.

A 3-manifold pair is a pair (N, C) where N is a 3-manifold with boundary and $C \subset \partial N$ is a compact subsurface. A *relative compact core* for a 3-manifold pair (N, C) is a 3-manifold pair (M, P) so that M is a compact core for N; $P \subset C$; and every component D of C contains a unique component Q of P and $Q \cap D$ is a compact core for D. By McCullough, [29] every 3-manifold pair with finitely generated fundamental group has a relative compact core. Moreover, if N admits a manifold compactification then any two relative compact cores for (N, C) are isotopic [27].

A *pared 3-manifold* is a 3-manifold pair (M, P), where M is a compact 3-manifold with boundary, and $P \subset \partial M$ is a collection of annuli and tori so that

1. Every component of P is incompressilbe in M and represents a maximal abelian subgroup of $\pi_1(M)$.
2. Any essential, properly embedded annulus with both boundary components in P is properly homotopic into P
3. Any essential torus in M is homotopic into P.

By Thurston, a pared 3-manifold is hyperbolizable, and associated to a pared 3-manifold (M, P) is a relative deformation space $AH(M, P)$ defined to be the subset of $AH(M)$ consisting of (conjugacy classes of) those representations $\rho : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ so that for any closed curve γ homotopic into P, $\rho(\gamma)$ is parabolic.

2.2 Cores and ends

Isometries of $\mathbb{H}^3$ extend continuously to conformal automorphisms of the Riemann sphere, $\widehat{\mathbb{C}}$, the boundary at infinity of $\mathbb{H}^3$. But while a discrete, finitely generated subgroup Γ of hyperbolic isometries acts properly discontinuously on $\mathbb{H}^3$ it need not act properly discontinuously on $\widehat{\mathbb{C}}$. We define the *domain of discontinuity* or *regular set* of Γ, to be the largest open subset of $\widehat{\mathbb{C}}$ upon which Γ acts properly discontinuously. Denoted by $\Omega(\Gamma)$, the quotient $\Omega(\Gamma)/\Gamma$ is a Riemann surface, of finite type by Ahlfors' finiteness theorem [3]. This quotient surface $\Omega(\Gamma)/\Gamma$ is called the *conformal boundary* or *boundary at infinity* of $\mathbb{H}^3/\Gamma$, and is denoted $\partial\mathbb{H}^3/\Gamma$.

When $\Gamma = \rho(\pi_1(S))$, $\rho \in AH(M)$, then we denote the domain of discontinuity by $\Omega(\rho)$, and of course the conformal boundary is now ∂N_ρ.

The *limit set* $\Lambda(\Gamma)$ for Γ is the complement in $\widehat{\mathbb{C}}$ of $\Omega(\Gamma)$. It is the smallest, closed, non-empty Γ-invariant subset of $\widehat{\mathbb{C}}$. The convex hull of $\Lambda(\Gamma)$ is also Γ-invariant, and the quotient is a convex submanifold of $\mathbb{H}^3/\Gamma$ called the *convex core*. The convex core is the smallest convex submanifold whose inclusion into $\mathbb{H}^3/\Gamma$ is a homotopy equivalence.

Again, when Γ is the image of (a representative of) an element of $\rho \in AH(M)$, then the limit set of Γ is denoted $\Lambda(\rho)$.

Set $N_\epsilon \subset N$ to be the *ϵ-thin* part of N: N_ϵ consists of those points in N which have injectivity radius less than ϵ. The complement in N is the *ϵ-thick* part of N. The Margulis lemma implies that if ϵ is chosen sufficiently small then every component of N_ϵ is either a regular neighbourhood of a short geodesic (a *Margulis tube*); a thickened annulus $S^1 \times \mathbb{R} \times \mathbb{R}^+$, called a *rank-1* or *annulus cusp*; or a thickened torus $S^1 \times S^1 \times \mathbb{R}^+$, called a *rank-2* or *torus cusp.* The fundamental group of an annulus cusp is an infinite cyclic group generated by a parabolic isometry. The fundamental group of a torus cusp is rank-2 free abelian, and is generated by two parabolic isometries sharing a common fixed point in $\widehat{\mathbb{C}}$. A discrete subgroup of $PSL_2(\mathbb{C})$ is *elementary* if it is (virtually) abelian.

