Argument Principle

Zeroes and Poles

For the moment, we shall consider a function $f(z)$ analytic in the *punctured disk*

$$\dot{D}_{z_0,R} = \{z\,|0 < |z - z_0| \le R\}\,.$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z - z_0)^n\,,$$
$$a_n = \frac{1}{2\pi i}\oint_{C_{z_0,r}} f(\zeta)\,(\zeta - z_0)^{-n-1}\,d\zeta.$$

- If $f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$, $f(z)$ may be extended by defining $f(z_0) = a_0$, and the resulting function is analytic in $|z - z_0| \le R$.
- If $f(z) = \sum_{n=N}^{\infty} a_n\,(z - z_0)^n$, $N \ge 0$, $a_N \ne 0$, $f(z)$ is said to have a *zero of order* N at $z = z_0$. Near $z = z_0$,
$$f(z) = (z - z_0)^N \cdot g(z),$$
where $g(z)$ is analytic in $|z - z_0| \le R$, $g(z_0) \ne 0$.
- If $f(z) = \sum_{n=-M}^{\infty} a_n\,(z - z_0)^n$, $M \ge 0$, $a_{-M} \ne 0$, $f(z)$ is said to have a *pole of order* M at $z = z_0$. Near $z = z_0$,
$$f(z) = (z - z_0)^{-M} \cdot g(z),$$
where $g(z)$ is analytic in $|z - z_0| \le R$, $g(z_0) \ne 0$.
- If $f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z - z_0)^n$, $a_n \ne 0$ for infinitely many negative n, then $f(z)$ is said to have an *essential singularity* at $z = z_0$.
- The coefficient of $(z - z_0)^{-1}$ is called the *residue* of $f(z)$ at $z = z_0$, and is written
$$\mathrm{Res}(f, z = z_0) = \mathrm{Res} f(z)|_{z=z_0} = \frac{1}{2\pi i}\oint_{C_{z_0,r}} f(\zeta)\,d\zeta.$$
- Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_{z_0,R} = \{z\,|0 < |z - z_0| \le R\}\,.$$
Then for r small and positive,
$$\oint_{C_{z_0,r}} f(\zeta)\,d\zeta = 2\pi i\,\mathrm{Res} f(z)|_{z=z_0}\,.$$
- Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_{z_0,R} = \{z\,|0 < |z - z_0| \le R\}\,.$$

Suppose that $f(z)$ has a zero of order $N > 0$, at $z = z_0$.
For z near z_0, $f(z) = (z - z_0)^N g(z)$, $g(z)$ analytic, and $g(z_0) \neq 0$. It follows that

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{N (z - z_0)^{N-1} g(z) + (z - z_0)^N g'(z)}{(z - z_0)^N g(z)} \\ &= N (z - z_0)^{-1} + \text{analytic}, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{Res}\left(\frac{f'}{f}, z = z_0\right) &= N \\ &= \text{order of zero at } z = z_0 \end{aligned}$$

Then for r small and positive,

$$\frac{1}{2\pi i} \oint_{C_{z_0,r}} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = N.$$

There is another interpretation of the number N. For the moment let $f_N(z) = (z - z_0)^N$. Follow the $\arg f_N(z)$ as $C_{z_0,r}$ is traversed in the counterclockwise direction. The change in argument of $(z - z_0)^N$, denoted by $\Delta_{C_{z_0,r}} \arg f_N(z)$ is exactly $2\pi N$. This is the first statement of the *Argument Principle*:

$$\begin{aligned} \frac{1}{2\pi} \Delta_{C_{z_0,r}} \arg f_N(z) &= N \\ &= \text{order of zero.} \end{aligned}$$

In the case $f(z)$ has a zero of order N at $z = z_0$, we expect that an antiderivative of the function $\dfrac{f'(z)}{f(z)}$ is $\log(f(z))$. This is the case locally, at least if we are near enough to a point z_1 on $C_{z_0,r}$. As the path $C_{z_0,r}$ is traversed counterclockwise, the logarithm of $f(z)$ may be defined locally in a continuous manner, but when we make one full revolution around the circle returning to z_1, the argument of $f(z)$ may have changed by a multiple of 2π. We have that

$$\begin{aligned} \oint_{C_{z_0,r}} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta &= i \cdot \Delta_{C_{z_0,r}} \arg f(z) \\ &= 2\pi i \cdot N. \\ &= 2\pi i \cdot \text{order of zero at } z = z_0. \end{aligned}$$

