Differentiability and Derivatives

A real valued function of a real variable \( x \) is differentiable at \( x \) with derivative \( f'(x) \) if

\[
f(x + \Delta x) = f(x) + f'(x)\Delta x + o(\Delta x).
\]

We consider several types of functions:

**Curves**: Let \( z(t) = x(t) + iy(t) \) be a path (a complex valued function of a real variable \( t \)). Then \( z(t) \) is differentiable at \( t \) with derivative \( z'(t) \) if

\[
z(t + \Delta t) = z(t) + z'(t)\Delta t + o(\Delta t).
\]

In this case \( z'(t) = \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} \). Moreover, the derivative can be calculated as the limit of difference quotients:

\[
\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}.
\]

**Complex functions of a complex variable** \( z \): Let \( f(z) \) be defined in an open set. Then \( f \) is differentiable at \( z \) with derivative \( f'(z) \) if

\[
f(z + \Delta z) = f(z) + f'(z)\Delta z + o(\Delta z).
\]

The derivative can be calculated as the limit of difference quotients:

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.
\]

While the formal definitions are the same, the requirements are quite different. If the variable, e.g., \( x \), is real, there are only two ways that \( \Delta x \to 0 \) – from the right (\( \Delta x > 0 \)) or from the left (\( \Delta x < 0 \)). In the complex variable case, \( |\Delta z| \to 0 \), but \( \Delta z \) is allowed to have random directions as \( \Delta z \to 0 \). The existence of a limit \( \lim_{\Delta z \to 0} \) is stronger than the concept \( \lim_{\Delta x \to 0} \).

If \( f \) is a differentiable function of the complex variable \( z \) in an open set or region, \( f(z) \) is also called an analytic or holomorphic function.

**Real [Complex] Functions of Two Variables** \( (x,y) \): There is another concept of differentiability of functions of two (or more) variables \( (x,y) \). For simplicity write \( P = (x,y) \) and \( \Delta P = (\Delta x, \Delta y) \). Then a real (or complex) valued function \( G \) is differentiable at \( P \) if

\[
G(P + \Delta P) = G(P) + \text{linear function of } \Delta P + o(\Delta P).
\]

---

1 See Knopp, *Elements of the Theory of Functions*, p. 101, and Knopp, *Theory of Functions, Part I*, p. 27. Thus if \( f(z) \) is analytic at a point \( z_0 \), \( f(z) \) is actually analytic in a neighborhood of \( z_0 \).
All linear functions of $\Delta P = (\Delta x, \Delta y)$ are of the form $a \cdot \Delta x + b \cdot \Delta y$. The numbers $a$ and $b$ can be calculated as the partial derivatives of $G$:

$$
a = \frac{\partial G}{\partial x} = \lim_{\Delta x \to 0} \frac{G(x + \Delta x, y) - G(x, y)}{\Delta x} ,
$$

$$
b = \frac{\partial G}{\partial y} = \lim_{\Delta y \to 0} \frac{G(x, y + \Delta y) - G(x, y)}{\Delta y} .
$$

The derivative of $G$ at $P$ is then the (complex) pair $< \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} >$ so that

$$G(P + \Delta P) = G(P) + \frac{\partial G}{\partial x} \cdot \Delta x + \frac{\partial G}{\partial y} \cdot \Delta y + o(|\Delta P|) .$$

The [complex] pair $< \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} >$ is called the [complex] gradient of $G$ and is written grad $G$ or $\nabla G$.

The formal expression $\frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$ is also called the differential $dG$ of the function $G(x, y)$. We write

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy .$$

**Back to Analytic (Holomorphic) Functions:** That a differentiable function $G$ of two real variables $P = (x, y)$ is a differentiable function of the complex variable $z$ or analytic (holomorphic) reduces to the statement:

If $P = (x, y)$, $z = x + iy$, there is a complex number $G'(z)$ such that

$$G'(z) \cdot (\Delta x + i \Delta y) = \frac{\partial G}{\partial x} \cdot \Delta x + \frac{\partial G}{\partial y} \cdot \Delta y .$$

The complex number $G'(z)$ can be calculated several ways:

- Let $\Delta y = 0$ and $\Delta x \neq 0$:

$$G'(z) = \frac{\partial G}{\partial x}$$

- Let $\Delta y \neq 0$ and $\Delta x = 0$:

$$iG'(z) = \frac{\partial G}{\partial y}, \text{ or } G'(z) = -i \frac{\partial G}{\partial y}$$

- $G'(z) = \frac{\partial G}{\partial z} \overset{\text{def}}{=} \frac{1}{2} \left( \frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y} \right)$

The formal expression $\frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$ is also called the differential $dG$ of the function $G(x, y)$. If $G$ is a differentiable function of the complex variable $z$, we can write formally:

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy = \frac{dG}{dz} dz = \frac{dG}{dz} dz .$$

---

The words holomorphic and analytic are used interchangeably.
The precise meaning of the statement \( dG = \frac{dG}{dz} \, dz \) is that

\[
G(z + \Delta z) - G(z) = \frac{dG}{dz} \cdot \Delta z + o(\Delta z).
\]

**Remarks on Analyticity and Partial Differential Equations (PDE)**

If \( G(x, y) \) is real differentiable,

\[
dG = \frac{\partial G}{\partial x} \, dx + \frac{\partial G}{\partial y} \, dy,
\]

\[
dx = \frac{1}{2} (dz + d\bar{z}),
\]

\[
dy = \frac{1}{2i} (dz - d\bar{z}),
\]

so that

\[
dG = \frac{1}{2} \left( \frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y} \right) \, dz + \frac{1}{2} \left( \frac{\partial G}{\partial x} + i \frac{\partial G}{\partial y} \right) \, d\bar{z}
\]

\[
= \frac{\partial G}{\partial z} \, dz + \frac{\partial G}{\partial \bar{z}} \, d\bar{z}.
\]

Thus the analytic functions are the real differentiable functions \( G(x, y) \) which satisfy the partial differential equation

\[
\frac{\partial G}{\partial z} = \frac{1}{2} \left( \frac{\partial G}{\partial x} + i \frac{\partial G}{\partial y} \right) = 0.
\]