A *horoball* based at infinity is a subset of $\mathbb{H}^3$ of the form $\tilde{H}_c = \{(z,t) \mid t > c\}$, for some $c > 0$. If $J =< z \mapsto z+1 >$ is a maximal parabolic subgroup of a Kleinian group Γ, then for c sufficiently large (greater than 1 suffices) the quotient $H_c = \tilde{H}_c/J = \tilde{H}_c/\Gamma$ is an annulus cusp in $\mathbb{H}^3/\Gamma$ for the subgroup J. If $J =< z \mapsto z+1, z \mapsto z+\tau >$, $\tau \notin \mathbb{R}$, is a maximal parabolic subgroup of Γ then, again for c sufficiently large, $H_c = \tilde{H}_c/J = \tilde{H}_c/\Gamma$ is a torus cusp in $\mathbb{H}^3/\Gamma$ for the subgroup J. Every parabolic elementary group is conjugate to an elementary group fixing infinity, and hence we can define horoballs and obtain models of cusps as above for arbitrary maximal parabolic subgroups of Γ.

In $\mathbb{H}^3$ let $\mathcal{H}$ be a Γ-invariant collection of horoballs based at the parabolic fixed points of Γ, so that for every $H \in \mathcal{H}$, $H/Stab_\Gamma(H) = H/\Gamma$. If $N = \mathbb{H}^3/\Gamma$, then set $N^0 = (\mathbb{H}^3 \setminus \mathcal{H})/\Gamma$. Set $C = \partial\mathcal{H}/\Gamma = Fr(N^0)$.

Let (M_0, P_0) be a relative compact core for (N^0, C). The components of $\partial M_0 \setminus P_0$ are in bijection with the *ends* of N^0, where a component of $\partial M_0 \setminus P_0$ corresponds to the component of $N^0 \setminus M_0$ which it bounds. An end of N^0 is *geometrically finite* if it has a neighbourhood disjoint from the convex core of N. Otherwise it is geometrically infinite. If every end of N^0 is geometrically finite then we say that N is geometrically finite, and in this case $\partial M_0 \setminus P_0$ is homeomorphic to ∂N.

The components of ∂N correspond to the *geometrically finite ends* of N^0, via the nearest-point retraction map. Thus we will often identify a component of $M_0 \setminus P_0$ with the component of ∂N "facing" it.

An *accidental parabolic* for a hyperbolic 3-manifold N is a simple closed curve in ∂N which can be homotoped in N to have arbitrarily short length, but is not homotopic in ∂N into a puncture. Equivalently, an accidental parabolic is a simple closed curve in ∂M_0 which is not homotopic into P_0 in ∂M_0, but is homotopic into P_0 in M_0.

3 The geometry of torus cusps

We prove our results by showing that geodesics that correspond to curves that are short on the conformal boundary cannot wrap around torus cusps. Thus we need to know that there are not arbitrarily "skinny" torus cusps. This section makes this notion explicit.

Lemma 2 *For $c \in \mathbb{C}$, $|c| > \sqrt{2}$, define two infinite cyclic parabolic groups*

$$\Gamma_1 =< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$$

and

$$\Gamma_2 =< \begin{pmatrix} 1 & 0 \\ c^2 & 1 \end{pmatrix} >$$

For $\mu \in \mathbb{C}$, with $0 \leq \operatorname{Re} \mu < 1$, and $\operatorname{Im} \mu > 0$ set

$$\delta_\mu = \begin{pmatrix} 0 & \frac{i}{c} \\ ic & -ic\mu \end{pmatrix}$$

so that $\delta_\mu \Gamma_1 \delta_\mu^{-1} = \Gamma_2$.

Then

1. *$\exists K_1 > 0$ so that if $\operatorname{Im} \mu > K_1$ then $\Gamma_\mu = \Gamma_1 *_{\delta_\mu}$ is geometrically finite;*
2. *Given $\epsilon > 0$ there exists $K_2 > 0$ so that if $\operatorname{Im} \mu > K_2$, then $\partial \mathbb{H}^3/\Gamma_\mu$ contains a simple closed curve α so that $< \alpha >= \Gamma_1$, and the length of γ in the Poincare metric is less than ϵ.*

Proof

The first statement is an application of the 2nd Klein-Maskit combination theorem [26], and the second statement is an exercise in the Poincare metric - simply compare the Poincare metric to the pseudo-hyperbolic metric.

□

Lemma 3 *Let $S_{1,1}$ denote the genus one surface with one boundary component, and let α be a non-peripheral, simple closed curve on $S_{1,1}$.*

Set $M = S_{1,1} \times [-1,1]$, and set $P = \mathcal{N}(\alpha) \times \{-1\} \subset \partial M$. For a constant $K > 0$, set

$$B = \{\rho \in AH(M,P) \mid \text{length}_{N_\rho}(\partial S^*_{1,1}) \leq K\}$$

Let $\rho \in B$ be a sequence so that the length of $\alpha \times \{1\}$ in ∂N_ρ is bounded below. Then there is a sequence of homeomorphisms $\phi_n : (M,P) \longrightarrow (M,P)$, whose restriction to $S_{1,1} \times \{1\}$ is a power of a Dehn twist about α, so that the sequence $\rho \circ (\phi_n)_*$ has a convergent subsequence.