Thus for the small circle $C_{z_0,r}$,

$$\begin{aligned} N &= \frac{1}{2\pi} \Delta_{C_{z_0,r}} \arg f(z) \\ &= \frac{1}{2\pi i} \oint_{C_{z_0,r}} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta \end{aligned}$$

- Let $f(z)$ be analytic in the *punctured disk*

$$\dot{D}_{z_0,R} = \{z\,|0 < |z - z_0| \leq R\}.$$

Suppose that $f(z)$ has a pole of order $M > 0$, at $z = z_0$.
Then for r small and positive,

$$\frac{1}{2\pi i} \oint_{C_{z_0,r}} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = -M.$$

Mimicking the discussion above for zeroes, we obtain for the small circle $C_{z_0,r}$

$$\begin{aligned} -M &= \frac{1}{2\pi} \Delta_{C_{z_0,r}} \arg f(z) \\ &= \frac{1}{2\pi i} \oint_{C_{z_0,r}} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta \end{aligned}$$

We are now ready to state the *Argument Principle*.

Theorem. *Let C be a simple closed path. Suppose that $f(z)$ is analytic and nonzero on C and meromorphic inside C.*

- *List the zeroes of f inside C as $z_1, \ldots, z_k$ with multiplicities $N_1, \ldots, N_k$, and let*

$$Z_C = N_1 + \ldots + N_k.$$

- *List the poles of f inside C as $w_1, \ldots, w_j$ with orders $N_1, \ldots, N_k$, an let*

$$P_C = M_1 + \ldots + M_j.$$

Then

$$\begin{aligned} Z_C - P_C &= \frac{1}{2\pi} \Delta_C \arg f(z) \\ &= \frac{1}{2\pi i} \oint_C \frac{f'(\zeta)}{f(\zeta)}\,d\zeta. \end{aligned}$$

Proof. calculate the integral two ways. First take a local antiderivative $\log(f(z))$ to obtain

$$\frac{1}{2\pi i} \oint_C \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \frac{1}{2\pi} \Delta_C \arg f(z).$$

Second take small circles around each z_i and w_i and the usual cuts from C to to the circles. In this way, obtain

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \sum_{i=1}^{j} \frac{1}{2\pi i} \oint_{C_{z_i,r}} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta + \sum_{i=1}^{k} \frac{1}{2\pi i} \oint_{C_{w_i,r}} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta \\ &= \sum_{i=1}^{j} N_i - \sum_{i=1}^{k} M_i \\ &= Z_C - P_C. \end{aligned}$$

Corollary. *Let C be a simple closed path. Suppose that $f(z)$ is analytic and nonzero on C and analytic inside C.*

- *List the zeroes of f inside C as $z_1, \ldots, z_k$ with multiplicities $N_1, \ldots, N_k$, an let*

$$Z_C = N_1 + \ldots + N_k.$$

Then

$$\begin{aligned} Z_C &= \frac{1}{2\pi} \Delta_C \arg f(z) \\ &= \frac{1}{2\pi i} \oint_C \frac{f'(\zeta)}{f(\zeta)}\, d\zeta. \end{aligned}$$

Briefly stated: Let C be a simple closed path. Suppose that $f(z)$ is analytic and nonzero on C and analytic inside C. Then

$$\frac{1}{2\pi} \Delta_C \arg f(z) = \text{number of zeroes inside } C \text{ – counting multiplicities.}$$

Indices and Winding Numbers

Let C be a simple closed path. Suppose that $f(z)$ is analytic and nonzero on C and meromorphic inside C. Then $w = f(z) = f(z(t))$ is a closed path (not necessarily simple). Call this path $f(C)$. As w traverses $f(c)$, the number of times the argument of w changes by a multiple of 2π is called the *index* or *winding number* of the path $f(C)$. The **Argument Principle** says that the winding number of $f(C)$ is $Z_C - P_C$.