Proof

Let $S_{0,1,2}$ be the surface of genus zero, with one boundary component and two punctures p_1 and p_2. Let $M' = (S_{0,1,2} \setminus \mathcal{N}(p_1) \cup \mathcal{N}(p_2)) \times [-1,1]$ and set $P' = \partial(\mathcal{N}(p_1) \cup \mathcal{N}(p_2)) \times [-1,1] \subset \partial M'$. The punctures determine two subgroups of $\pi_1(M)$, G_1 and G_2, where $G_i = \pi_1(\partial\mathcal{N}(p_i))$. If β is the boundary component of $S_{0,1,2}$, then for $K > 0$ the set

$$B' = \{\rho \in AH(M',P') \mid \text{length}_{N_\rho}(\beta^*) \leq K\}$$

is compact.

Any $\rho \in B$ is uniquely determined as a pair (ρ', δ), where $\rho' \in B'$, and $\delta \in PSL_2(\mathbb{C})$ is an element conjugating $\rho'(G_1)$ to $\rho'(G_2)$, and the assignment is continuous with respect to the appropriate topologies. By conjugating ρ we may assume that $\rho'(G_1) = \Gamma_1$, and $\rho'(G_2) = \Gamma_2$, using the notation of lemma 2. Then δ will have the form δ_μ for some $\mu \in \mathbb{H}$. Let $\mathrm{D} \subset \mathbb{H}$ parameterize the set of such δ. The Dehn twist about α acts on D as a real translation, and a fundamental set D_0 for the action is contained in the set $\{\mu \in \mathbb{H} \mid 0 \leq \Re(\mu) \leq 1\}$.

Thus if ρ_n is a sequence in B, there is an associated sequence (ρ'_n, δ_n). The sequence ρ'_n converges, up to subsequence, and so ρ_n converges provided that δ_n has a convergent subsequence.

There is a sequence of powers of the Dehn twist about α which translate the δ_n into a sequence δ'_n lying in D_0. Moreover, by lemma 2 the set $\{\delta'_n\}$ avoids a neighbourhood of ∞ and hence lies in a compact set. Thus the sequence δ'_n has a convergent subsequence. It follows that there is a sequence of homeomorphisms $\phi_n : (M,P) \longrightarrow (M,P)$, which restrict to powers of Dehn twists about α on $S_{1,1} \times \{1\}$, so that $\rho_n \circ (\phi_n)_*$ converges, as required.

□

Lemma 4 *Let $S_{0,4}$ denote the genus zero surface with four boundary components, and let α be a non-peripheral, simple closed curve on $S_{0,4}$. Set $M = S_{0,4} \times [-1,1]$ and set $P = \mathcal{N}(\alpha) \times \{-1\} \subset \partial M$. For a constant $K > 0$ set*

$$B = \{\rho \in AH(M,P) \mid \forall \gamma \subset \partial S_{0,4}, \text{length}_{N_\rho}(\gamma *) \leq K\}$$

Let $\rho_n \in B$ be a sequence so that the length of $\alpha \times \{1\}$ in ∂N_ρ is bounded below. Then there is a sequence of homeomorphisms $\phi_n : (M,P) \longrightarrow (M,P)$, whose restriction to $S_{0,4} \times \{1\}$ is a power of the Dehn twist about α, so that $\rho_n \circ (\phi_n)_*$ has a convergent subsequence.

Proof

Let $S_{0,2,1}$ denote the surface of genus 0, with two boundary components and one puncture, p. Let $M' = (S_{0,2,1} \setminus \mathcal{N}(p)) \times [-1,1]$, and set $P' = \partial\mathcal{N}(p) \times [-1,1] \subset \partial M'$. If β_1 and β_2 are the boundary components of $S_{0,2,1}$, then for any $K > 0$ the set

$$B' = \{\rho \in AH(M,P) \mid \text{length}_{N_\rho}(\beta_i^*) < K, i = 1,2\}$$

is compact.

The puncture p determines a subgroup $G = \pi_1(\partial\mathcal{N}(p))$, and $\pi_1(M)$ is isomorphic as an abstract group to the amalgamation $\pi_1(M') *_G \delta\pi_1(M')\delta^{-1}$, where δ commutes with G. Hence any $\rho \in B$ is uniquely determined as a triple (ρ_1, ρ_2, δ), where $\rho = \rho_1 *_{\rho_0} \delta\rho_2\delta^{-1}$, for $\rho_i \in B'$, $\rho_0 = \rho_1|_G = \rho_2|_G$, and δ in the centralizer of $\rho_0(G)$ in $PSL_2(\mathbb{C})$.

Thus, by compactness of B', any sequence ρ_n in B has a convergent subsequence, provided that the associated sequence δ_n in the centralizer of, say, $< z \mapsto z+1 >$, converges in $PSL_2(\mathbb{C})$. As in lemma 3 the set of possible δ is parameterized by a subset D of $\mathbb{H}$, and Dehn twists about α act on this space, with fundamental set D_0 contained in $\{\mu \in \mathbb{H} \mid 0 \leq \Re\mu \leq 1\}$. Thus there is a sequence of Dehn twists which translate the sequence δ_n to a sequence δ'_n contained in D_0. But similar to lemma 2 if such a sequence δ'_n diverges, then the length of $\alpha \times \{1\}$ in ∂N_ρ would tend to zero, contradicting our hypothesis.

□

Let F be an oriented (but not necessarily connected) surface without boundary that is not a sphere. A *compression body* is either a handlebody or a compact 3-manifold with boundary obtained by adding handles $D^2 \times I$ to $F \times [-1, 1]$ via attaching maps $f : D^2 \times \{-1, 1\} \longrightarrow F \times \{1\}$.

By a result of Scott [36] any 3-manifold with finitely generated fundamental group has a compact core (a *Scott core*); that is, a compact submanifold whose inclusion is a homotopy equivalence. Given a compact 3-manifold, M, by Bonahon [10] there is a compression body neighbourhood of the compressible boundary components of ∂M, and the compression body is unique up to isotopy in M. Removing the compression body neighbourhood and taking the closure in M gives the *maximal incompressible core* of M, see [16].

We note that the following theorem cannot be strengthened to include the case when M is a handlebody - the case when the maximal incompressible core is empty, or when K is a compression body constructed by adding handles to a union of thickened tori.

If M is a compact 3-manifold with incompressible boundary, we let $\partial_0 M$ be the non-torus boundary components of M.

A hyperbolic structure on the interior of M induces a conformal structure on the torus components of ∂M, described by its *modulus*. Let T be a torus in ∂M and N a hyperbolic structure on the interior of M. Then the fundamental group of T as a subgroup of $\pi_1(N)$ is free-abelian of rank 2, and hence is conjugate to $< z \mapsto z+1, z \mapsto z+\tau >$ for some $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$. The *modulus* $\operatorname{mod}(T)$ of T is the modulus of a rectangle which forms a fundamental domain for the action of $\pi_1(T)$ on $\mathbb{C}$: $\operatorname{mod}(T) = \operatorname{Im} \tau$.

Proposition 5 *Let K be a compact orientable hyperbolizable 3-manifold with maximal incompressible core M. Let α be a simple closed curve in M which is isotopic to a simple closed curve in $\partial_0 M$. Set $\hat{K} = K \setminus \mathcal{N}(\alpha)$, and let $T = \partial \mathcal{N}(\alpha)$. Then there is a constant $L = L(K)$ so that for all $\rho \in AH(\hat{K})$, the modulus of T in N_ρ is at least L.*

Proof

Let S be the component of $\partial_0 M$ into which α is isotopic. For any $\rho \in AH(\hat{K})$ there is an induced representation in $AH(S \times [-1, 1] \setminus \mathcal{N}(\alpha))$, and the modulus of the torus in this structure is the same as in N_ρ. Thus we may assume that $\hat{K} = S \times [-1, 1,] \setminus \mathcal{N}(\alpha))$, for S a surface of negative Euler characteristic.

Moreover, since the modulus of the torus T is a continuous function on $AH(\hat{K})$, it suffices to establish the theorem for $MP(\hat{K})$, the interior of $AH(\hat{K})$.

For any $\rho \in MP(\hat{K})$ there is path of simplicial hyperbolic surfaces $h_t : S \longrightarrow N_\rho$ in the homotopy class of the inclusion $S \longrightarrow S \times \{1\}$. For any t there is a collection of curves $\beta_1(t), \ldots, \beta_n(t)$ defining a pair-of-pants decomposition of S such that the length (in the metric induced by h_t) of each $\beta_j(t)$ is bounded above by a constant depending only on the Euler characteristic of S.

Let t be such that some $\beta_j(t)$ must be in the homotopy class of α, and denote this curve by α.

If α is contained in the boundary of exactly one pair-of-pants then α is contained in a subsurface S_1 of S homeomorphic to $S_{1,1}$, the genus one surface with one boundary component. If α is contained in the boundary of two pairs of pants, then α is contained in a subsurface S_2 of S homeomorphic to $S_{0,4}$, the genus zero surface with four boundary components. For $i = 1, 2$ let $\hat{K}_i = S_i \times [-1, 1] \setminus \mathcal{N}(\alpha)$, and consider T as a torus in $\hat{K}$, $\hat{K}_1$, and $\hat{K}_2$.

Hence from representation $\rho \in MP(\hat{K})$ we obtain a geometrically finite representation ρ' in $AH(\hat{K}_i)$, for $i = 1$ or 2, where the boundary of S_i is represented by curves of uniformly bounded length, and the modulus of T in $N_{\rho'}$ is equal to the modulus of T in N_ρ. Moreover, a compact core for $N_{\rho'}$ embeds in N_ρ under the obvious covering map.

Let ρ_i be a sequence in $MP(\hat{K})$ for which the modulus of T in N_{ρ_i} tends towards a minimum. We wish to show that this minimum cannot be zero. As in the discussion above, we obtain a sequence $\rho'_i \in AH(\hat{K}_{j_i})$ $j_i = 1$ or 2, with the modulus of T tending towards the same minimum. By passing to a subsequence, re-labelled as ρ'_i, we may assume that j_i is constant, k. We remark that we may assume that the length of α on the conformal boundary of $N_{\rho'_i}$ is bounded below (if α is represented in the conformal boundary), as otherwise the modulus of T is unbounded.

We claim that the sequence $\{\rho'_i\}$ converges algebraically. The sequence of representations $\{\rho'_i|_{\pi_1(S_k \times \{1\})}\}$ has a convergent subsequence by an application of either lemma 3 or lemma 4. Since $\pi_1(\hat{K}_k)$ is generated by $\pi_1(S_k)$ and $\pi_1(T)$ it is sufficient to show that ρ'_i converges on $\pi_1(T)$. We normalize so that the representation ρ'_i restricted to the fundamental group of the torus boundary component of $\hat{K}_k$ has image $< z \mapsto z + 1, z \mapsto z + \tau_i >$. We may assume that the real part of τ_i is at least zero and at most 1, and that the imaginary part of τ_i is non-negative. The assumption that the modulus of T in $N_{\rho'_i}$ is approaching a minimum implies that the imaginary part of τ_i eventually lies in a compact set. In particular we can can conclude that (up to subsequence) $\{\tau_i\}$ converges algebraically to say τ_∞ where the imaginary part of τ_∞ is the minimum modulus value. Together, algebraic convergence of $\{\rho'_i\}$ restricted to $\pi_1(S_k \times \{1\})$ and algebraic convergence of $\{\rho'_i\}$ restricted to the fundamental group of the boundary torus of $\hat{K}_k$ implies algebraic convergence of the sequence of full representations $\{\rho'_i : \pi_1(\hat{K}_k) \to PSL_2(\mathbb{C})\}$.

Each of the approximates is discrete and faithful, and hence so is the algebraic limit. In particular the restriction of the algebraic limit to $\pi_1(T)$ is a faithful representation of $\mathbb{Z} \oplus \mathbb{Z}$, which implies that the imaginary part of τ_∞, which is the modulus of T, is non-zero.

□

4 Non-wrapping

In this section we prove some lemmas that will be needed in the proof of Theorem 11. We begin by setting some notation that will be used throughout. Let N_ρ and $\hat{N}$ be as in the previous sections and $\pi : N_\rho \longrightarrow \hat{N}$ the covering map. Q will denote an annulus cusp in N_ρ which covers a torus cusp $T = \pi(Q)$. We suppose that there is an accidental parabolic δ corresponding to Q in ∂N_ρ and conjugate $\Gamma = \rho(\pi_1(S))$ so that $\pi_1(Q) =< z \mapsto z + 1 >$. With this normalization, Q belongs to the family of cusp neighbourhoods $\{H_c\}$, where H_c is covered by the horoball $\{(z, t) \mid t \geq c\}$.

Lemma 6 *Let γ be a simple closed curve in ∂N_ρ. If l denotes the length in N_ρ of the geodesic representative γ^* of γ, then*

$$\gamma^* \cap H_c = \emptyset$$

where $c = \sinh(l/2)$.

Proof

Lift γ^* to a geodesic $\tilde{\gamma}$ in $\mathbb{H}^3$. There are two disjoint half-spaces, C_1 and C_2, of equal Euclidean radius r, so that the intersection of $\tilde{\gamma}$ with the complement of C_1 and C_2 is a fundamental domain β for the action of $\rho([\gamma])$ on $\tilde{\gamma}$. If we consider $\tilde{\gamma}$ as lying in a copy of the upper-half-space, then we can parameterize β by $\theta \mapsto (R\cos(\theta), R\sin(\theta))$, $0 < \theta_1 \leq \theta \leq \pi - \theta_1$, for some $0 < \theta_1 < \pi/2, R > 0$. By bounding R we will obtain our result.

Elementary hyperbolic geometry gives us that

$$\frac{l}{2} = \ln\left[\frac{\sin(\theta_1)}{1-\cos(\theta_1)}\right]$$

Hence

$$\cos(\theta_1) = \frac{e^l - 1}{e^l + 1}$$

On the other hand, since $\tilde{\gamma}$ meets the boundary of C_i, $i = 1, 2$, in a right angle, the radius r of C_i is related to R by $R = r\cot(\theta_1)$. No Kleinian group containing $z \mapsto z+1$ can have $r > 1$, so $R \leq \cot(\theta_1)$.

Equivalently,

$$R \leq \sinh(l/2)$$

□

Lemma 7 *Let $T = \pi(H_c)$ be a cusp neighbourhood in $\hat{N}$ and suppose that $\operatorname{mod}(T) > 0$. Let γ^* be a geodesic in N_ρ whose projection to the conformal boundary is a curve $\gamma \subset \partial N_\rho$ transverse to δ. Suppose that $\pi(\gamma^*) \cap H_c = \emptyset$ and let l be the length of γ^*.*

If $\pi(\gamma^*)$ wraps $n > 0$ times around H_c then

$$e^{l/2}\tanh(l/2) = \frac{e^l - 1}{e^l + 1} \geq \frac{n \operatorname{mod}(T)}{2c}$$

.

Proof

As in the proof of lemma 6, if we parameterize the fundamental domain β by $\theta \mapsto R(\cos(\theta), \sin(\theta))$, $0 < \theta_1 \leq \theta \leq \pi - \theta_1$, $\theta_1 < \pi/2$, $R > 0$, then we have $n \operatorname{mod}(T) \leq 2R\cos(\theta_1)$, and $l = \ln\left[\frac{1+\cos(\theta_1)}{1-\cos(\theta_1)}\right]$. The result follows after noting that by assumption $R < c$.

□

5 The main results

For the remainder of the paper we specialize to the case where $M = S \times I$, for S a surface of negative Euler characteristic.

5.1 Geometric convergence and relative compact carriers

We say that a sequence of hyperbolic 3-manifolds with basepoint (N_n, x_n) converges *geometrically* to a hyperbolic 3-manifold with basepoint (N, x) provided that there exists two sequences of numbers, $R_n > 0, K_n \geq 1$, and K_n-biLipshitz diffeomorphisms $f_n : B(x, R_n) \longrightarrow N_n$, where $f_n(x) = x_n$ and $R_n \to \infty$, $K_n \to 1$, as n tends to infinity. (Here $B(x, R)$ is the ball in N centered at x of radius R). We call N the *geometric limit* of the sequence N_n, suppressing the choice of basepoints.

The above definition of geometric convergence is equivalent to convergence of the fundamental groups of the hyperbolic manifolds in the Hausdorff topology on closed subsets of $PSL_2(\mathbb{C})$. In this setting, a sequence of Kleinian groups Γ_n converges geometrically to Γ provided that:

1. Every $\gamma \in \Gamma$ is the limit of a sequence $\{\gamma_n \mid \gamma_n \in \Gamma_n\}$;
2. Every accumulation point of any sequence $\{\gamma_n \mid \gamma_n \in \Gamma_n\}$ is an element of Γ.

For a sequence $\rho_n \in AH(M)$ converging algebraically to $\rho \in AH(M)$ we say that the sequence ρ_n converges geometrically provided that the images $\rho_n(\pi_1(S))$ converge geometrically.

Jørgensen and Marden [21] proved that every algebraically convergent sequence has a subsequence which converges geometrically. Thus the algebraic limit covers the geometric limit. The purpose of this paper is to study this covering. In particular we want to show that a certain condition on the accidental parabolics in the algebraic limit ensures that there is a compact core for the algebraic limit which embeds in the geometric limit under the covering map. We rely strongly on work by Anderson-Canary [5], and Anderson-Canary-McCullough ([7]). In the former citation, Anderson and Canary show that when the algebraic limit contains no accidental parabolics then there is such a compact core. The track of a homotopy between an accidental parabolic and the paring locus of a relative compact core for the algebraic limit provides an essential, properly embedded annulus. Decomposing the relative compact core along these annuli (one for each accidental parabolic) produces a set of sub-manifolds whose fundamental groups inject into the fundamental group of the algebraic limit. Ideally these submanifolds would embed in the geometric limit under the covering map, and then we would proceed to argue that our condition on the accidental parabolic allows us to sew together the pieces in both the algebraic and geometric limits to produce the desired core. In practice it is not so straightforward as this and we need the technology of *relative compact carriers* introduced by Anderson, Canary and McCullough in [7].

Let Γ be a Kleinian group and let $\mathcal{H}$ be a maximal collection of Γ-invariant horoballs based at the fixed points of parabolic elements of Γ. Let Γ_1 be a subgroup of Γ, and let $\mathcal{H}_1$ be a maximal collection of Γ_1-invariant horoballs based at the fixed points of the parabolic elements of Γ_1 with $\mathcal{H}_1 \subset \mathcal{H}$. Set $N_1^0 = (\mathbb{H}^3 \setminus \mathcal{H}_1)/\Gamma_1$, and $N^0 = (\mathbb{H}^3 \setminus \mathcal{H})/\Gamma$. Let $q : \mathbb{H}^3/\Gamma_1 \longrightarrow \mathbb{H}^3/\Gamma$ be the covering map. If R' is a relative compact core for N_1^0 and q is injective on R' then we say that $R = q(R')$ is a *relative compact carrier* for Γ_1.

5.2 Decomposing the relative compact core: the analysis of Anderson, Canary and McCullough

Let ρ_n converge algebraically to ρ and geometrically to $\hat{\Gamma}$. Set $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ and let $\pi : N_\rho \longrightarrow \hat{N}$ be the covering map. Let $\hat{\mathcal{H}}$ be a maximal collection of $\hat{\Gamma}$-invariant horoballs centered at the fixed points of parabolic subgroups of $\hat{\Gamma}$; let $\mathcal{H} \subset \hat{\mathcal{H}}$ be the analogous collection for $\Gamma = \rho(\pi_1(S))$. Set $\hat{N}^0 = (\mathbb{H}^3 \setminus \hat{\mathcal{H}})/\hat{\Gamma}$ and $N_\rho^0 = (\mathbb{H}^3 \setminus \mathcal{H})/\Gamma$; let (M_0, P_0) be a relative compact core for N_ρ^0.

An accidental parabolic gives rise to an essential, properly embedded annulus A in M_0, so that one component of ∂A is in P_0 and the other is in $\partial M_0 \setminus P_0$. Let $\mathcal{A}$ be a maximal, pair-wise disjoint collection of such annuli. Decompose M_0 by removing an open regular neighbourhood $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ from M_0. The fundemamental groups of the components of $M_0 \setminus \mathcal{N}(\mathcal{A})$ defines a collection $\Gamma_1, \ldots, \Gamma_k$ of subgroups of $\Gamma = \rho(\pi_1(S))$.

Lemma 8 *(pages 32-35 of [7])*
There exists a pair-wise disjoint collection of relative compact carriers $\{\hat{Y}_1, \ldots, \hat{Y}_k\}$ *for* $\{\Gamma_1, \ldots, \Gamma_k\}$ *in* $\hat{N}$.

It is clear from the definition that there is a disjoint collection of relative compact carriers $\{Y_1, \ldots, Y_k\}$ in N_ρ, so that $\pi(Y_i) = \hat{Y}_i$, $i = 1, \ldots, k$.

We wish to piece together these relative compact carriers to produce a compact core for N_ρ, and we wish to do so by adding solid tori in cusp neighbourhoods of N_ρ, for in these regions we have control over the covering map into $\hat{N}$. Thus, following [7], we enumerate the cusps (the components of $Fr(N_\rho^0)$) as

$$Q_1, \ldots, Q_k, Q_{k+1}, \ldots, Q_l$$

where

1. For $1 \leq i \leq k$ Q_i is an infinite annulus which covers a torus cusp in $\hat{N}$
2. For $k+1 \leq i \leq l$ Q_i is an infinite annulus which covers a rank-1 cusp in $\hat{N}$.

Note that in every case $(\bigcup Y_i) \cap Q_j$ has at most two components, since we have specialized to the surface bundle case. In fact, this is the only point at which we need to assume that M has that form.

If C is a component of $\mathcal{H}/\Gamma$ there is a homeomorphism $r : C \longrightarrow \partial C \times [0, \infty)$ whereby $r(x) = (y, t)$ if y is the nearest point to x on ∂C and t is the hyperbolic distance between x and y. For a submanifold Z of ∂C let $\mathcal{C}(Z) = r^{-1}(Z \times [0, 1])$.

For each i, let C_i be the minimal annulus in Q_i containing $(\bigcup Y_j) \cap Q_i$. Set $M_1 = (\bigcup Y_i) \cup \bigcup \mathcal{C}(C_j)$.

Let γ_i denote the accidental parabolic in N_ρ corresponding to C_i. If $\Gamma_j = \pi_1(Y_j)$, then there is j, k so that Γ_{γ_i} is an amalgram of Γ_j and Γ_k. If $\pi(C_i)$ is a torus with fundamental group $< \rho(\gamma_i), \delta_i >$ then can be written $\hat{\Gamma} = \hat{\Gamma}_0 *_{\delta_i}$. With this we recall the definition of the wrapping of γ_i in $\hat{N}$.

The *wrapping of* γ_i in $\hat{N}$ is the smallest n such that π_{γ_i} factors through $\pi : \mathbb{H}^3/\Gamma_{\gamma_i} \longrightarrow \bar{N}$ and $\bar{\pi} : \bar{N} \longrightarrow \hat{N}$ so that

1. There is a compact core K for $\mathbb{H}^3/\Gamma_{\gamma_i}$ so that $\pi|_K$ is an embedding
2. $\pi_1(\bar{N}) = \hat{\Gamma}_0 *_{\delta_i^{n+1}}$.

Reorder the cusps $Q_1, \ldots, Q_k$ by

$$Q_1, \ldots, Q_s, Q_{s+1}, \ldots Q_k$$

so that

1. For $1 \leq i \leq s$, γ_i has wrapping $p_i > 0$ in $\hat{N}$.

2. For $s+1 \leq i \leq k$, γ_i has wrapping 0 in $\hat{N}$.

Note that if $(\bigcup Y_i) \cap C_j$ has only one component then γ_i has wrapping 0 in $\hat{N}$.

Proposition 9 *(pages 36-37 of [7])*
M_1 is a compact core for N_ρ. The covering map $\pi : N_\rho \longrightarrow \hat{N}$ is injective on $M_1 \setminus \bigcup_{i=1}^{s} \mathcal{C}(C_i)$.

5.3 Proof of main theorem

If $\bigcup Y_j \cap C_i$ has two components then the complement $C_i \setminus \bigcup Y_j \cap C_i$ is an annulus whose core curve corresponds to an accidental parabolic γ_i in ∂N_ρ. Let $\mathrm{t}(\gamma_i)$ be the infimum of lengths (in the Poincare metric) of simple closed curves in ∂N_ρ which meet γ_i transversely. The main theorem follows from Proposition 9 and the following proposition.

Theorem 10 *There is a constant $L = L(S) > 0$ so that if γ is an accidental parabolic in ∂N_ρ and $\mathrm{t}(\gamma) \leq L$ then γ has wrapping 0.*

We will prove the following proposition, from which Proposition 10 follows as an immediate corollary.

Theorem 11 *Let n_0 be a non-negative integer. Then there is a constant $L = L(n_0, S)$ so that if γ is an accidental parabolic in ∂N_ρ and $\mathrm{t}(\gamma) \leq L$ then γ has wrapping at most n_0.*
The constant L grows logarithmically with n_0.

Proof
Let $\hat{N}$ be any geometric limit associated to a sequence in $AH(S)$ converging to ρ, and adopt the notation used in the previous section. Let $L = L(\pi_1(S)) > 0$ be the constant given by Proposition 5. Suppose that γ is an accidental parabolic in N_ρ and that γ has wrapping n in $\hat{N}$.

Let δ be a curve in N_ρ which is transverse to the accidental parabolic γ. The representative of δ on the boundary of the convex core of N_ρ has length at most K_0 times the length of δ, for a universal K_0 (see [19]). Thus the length of the geodesic representative δ^* of δ in N_ρ is at most $l = K_0 length_{\partial N_\rho}(\delta)$.

By Lemma 6, the geodesic δ^* fails to intersect the cusp neighbourhod corresponding to H_c, with $c = \sinh(l/2)$.

By Lemma 7, the length l of δ^* satisfies the inequality

$$\frac{e^l - 1}{e^l + 1} \geq \frac{nL}{2\sinh(l/2)}$$

Thus by choosing $length_{\partial N_\rho}(\delta)$ sufficiently small we can force n to be less than or equal to n_0.

□

The following follows immediately from proposition 9.

Corollary 12 *There is a constant $L = L(S)$ so that if $t(\gamma) < L$ for all accidental parabolics γ in ∂N_ρ, then ρ does not wrap.*

Note that when S is the once-punctured torus, having a short curve transverse to an accidental parabolic is the same as having a long accidental parabolic.

Corollary 13 *Let S be the once-punctured torus. There is a constant C so that $\rho \in AH(S)$ does not wrap provided that for any accidental parabolic $\gamma \subset \partial N_\rho$ we have $length_{\partial N_\rho}(\gamma) > C$.*

By [20] non-wrapping implies non-self-bumping. Hence we obtain our final corollaries.

Corollary 14 *Let S be the once-punctured torus. There is a constant C so that if $\rho \in AH(S)$ is such that any accidental parabolic has length at at least C then ρ is not a point of self-bumping.*

Corollary 15 *Let S be the once-punctured torus. Then the points of non-local-connectivity in $\partial AH(S)$ are not dense.*

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